Fluctuations of 1 lead to a change of the macroscopic ground state: a static stochastic magnetic structure appears, which can be investigated by electron-optical methods. Use of such supplementary methods can appreciably increase the informational content of the SWS method.

The greatest difficulty of the theory of SWS lies in allowance for the magnetic dipole fields. This problem has so far been solved in analytical form only for fluctuations of α;[2] for fluctuations of β and 1, the shift of the uniform FMR frequency has been calculated.

Expressions obtained without allowance for the magnetic dipole fields are useful only for qualitative interpretation of experiment.

[1)]As has already been mentioned,[7] fluctuations of β and 1 describe not only inhomogeneity of the crystallographic anisotropy, but also any inhomogeneities whose effect can be approximately described by a term $\beta(\mathbf{M})^2$ in the phenomenological Hamiltonian (internal elastic stresses, inclusions, etc.).

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State density in a one-dimensional disordered system in the two-band approximation

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An exact solution is obtained of the problem of the density of one-electron states for two models of a disordered semiconductor that is described by a system consisting of two first-order equations and corresponding to the two-band approximation. In the first model, where the disorder is produced by a random impurity potential of a definite type, the state density has no singularities when the gap collapses. In the second model, where the fluctuation parameter is the gap width, the state density can have a singularity at the center of the forbidden band, when the fluctuations of the gap are large enough. Asymptotic formulas are obtained for the state densities in characteristic sections of the spectrum and at the most interesting values of the parameters.

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INTRODUCTION

The interest presently shown in the spectra of one-dimensional disordered systems, with allowance for the band structure, is natural. Up to now, the equivalent-mass approximation was used most treatments of the electron spectrum or of the equivalent concept of the structure of the unperturbed (ordered) crystal for other types of excitations (see, e.g., Refs. 1–5 and the bibliographies therein). Yet situations exist when allowance for the periodicity of the initial ordered system, meaning also the band structure of the bare spectrum, is important. This problem arises, in particular, when an attempt is made to explain some observed singularities of physical quantities (for example, the low-temperature behavior of the magnetic susceptibility[6]) in quasi-one-dimensional compounds.[7-9] Since full allowance for the band structure entails great difficulties, it is natural to turn to the simplest case of two broad resolved bands with a narrow gap between them. In this case (see Ref. 10) the spectrum and the states of the quasiparticles in the vicinity of the gap in the presence of a random potential are described by a system of two first-order equations of the Dirac type.

This paper consists of two parts. In the first (Secs. 1 and 2) we derive a number of relations that hold for an arbitrary random potential, and calculate the state density within the framework of the indicated equations for a model in which the potential is a sequence of rec-

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tangular barriers of fixed height with an exponential distribution of the random lengths of the barriers and of the distances between them.[1)] In the second part we solve the analogous problem for a somewhat different model, in which the fluctuating parameter is the gap width. Such a model was considered in Ref. 8 for the case when the gap fluctuation is Gaussian white noise.

1. DERIVATION OF THE BASIC EQUATIONS

1. The system of equations that define the electron spectrum takes in the considered approximation the form[10]

$$\begin{aligned} -i\frac{d\psi_1}{dx}+(v+\Delta-E)\psi_2=0,\\ -i\frac{d\psi_2}{dx}+(v-\Delta-E)\psi_1=0. \end{aligned} \tag{1}$$

Here v is the impurity potential, E is the energy reckoned from the center of the gap, Δ is the half-width of the gap (measured in units of $\hbar p/m$, where m is the electron mass),

$$p=\int u_{c0}^{*}(x)\frac{\hbar}{i}u_{v0}'(x)\,dx,$$

and $u_{c,v0}(x)$ are the Bloch amplitudes in the upper and lower bands, respectively.

A similarity transformation reduces the system (1) to

$$\begin{pmatrix} v-i\,d/dx & \Delta \\ \Delta & v+i\,d/dx \end{pmatrix}\varphi=E\varphi. \tag{2}$$

Introducing in lieu of the complex vector φ two real vectors f and g in accord with the formulas[8]

$$f_1=\mathrm{Re}(\varphi_1+\varphi_2),\quad g_1=-\mathrm{Im}(\varphi_1+\varphi_2),$$
$$f_2=\mathrm{Im}(\varphi_1-\varphi_2),\quad g_2=\mathrm{Re}(\varphi_1-\varphi_2),$$

we obtain the following system of equations:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{df}{dx}+\begin{pmatrix} v+\Delta & 0 \\ 0 & v-\Delta \end{pmatrix}f=Ef \tag{3}$$

for the function f, and an identical system for the function g. This means that the state density of the initial problem (1) is equal to double the state density for (3). This system of equations will be considered on the segment $[0,L]$ with zero boundary conditions for the component f_1, followed by a transition to the limit as $L\to\infty$.

We introduce in (f_1,f_2) space the polar coordinates

$$r^2=f_1^2+f_2^2,\quad \mathrm{ctg}\,\varphi=-f_2/f_1,$$

and to define uniquely the phase shift $\varphi(x)$ we must stipulate that it be a continuous function of x. From (3) we find that $\varphi(x)$ satisfies the equation

$$\frac{d\varphi}{dx}=\Phi_E(v(x),\varphi),\quad \Phi_E(v,\varphi)=E-v+\Delta\cos 2\varphi \tag{4}$$

and the initial condition $\varphi(0)=0$, while the eigenvalues of the problem are determined with the aid of the relations

$$\varphi(E,L)=m\pi, \tag{5}$$

where m is an integer.

