States with minimum phase uncertainty

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(Submitted 24 March 1978)
Zh. Eksp. Teor. Fiz. 75, 1584–1588 (November 1978)

The uncertainties of the phase ($\Delta\varphi$) and of the number of photons (Δn) are connected by the relation $\Delta n\Delta\varphi \geq 1/2$. For coherent states, $\Delta\varphi \sim 1/2\sqrt{n}$ and $\Delta n \sim \sqrt{n}$. The laws of quantum mechanics do not forbid "supercoherent states," which at a given energy (at a given number of photons) have a lower phase uncertainty than the coherent state, namely $\Delta\varphi \sim 1/n$ and $\Delta n \sim n$. The class of such states is indicated and the normalized wave functions are obtained in the energy representation.

PACS numbers: 42.50. + q

The uncertainty Δn of the number of photons and the uncertainty $\Delta\varphi$ of the phase satisfy the inequality[1–4]

$$\Delta n\Delta\varphi \geqslant 1/2, \qquad (1)$$

and a state in which

$$\Delta n \sim n, \quad \Delta\varphi \sim 1/n. \qquad (2)$$

is not forbidden. An attempt to find the state (2) was made by Siegman[4] but without success. The problem was first formulated by Serber and Townes,[5] and is reflected in the article of Braginskiĭ and Vorontsov.[6] The procedure of producing states with low phase uncertainty $\Delta\varphi \ll 1/2\sqrt{n}$ was discussed in Refs. 7 and 8. In the present note we indicated the class of such states.

In the energy representation, the state is characterized by complex amplitudes a_k:

$$\psi=\sum_k a_k\psi_k, \quad H\psi_k=E_k\psi_k, \quad E_k=\hbar\omega k, \quad k=0,1,2\ldots;$$

$$\sum_k |a_k^2|<\infty. \qquad (3)$$

From the commutation relations for the phase operator $\hat{e}^{i\varphi}$ and the photon-number of operator $\hat{n}$

$$[\hat{e}^{i\varphi}\hat{n}]_-=\hat{e}^{i\varphi}, \qquad (4)$$

we have for the eigenfunction of the phase operator[4]

$$\Delta\varphi=0, \quad a_{k+1}=e^{i\varphi}a_k, \quad a_k=e^{ik\varphi}a_0,$$
$$\sum_k |a_k^2|=\infty, \qquad (5)$$

i.e., the exact eigenfunction of the phase operator is not normalizable. Our purpose is to find normalizable wave functions.

We are interested in functions in which, according to (2), $\Delta\varphi \sim 1/n$. We make the assumption, subsequently justified by the result, that this function differs from (5) only by amplitude factors. We put

$$a_k=e^{ik\varphi}u_k, \quad \operatorname{Im} u_k=0, \quad \operatorname{Re} u_k \geqslant 0. \qquad (6)$$

The mean value of the phase operator is[4]

$$\langle e^{i\varphi}\rangle=\sum a_k^* a_{k+1}=e^{i\varphi}\sum u_k u_{k+1}, \qquad (7)$$

and quantities u_k must satisfy the relations

$$\sum u_k^2=1, \quad \sum k u_k^2<\infty, \quad \sum k^2 u_k^2<\infty. \qquad (8a)$$

For the phase uncertainty we have[4]

$$\tfrac{1}{2}\langle\Delta\varphi^2\rangle=1-|\langle e^{i\varphi}\rangle|=1-\sum u_k u_{k+1}, \quad \Delta\varphi \ll 1. \qquad (9)$$

We note that the determination of the phase encounters yet unsolved difficulties in the classical[9] and quantum formulation of the problem,[10] which seem to have a common nature. We wish to note that the solution proposed by Carruthers and Nietto[10] for the problem is in our opinion unsatisfactory. They introduced phase operators for which $\varphi \to \varphi + 2\pi$ is an identity transformation. In the classical region, a phase jump of 2π leads to a new state; such jumps lead to observable effects, for example to frequency errors in radio circuits with automatic phase control.[11] Therefore the phase operators introduced in Ref. 10 do not permit a limiting transition to the classical case. We consider the case $n \gg 1$, $\Delta\varphi \ll 2\pi$, when there are no jumps, therefore the difficulties indicated in Refs. 9 and 10 and encountered in the determination of the phase are of no importance to us.

Inasmuch as a small phase uncertainty requires a large number of photons, we assume k to be a continuous variable and change from summation to integration

$$u_k \to u(k), \quad \int_0^\infty u^2(k)\,dk=1. \qquad (8b)$$

We put

$$u(k)=\alpha^{1/2}w(\xi), \quad \xi=\alpha k, \qquad (10)$$

where α is a small parameter of the problem. We then get from (8)

$$\int_0^\infty u^2(k)\,dk=\int_0^\infty w^2(\xi)\,d\xi=1,$$
$$M\xi^q=\int_0^\infty \xi^q w^2(\xi)\,d\xi \quad (q=1,2). \qquad (11)$$

The assumptions (8b) and (10) make it easy to obtain the class of distributions for which (2) is valid; we believe that this class is not the only one. We obtain the number of photons n and the uncertainty Δn:

$$\langle n\rangle=\int_0^\infty k u^2(k)\,dk=\frac{1}{\alpha}M\xi, \quad n \sim \frac{1}{\alpha} \qquad (12a)$$

$$\langle n^2\rangle=\int_0^\infty k^2 u^2(k)\,dk=\frac{1}{\alpha^