Differentiation of (4) with respect to E and integration of the resultant linear equation show that $\partial\varphi(E,x)/\partial E>0$ for any $x>0$. This fact, which is the main point of the described formalism, leads directly together with Eqs. (5) for the eigenvalues, to the formula

$$n_L(E_0,E)=\frac{1}{\pi L}(\varphi(E,L)-\varphi(E_0,L)),$$

where $n_L(E_0,E)$ is the number of states per unit length in the interval $[E_0,E]$. Going to the limit as $L\to\infty$ in this formula and using the self-averaging of the limiting number of states, i.e., the fact that the limit

$$n(E_0,E)=\lim_{L\to\infty}n_L(E_0,E)$$

is not random, we find that

$$n(E_0,E)=\lim_{L\to\infty}\frac{\langle\varphi(E,L)\rangle-\langle\varphi(E_0,L)\rangle}{\pi L}. \tag{6}$$

Replacing $\varphi(E,L)$ in this expression by

$$\int_0^L\Phi_E(v(x),\varphi(x))\,dx$$

and recognizing that as $x\to\infty$ the probability density $P_E(v,\varphi,x)$ tends to a stationary limit $P_E(v,\varphi)$, we get

$$\pi n(E_0,E)=\langle\Phi_E(v,\varphi)\rangle_{st}-\langle\Phi_{E_0}(v,\varphi)\rangle_{st}, \tag{7}$$

where $\langle\cdots\rangle_{st}$ denotes averaging over the stationary distribution $P_E(v,\varphi)$.

We introduce now the probability density $P_E(\phi,x)$ referred to the interval $[0,\pi]$ of the phase at the point x:

$$P_E(\phi,x)=\langle\delta_p(\varphi(E,x)-\phi)\rangle, \tag{8}$$

where

$$\delta_p(\varphi)=\sum_n\delta(\varphi-n\pi)$$

is a periodic δ function. Differentiating both sides of this equation with respect to x and using (4), we get

$$\frac{\partial P_E(\phi,x)}{\partial x}+\frac{\partial J_E(\phi,x)}{\partial\phi}=0, \tag{9}$$

where

$$J_E(\phi,x)=\langle\delta_p(\varphi(E,x)-\phi)\,\Phi_E(v(x),\varphi(x))\rangle \tag{10}$$

is the probability flux, and the relation (9) itself is simply the continuity equation and expresses its conservation–constancy of the normalization of $P_E(\varphi,x)$ with the "time" x:

$$\int_0^\pi P_E(\phi,x)\,d\phi=1.$$

As $x\to\infty$, the flux $J_E(\phi,x)$ tends to its limiting value $J(E)$, which is independent of both x (by virtue of the produced spatial homogeneity) and ϕ (according to Eq. (10):

$$\pi J(E)=\lim_{x\to\infty}\int_0^\pi J_E(\phi,x)\,d\phi=\langle\Phi_E(v,\varphi)\rangle_{st}.$$

This relation, together with (7), makes it possible to express $n(E_0,E)$ directly in terms of the stationary probability flux $J(E)$:

$$n(E_0,E)=J(E)-J(E_0). \tag{11}$$

Formulas (6) and (7) obtained above for the number of

states are not always the most convenient for actual calculations, since the first contains explicitly the operation of taking the limit, while the second calls for knowledge of the stationary probability density $P_E(v,\varphi)$ followed by calculations of integrals of the type $\int d\varphi dv P_E(v,\varphi)\Phi_E(v,\varphi)$. In addition, it follows from the definition of the probability flux (10) that

$$J(E)=\lim_{x\to\infty} J_E(0,x)=\langle\delta_p(\varphi)\Phi_E(v,\varphi)\rangle_{st}=\int dv\,P_E(v,0)\,\Phi_E(v,0). \qquad (12)$$

Relations (11) and (12) lead to a formula simpler than (7)

$$n(E_0,E)=\langle\Phi_E(v,0)\rangle_{st}-\langle\Phi_{E_0}(v,0)\rangle_{st}, \qquad (13)$$

which makes it possible to calculate the number of states if the stationary distribution is known $P_E(v,\varphi)$ only at $\varphi=0$. This is precisely the formula we shall use hereafter to obtain most results.

2. We assume that the potential $v(x)$ is given by

$$v(x)=v_0 s(x), \qquad (14)$$

where the random process $s(x)$ is equal to zero or unity on intervals x_0 and x_1 having probability densities $a_i^{-1}\exp(-a_i^{-1}x_i)$, where $i=0$ or 1. In this case the stationary distribution

$$P_E(v,\varphi)=P_E(v_0 s,\varphi)\equiv P_s(\varphi)$$

satisfies the Fokker-Planck equation (see Ref. 11)

$$\frac{\partial}{\partial\varphi}[\Phi_s(\varphi)P_s(\varphi)]-a_s^{-1}P_s(\varphi)+a_{1-s}^{-1}P_{1-s}(\varphi)=0,\quad s=0,1, \qquad (15)$$

where

$$\Phi_s(\varphi)\equiv\Phi_E(v_0 s,\varphi)=E-v_0 s+\Delta\cos 2\varphi, \qquad (16)$$

and satisfies in addition the conditions of π-periodicity and normalization

$$P_s(\pi)=P_s(0),\quad \sum_{s=0,1}\int_0^{\pi} d\varphi P_s(\varphi)=1.$$

Before we proceed to solve (15), it will be useful to take into account some symmetry properties of the spectrum of the system (3) with the potential (14). It is easy to verify that each solution (f_1,f_2) of the system, corresponding to the realization $s(x)$, the energy E, the potential amplitude v_0, and the boundary conditions $f_1|_{0,L}=0$, corresponds to a solution (f_2,f_1) of the same system, corresponding either to a realization $s(x)$, an energy $-E$ and an amplitude $-v_0$, or else to a realization $1-s(x)$, an energy v_0-E, and an amplitude v_0 with boundary conditions $f_2|_{0,L}=0$.

Inasmuch as in the limit as $L\to\infty$ the spectrum does not depend on the form of the boundary conditions, the foregoing leads to the following symmetry relations for the limiting state density $\rho(E,v_0,a_0,a_1)$:

$$\rho(E,v_0,a_0,a_1)=\rho(-E,-v_0,a_0,a_1)=\rho(-E+v_0,v_0,a_1,a_0). \qquad (17)$$

It suffices therefore to find the number of states $n(v_0/2,E)$ for the case $v_0>0$ and $E>v_0/2$ (the hatched region in Fig. 1), since the spectrum in the remaining region of the parameters (v_0,E) can be easily obtained with the aid of relations (17).

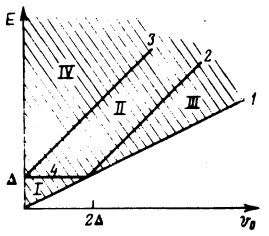

FIG. 1. Equations of the straight lines 1) $E=v_0/2$, 2) $E=v_0-\Delta$, 3) $E=v_0+\Delta$, 4) $E=\Delta$.

2. INVESTIGATION OF THE STATE DENSITY

1. The procedure for solving (15) is quite standard, although somewhat cumbersome. We present therefore only the final formulas for the number of tests $n(E_0,E)$ or flux $J(E)$ in various ranges of the energy of the amplitude of the potential (see Fig. 1). It is convenient in this case to define the following functions:

$$Q_s(\varphi)=\frac{1}{a_s[(E-v_0 s)^2-\Delta^2]^{1/2}}\operatorname{arcctg}\left(\left(\frac{E-v_0 s+\Delta}{E-v_0 s-\Delta}\right)^{1/2}\operatorname{ctg}\varphi\right),$$
$$T_s(\varphi)=\frac{1}{2a_s[(E-v_0 s)^2-\Delta^2]^{1/2}}\ln\left|\frac{\operatorname{ctg}\varphi+\operatorname{ctg}\varphi_s}{\operatorname{ctg}\varphi-\operatorname{ctg}\varphi_s}\right|, \qquad (18)$$

where

$$\operatorname{ctg}\varphi_s=\left(\frac{\Delta+v_0 s-E}{\Delta-v_0 s+E}\right)^{1/2},\quad s=0,1.$$

A distinction must be made between the following cases:

I. $v_0<2\Delta$, $E<\Delta$. Under these conditions a gap exists in the spectrum of the system, $n(v_0/2,E)=0$, since the amplitude of the potential is less than the width of the bare gap.

II. $|E-v_0|<\Delta$, $E>\Delta$. In this case we have

$$\frac{a_0 a_1}{(a_0+a_1)J(E)}=\int_{S_1}\frac{\exp(R(\varphi')-R(\varphi))}{\Phi_0(\varphi)|\Phi_1(\varphi')|}d\varphi\,d\varphi'$$
$$+e^{-R(\pi)}\int_{S_2}\frac{\exp[R(\varphi')-R(\varphi)]}{\Phi_0(\varphi)|\Phi_1(\varphi')|}d\varphi\,d\varphi', \qquad (19)$$

where $R(\varphi)=Q_0(\varphi)+T_1(\varphi)$, and the integration regions S_1 and S_2 are indicated on Fig. 2.

Interest attaches here primarily to the behavior of the number of states in the case $v_0<2\Delta$, $E\to\Delta+0$. It is easy to verify that the main contribution to $J^{-1}(E)$ is made by the integral over the square $\varphi_1<\varphi<\pi/2$, $\pi/2<\varphi'<\pi-\varphi_1$, and the corresponding asymptotic form of $n(v_0/2,E)$ is

$$n(v_0/2,E)\approx\frac{\exp[-\pi/a_0(E^2-\Delta^2)^{1/2}]}{(a_0+a_1)\Gamma^2(q+1)W^2_{-q,-1/2}(p)},\quad E\to\Delta+0,$$
$$q^{-1}=2a_1|v_0(2\Delta-v_0)|^{1/2},\quad p=(a_0\Delta)^{-1}((2\Delta-v_0)/v_0)^{1/2}. \qquad (20)$$

where the symbols for the special functions here and below are the same as in the book of Gradshteĭn and Ryzhik.[12]

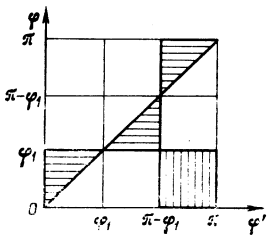

FIG. 2. The regions S_1 and S_2 are hatched horizontally and vertically, respectively.

We see thus that at $v_0 < 2\Delta$ the spectrum of the system has a gap whose boundaries are the true fluctuation boundaries[1,13] with the exponential asymptotic form of $n(v_0/2, E)$ typical of these boundaries. In the limiting case $v_0 \to 2\Delta - 0$ the gap collapse, but the character of the asymptotic number of states in the vicinity of the point $E = \Delta$ remains the same as before:

$$n(\Delta, E) = \frac{2\Delta}{\pi} \frac{(a_0 a_1)^{1/2}}{a_0 + a_1} \exp\left[-\frac{\pi}{a_0 (E^2 - \Delta^2)^{1/2}} + \frac{2}{\Delta (a_0 a_1)^{1/2}} \right], \quad E \to \Delta + 0. \tag{21}$$

We note that this result can be obtained both from the exact formula (19) at $v_0 = 2\Delta$, and as a result of taking the limit $v_0 \to 2$ in (20).

III. $\Delta < v_0/2 < E < v_0 - \Delta$. Under these conditions we arrive at the formula

$$\frac{a_0 a_1 (e^{R(\pi)} - 1)}{(a_0 + a_1) J(E)} = \int_0^{\pi} d\varphi \int_{\varphi}^{\pi} d\varphi' \frac{\exp[R(\varphi') - R(\varphi)]}{\Phi_0(\varphi) \Phi_1(\varphi')} + e^{R(\pi)} \int_0^{\pi} d\varphi \int_0^{\varphi} d\varphi' \frac{\exp[R(\varphi') - R(\varphi)]}{\Phi_0(\varphi) \Phi_1(\varphi')}, \tag{22}$$

where now $R(\varphi) = Q_0(\varphi) - Q_1(\varphi)$.

We see immediately that at the point $E = v_0/2$, which is singled out by the symmetry properties of the spectrum, no singularity appears in the number of states. It is meaningful, however, to examine $n(v_0/2, E)$ in the vicinity of this point, when the gap has just collapsed, i.e., at $v_0 - 2\Delta \ll \Delta$, $E \to v_0/2 + 0$. The main contribution is then due to the integral over the square $\pi/2 < \varphi' < \pi$, $0 < \varphi < \pi/2$. Introducing the symbol

$$\delta = \left(\frac{v_0 - 2\Delta}{v_0 + 2\Delta} \right)^{1/4} \ll 1,$$

we obtain from (22)

$$\rho\left(\frac{v_0}{2} + 0 \right) \approx \frac{c\pi}{8\delta^3} \frac{a_0 a_1 \Delta}{a_0 + a_1} \frac{\exp(-c\pi/2\delta)}{K_1^2 (1/\Delta (a_0 a_1)^{1/2})}, \tag{23}$$

where $c = \min(1/a_0\Delta, 1/a_1\Delta)$, and K_1 is a MacDonald function. We note that as $\delta \to 0$ this formula is valid outside an exponentially small vicinity of the point $E = v_0/2$, namely, at

$$\exp\left(-\frac{c\pi}{2\delta} \right) \ll \frac{2E - v_0}{2\Delta\delta^3} \ll 1.$$

In addition, by virtue of the symmetry relations (17), the state density to the left of $E = v_0/2$ turns out to be different:

$$\rho\left(\frac{v_0}{2} - 0 \right) \approx \frac{c'\pi}{8\delta^3} \frac{a_0 a_1 \pi}{a_0 + a_1} \frac{\exp(-c'\pi/2\delta)}{K_1^2 (1/\Delta (a_0 a_1)^{1/2})}, \tag{24}$$

where $c' = \max(1/a_0\Delta, 1/a_1\Delta)$.

Depending on the relation between c and c', the state density in the vicinity of the point $v_0/2$ either remains practically unchanged (if $2c > c'$) or jumps very rapidly in the case $2c < c'$, with a derivative $\rho' \Delta \sim \delta^{-6} \gg 1$. In either case, however, as $\delta \to 0$ the state density is exponentially small to the left and to the right of $v_0/2$.

In the limiting case $a_0 = a_1 \to 0$, $v_0 \to \infty$ $v_0^2 a_0/4 = B_0 = \text{const}$, the initial potential (14) becomes a Gaussian white noise. Making the indicated limiting transition in (22) and simultaneously shifting the energy by an amount equal to the average potential ($E \to v_0/2 + E$), we get

$$n(0, E) = B_0 \left(1 - \exp\left[-\frac{2\pi E}{B_0} \right] \right) \Big/ 2 \int_0^{2\pi} d\varphi \int_{\varphi - 2\pi}^{\varphi} d\varphi' \exp \frac{R(\varphi') - R(\varphi)}{B_0}, \tag{25}$$

where $R(\varphi) = E\varphi + \Delta \sin\varphi$.

In the limiting case $E \to 0$ or $E \to \infty$ we have

$$n(0, E) = \pi E \left[\int_0^{2\pi} d\varphi\, dt \exp\left\{ \frac{\Delta}{B_0} (\sin(\varphi + t) - \sin\varphi) \right\} \right]^{-1}, \quad E \to 0,$$

$$n(0, E) \approx \frac{B_0}{4\pi} \left(1 - \frac{\Delta^2}{2E^2} \right), \quad E \to \infty.$$

IV. $E > v_0 + \Delta$. In this case $J(E)$ is determined by formula (22), in which, however, $R(\varphi) = Q_0(\varphi) + Q_1(\varphi)$. It is of interest here to trace the asymptotic behavior of $n(v_0/2, E)$ as $E \to \infty$. The corresponding formula is

$$n\left(\frac{v_0}{2} E \right) \approx \frac{E}{\pi} \left(1 - \frac{\langle v \rangle}{E} \right), \quad E \to \infty, \tag{26}$$

where

$$\langle v \rangle = \frac{a_1 v_0}{a_0 + a_1}. \tag{27}$$

2. A number of the asymptotic forms presented above are valid not only for the model considered, but in a much larger class of cases. We shall discuss now some other more general methods of obtaining such asymptotic forms.

a) The first is the asymptotic number of states at high energies. This number can be obtained with the aid of (6), in which $\varphi(E, L)$ is obtained from Eq. (4) by using perturbation theory in the parameter $\Delta/(E - \langle v \rangle)$. Carrying out this expansion accurate to terms of fourth order in the indicated small parameter, inclusive, we get

$$n(E) \approx \frac{1}{\pi} ((E - \langle v \rangle)^2 - \Delta^2)^{1/2} \left(1 - \frac{\Delta^2 B(0)}{(E - \langle v \rangle)^4} \right), \tag{28}$$

where

$$B(x) = \langle v(x) v(0) \rangle - \langle v \rangle^2 = \frac{v_0^2 a_0 a_1}{(a_0 + a_1)^2} \exp\left[-|x| \frac{a_0 + a_1}{a_0 a_1} \right]. \tag{29}$$

We call attention to the fact that Eq. (28) is even more accurate than the asymptotic form obtained from the exact solution.

b) Next, the asymptotic form (20) of $n(v_0/2, E)$ near the true fluctuation boundary can also be obtained with logarithmic accuracy by another method that can be used in a much more general situation. We have in mind the method proposed by Eggarter[14] and developed further by us.[13] The main point of this method is the proof of a certain statement concerning the character of the behavior of the phase in the case when the energy is extremely close to the true fluctuation boundary. In the considered model[2] this statement can be easily proved and is formulated as follows.

Let

$$v_0 < 2\Delta, \quad \alpha = \left(\frac{E - \Delta}{\Delta} \right)^{1/4} \ll 1.$$

Then, if

$$\varphi(0)<\varphi_{cr}=\pi/2-\alpha,$$
$$r<r_{cr}=\frac{\pi}{(E^2-\Delta^2)^{1/2}}\left[1-\frac{2}{\pi}\operatorname{arctg}\left(\left(\frac{E-\Delta}{E+\Delta}\right)^{1/2}\operatorname{ctg}\alpha\right)\right],$$
$$\rho>\rho_{cr}=\frac{1}{[\Delta^2-(E-v_0)^2]^{1/2}} \tag{30}$$
$$\times\ln\left\{\left|\operatorname{ctg}\alpha+\left(\frac{\Delta+E-v_0}{\Delta-E+v_0}\right)^{1/2}\right|\Big/\left|\operatorname{ctg}\alpha-\left(\frac{\Delta+E-v_0}{\Delta-E+v_0}\right)^{1/2}\right|\right\},$$

then $\varphi(x)<\pi$ for all x from the interval $[0, +r]$ and $\varphi(r+\rho)<\varphi_{cr}$.

Hence, using (13) and reasoning as in Refs. 13 and 14, we find that at $v_0<2\Delta$, $E\to\Delta+0$ we have

$$\ln n(E)\approx\ln\int_\zeta^\infty f_0(x)\,dx, \tag{31}$$

where the lower integration limit is $\zeta=\pi/(E^2-\Delta^2)^{1/2}$ and $f_0(r)$ is the density of the probabilities of the distance between barriers. For a power-law decrease of the probability we obtain the more accurate result

$$n(E)\approx a_0^{-1}\int_\zeta^\infty f_0(x)\,dx,\quad a_0=\int_0^\infty x f_0(x)\,dx. \tag{32}$$

We note that these results are valid also at $v_0=2\Delta$, since the statement (30) remains valid also in this case.

c) As was demonstrated for a Schrödinger equation with the potential (14) in Ref. 11 and with a potential $\sim\Sigma\delta(x-x_j)$ in Ref. 4, the state density in some vicinity of the point $E=\langle v\rangle$ is of the same form as if the potential were a Gaussian white noise. A similar situation arises in the present case, and furthermore for a large class of random potentials[3] $v(x)$.

To demonstrate this, we consider the equation for the quantity $z=-\cot\varphi=f_2/f_1$:

$$\frac{dz}{dx}=\Delta(z^2-1)+[E-v(x)](z^2+1)$$

and make the change of variables $x=t/\varkappa$, $E=\langle v\rangle+\varepsilon$, $v(x)=\langle v\rangle+\xi(t)$, after which we get

$$\frac{dz}{dt}=\frac{\Delta}{\varkappa}(z^2-1)+\frac{\varepsilon}{\varkappa}(z^2+1)-\frac{1}{\varkappa}\xi(t)(z^2+1). \tag{33}$$

In terms of these variables, the correlation function of the potential $\xi(t)$ takes the form [see (29)]

$$\langle\xi(t)\xi(0)\rangle=B(t/\varkappa)$$

and in the case $\varkappa r_c\ll1$, where r_c is the characteristic distance over which $B(x)$ varies, is a δ-like function. When the indicated inequality is satisfied, the potential in (33) is therefore quite similar to a Gaussian white noise with a pair correlator of the type

$$\langle\xi(t)\xi(0)\rangle\approx2D\delta(t),\quad D=\frac{1}{2\varkappa}\int B(x)\,dx. \tag{34}$$

We now choose the parameter $\varkappa$ such as to satisfy the condition $D\sim1$. Then

$$\varkappa\sim B_0=\int B(x)\,dx\sim k_0^2 r_c,$$

where k_0 is the characteristic amplitude of the potential $\xi(t)$. In the case of the Gaussian white noise which enters in (33) at the very outset, the amplitude of the potential is large compared with the coefficients $\Delta/\varkappa$ and $\varepsilon/\varkappa$. Therefore when the conditions

$$\Delta\ll k_0,\quad \varepsilon\ll k_0 \tag{35}$$

are satisfied the relations between the different terms of the right-hand side of (33) will be the same as for a Gaussian white noise. Since, however, the behavior of the phase shift φ, and hence also of z, determines uniquely the spectrum of the system (3), the foregoing means that for any potential satisfying the conditions $k_0 r_c\ll1$ and $\Delta\ll k_0$ the number of states $n(\langle v\rangle,\langle v\rangle+E)$ in a region of width $|E-\langle v\rangle|\ll k_0$ in the vicinity of the point $\langle v\rangle$ will take a form corresponding to a Gaussian white noise and determined by formula (25), which is obtained as the limiting case of a certain exactly solvable model.

d) Among the various limiting cases in the model of Sec. 1, interest attaches to the situation wherein the average lengths of the barriers and of the intervals between them increase, $a_0, a_1\to\infty$, $a_0/a_1=$ const, a situation corresponding to a very smoothly varying potential. In this case the number of states takes the asymptotic form

$$n(E)=\frac{1}{\pi}\left[\frac{a_0}{a_0+a_1}(E^2-\Delta^2)^{1/2}+\frac{a_1}{a_0+a_1}((E-v_0)^2-\Delta^2)^{1/2}\right], \tag{36}$$

which can be easily explained. In fact if a_0 and a_1 tend to infinity simultaneously, the potential remains practically constant over very long sections, and therefore leads simply to a random energy shift, as in the case of the Schrödinger equation.[15,11]

3. FLUCTUATING-GAP MODEL

1. In this section we consider a somewhat different model defined by the system of equations

$$\begin{pmatrix}-i\,d/dx & \Delta(x)\\ \Delta(x) & i\,d/dx\end{pmatrix}\varphi=E\varphi. \tag{37}$$

Comparison with (2) shows that this model can be formally obtained from (2) if $v=0$, and the half-width Δ of the gap is a random function of the coordinate. Such a model was proposed in Ref. 8 and was investigated for the case when $\Delta(x)=\langle\Delta\rangle+\xi(x)$, where $\xi(x)$ is a Gaussian white noise, with $\langle\xi(x)\xi(0)\rangle=2D\delta(x)$. One of the main results of Ref. 8 was the absence of a gap from the spectrum at arbitrarily small intensity D, and a singularity of the state density at the center of the gap in the case of sufficiently large fluctuations of the gap $D>\langle\Delta\rangle$. We consider below another variant of this model, in which

$$\Delta(x)=\Delta_0+\eta(x),\quad \eta(x)=(\Delta_1-\Delta_0)s(x),$$

where $s(x)$ is the random process used in Secs. 1 and 2 to describe the impurity potential $v(c)$ [see (14)].

It is easily seen that the phase formalism developed in Sec. 1 can be applied in its entirety also to the case considered here. The difference from the formulas of Sec. 1 is only that now the Fokker-Planck equation (15) contains a somewhat different function

$$\Phi_1(\varphi)=E+\Delta_1\cos2\varphi \tag{38}$$

and the symmetry conditions (17) are altered:

$$\rho(E)=\rho(-E). \tag{39}$$

In addition, $Q_s(\varphi)$ and $T_s(\varphi)$ in the formulas of the present section must be taken to mean the functions

$$Q_s(\varphi)=\frac{1}{a_s(E^2-\Delta_s^2)^{1/2}}\operatorname{arcctg}\left[\left(\frac{E+\Delta_s}{E-\Delta_s}\right)^{1/2}\operatorname{ctg}\varphi\right],$$
$$T_s(\varphi)=\frac{1}{2a_s(\Delta_s^2-E^2)^{1/2}}\ln\left|\frac{\operatorname{ctg}\varphi+\operatorname{ctg}\varphi_s}{\operatorname{ctg}\varphi-\operatorname{ctg}\varphi_s}\right|, \tag{40}$$
$$\operatorname{ctg}\varphi_s=\left(\frac{\Delta_s-E}{\Delta_s+E}\right)^{1/2}.$$

In accord with relation (39), we consider henceforth throughout the number of states $n(0,E)\equiv n(E)=J(E)$ (it turns out that in this model we always have $J(0)=0$). Just as before, Eq. (15) with $\Phi_s(\varphi)$ taken from (38) can be integrated in quadratures, and this leads to different equations for $n(E)$, depending on the energy E and on the gap parameter Δ_s. We write down and investigate below only those which correspond to the most interesting behavior of the number of states.

I. $0<E<\Delta_0<\Delta_1$. In this case $n(E)=0$, i.e., the spectrum of the system has a gap. Its existence follows both from the exact equations (4) and (15), and from general operator considerations. It precisely at this point that the model of gap fluctuation in the form $\eta(x)=(\Delta_1-\Delta_0)s(x)$ differs substantially from that considered in Ref. 8, where there was no gap in the spectrum at all values of the problem parameters.

II. $0<\Delta_0<E<\Delta_1$. For the number of states we obtain formula (19) in which $B(\varphi)=Q_0(\varphi)+T_1(\varphi)$. It is of interest to investigate the behavior of $n(E)$ near the true fluctuation limit $E\to\Delta_0+0$. The main contribution is made then by integration over the same square as in part II of Sec. 2, and the asymptotic form itself is

$$n(E)=\frac{\exp(-\pi/a_0(E^2-\Delta_0^2)^{1/2})}{(a_1+a_0)\Gamma^2(q+1)W^2_{-q,-1/2}(p)}, \tag{41}$$

where

$$p=\frac{1}{a_0\Delta_0}\left(\frac{\Delta_1+\Delta_0}{\Delta_1-\Delta_0}\right)^{1/2},\quad q=\frac{a_0\Delta_0}{a_1(\Delta_0+\Delta_1)}.$$

In contrast to the result (20) of Sec. 2, it is impossible to go to the limit $\Delta_0\to 0$ of gap collapse, thus indicating that the asymptotic form changes at $\Delta_0=0$. In fact, putting $\Delta_0=0$ as $E\to+0$ in (19), we obtain

$$n(E)\approx\frac{\exp[(a_0\Delta_1)^{-1}]}{(a_0+a_1)\Gamma^2((2a_1\Delta_1)^{-1})}\frac{\exp(-\pi/2a_0E)}{\exp\{(a_1\Delta_1)^{-1}\ln(a_0E)\}}. \tag{42}$$

We note that in the limiting case $\Delta_1\gg\Delta_0$ and $E\gg\Delta_0$, when the spectrum is insensitive to the existence of the lower band, the exact formula (19) goes over, when (38) and (40) are taken into account, into the equation for the number of states in a Schrödinger equation with a potential in the form (14) formula (8) of Ref. 11).

III. $0<E<-\Delta_0<\Delta_1$. The number of states is determined by the formula

$$\frac{a_0a_1}{(a_0+a_1)n(E)}=\iint_S\frac{\exp[R(\varphi')-R(\varphi)]}{|\Phi_0(\varphi)\Phi_1(\varphi')|}d\varphi\,d\varphi', \tag{43}$$

where $R(\varphi)=T_1(\varphi)-T_0(\varphi)$, and the integration region S is indicated in Fig. 3.

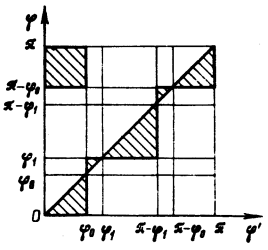

FIG. 3. The integration region S is hatched.

In this case greatest interest attaches to the behavior of the state density as $E\to 0$. From (43) we obtain at $E\ll-\Delta_0$ and $\langle\Delta\rangle>0$ the asymptotic expression

$$n(E)=\frac{4D}{\pi^2}\{J_\nu^2(\varepsilon)+N_\nu^2(\varepsilon)\}^{-1},$$
$$\varepsilon=\frac{E}{2E_0},\quad \nu=\frac{\langle\Delta\rangle}{2D},\quad \langle\Delta\rangle=\frac{\Delta_0a_0+\Delta_1a_1}{a_0+a_1}, \tag{44}$$
$$D=\frac{a_0a_1|\Delta_0|\Delta_1}{a_0+a_1},\quad E_0^{-1}=\frac{1}{a_1\Delta_1^2}+\frac{1}{a_0\Delta_0^2},$$

which is similar to formula (32) of Ref. 8 and coincides with the latter following the limiting transitions $\Delta_1\to+\infty$, $\Delta_0\to-\infty$, $a_1=a_2\to 0$, $\langle\Delta\rangle=\text{const}$, and $(\Delta_1+\Delta_0)^2a_1=\text{const}$, which transform the gap fluctuations into a Gaussian white noise. Moreover, the same limiting transition, when performed in the exact equation (3) which is valid in this case for all values of the energy E, also transforms it into (44) with the parameters D, ν, and ε from Ref. 8.

At $E\ll E_0$ it follows from (44) that the state density in the case $\nu>0$ takes the form

$$\rho(E)\approx\frac{2D\nu}{E_0\Gamma^2(\nu)}\left(\frac{E}{4E_0}\right)^{2\nu-1}. \tag{45}$$

Therefore in the case of a small gap overlap $\nu>1/2$ the state density at the center of the band tends to zero. With increasing overlap (with increasing $|\Delta_0|$) ν decreases and at $\Delta_0=-a_1\Delta_1/a_0(a_1\Delta_1+1)$ it is exactly equal to $1/2$, and the state density takes in this case the form

$$\rho(0)=D/\pi E_0=\text{const}.$$

Finally, if the overlap is strong $0<\nu<1/2$, the state density has a singularity that is described as before by formula (45). In the limiting case $\nu=0$ this singularity is somewhat altered

$$\rho(E)\approx 2D\Big/E\Big|\ln^3\frac{E}{4E_0}\Big|. \tag{46}$$

This close similarity between formulas (44)–(46) and the results of Ref. 8 suggest that the onset of the singularity of the state density at the center of the band is due not so much to the concrete probability properties of $\Delta(x)$ as to the fluctuating-gap model itself, since the point $E=0$ is singled out from the very outset by the symmetry of the model.

Favoring this point of view are also the result of an investigation, in the same range of parameters, of the damping coefficient $\gamma(E)$ of the envelope of the wave function (reciprocal localization length), defined by the formula

$$\gamma(E)=\left\langle\lim_{x\to\infty}\frac{\ln r(x)}{x}\right\rangle,\quad r^2=f_1^2+f_2^2.$$

Elementary transformation reduce this expression, in the case of the model (31), to the form

$$\gamma(E) = -\langle \Delta \cos 2\varphi \rangle_{st}$$

whence we get in the limit as $E \to 0$

$$\gamma(E) = D_1 \frac{E\rho(E)}{n(E)}, \tag{47}$$

where

$$D_1 = \frac{E_0(\Delta_1 + |\Delta_0|)a_0}{\Delta_1(a_1+a_0)}, \quad \rho(E) = \frac{dn(E)}{dE},$$

and the number $n(E)$ of the states is given by (44) (after a limiting transition that transforms the gap fluctuations into a Gaussian white noise, formula (47), just like (44), describes the entire range of values $0 < E < \infty$).

It follows from (47), in particular, that at $\nu = 0$ the damping coefficient vanishes at the point $E = 0$:

$$\gamma(E) \approx 2D_1 \Big/ \left| \ln \frac{E}{2E_0} \right|, \quad \nu = 0 \tag{48}$$

in contrast to $\nu > 0$, when

$$\gamma(0) = 2D_1\nu. \tag{49}$$

We note, however, that here the damping coefficient vanishes at the center of a formerly forbidden band, not an allowed one as, e.g., in Ref. 16.

IV. $0 < \Delta_0 < \Delta_1 < E$. Here $n(E)$ is determined by formula (22) with $\Phi_s(\varphi)$ from (38) and $R(\varphi) = Q_0(\varphi) + Q_1(\varphi)$. As $E \to \infty$ we get

$$n(E) \approx \frac{E}{\pi}\left(1 - \frac{\langle \Delta^2 \rangle}{2E^2}\right), \quad \langle \Delta^2 \rangle = \frac{\Delta_0^2 a_0 + \Delta_1^2 a_1}{a_0 + a_1}. \tag{50}$$

2. The content of this subsection is quite similar to that of subsection 2 of the preceding section. We shall therefore only list briefly the corresponding results.

a) The high-energy asymptotic number of states, calculated by perturbation theory from (4) with subsequent utilization of (6), is

$$n(E) = \pi^{-1}(E^2 - \langle \Delta \rangle^2)^{1/2}(1 - B(0)/2E^2), \tag{51}$$

where

$$B(x) = \langle \Delta(x)\Delta(0) \rangle - \langle \Delta \rangle^2 = \langle \Delta_1 - \Delta_0 \rangle^2 \frac{a_0 a_1}{a_0 + a_1} \exp\left[- |x| \frac{a_1 + a_0}{a_1 a_0} \right] \tag{52}$$

and turns out, just as (28) to be more accurate than formula (50) obtained from the exact solution.

b) The analog of the statement (30) in the fluctuating-gap model is formulated in the following manner. Let

$$\alpha = ((E - \Delta_0)/\Delta_0)^{1/4} \ll 1.$$

Then, if

$$\varphi(0) < \varphi_{cr} = \pi/2 - \alpha$$

$$r < r_{cr} = \frac{\pi}{(E^2 - \Delta_0^2)^{1/2}}\left[1 - \frac{2}{\pi} \operatorname{arctg}\left(\left(\frac{E - \Delta_0}{E + \Delta_0} \right)^{1/2} \operatorname{ctg} \alpha \right) \right], \tag{53}$$

$$\rho > \rho_{cr} = \frac{1}{(\Delta_1^2 - E^2)^{1/2}} \ln \left\{ \left| \operatorname{ctg} \alpha + \left(\frac{\Delta_1 + E}{\Delta_1 - E} \right)^{1/2} \right| \Big/ \left| \operatorname{ctg} \alpha - \frac{\Delta_1 + E}{\Delta_1 - E}^{1/2} \right| \right\},$$

we get $\varphi(x) < \pi$, $0 < x < r + \rho$, and $\varphi(r + \rho) < \varphi_{cr}$.

Hence, reasoning as before, we arrive at formulas (31) and (32) for the number of states, but Δ in their right hand sides must now be replaced by Δ_0.

c) Under the conditions

$$\langle \Delta \rangle \ll B_0, \quad |E - \langle \Delta \rangle| \ll B_0 \quad k_0 r_c \ll 1, \tag{54}$$

where the parameters k_0, r_c, and B_0 are defined in the same manner as in Sec. 2 but relative to the correlation function (52), the number of states $n(E)$ in the vicinity of the point $\langle v \rangle$, with a width $|E - \langle v \rangle| \ll B_0$, takes for any potential $\Delta(x)$ a form corresponding to a Gaussian white noise and defined by formula (44), in which $E_0 = D = B_0/2$, and $v = \langle \Delta \rangle / B_0$.

d) For an extremely smooth potential $a_0, a_1 \to \infty$, $a_0/a_1 = \text{const}$ we get from (19) and (22)

$$n(E) = \frac{1}{\pi}\left[\frac{a_0}{a_0 + a_1}(E^2 - \Delta_0^2)^{1/2} + \frac{a_1}{a_0 + a_1}(E^2 - \Delta_1^2)^{1/2} \right]. \tag{55}$$

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[1)] In the effective mass approximation (the Schrödinger equation), such a model was proposed and considered in Ref. 11.
[2)] We refer here to the system (3) with potential (14), but now with aribtrary probability densities of the barrier lengths $f_1(x)$ and of the distances between them $f_0(x)$.
[3)] The same holds ture for the Schrödinger equation.

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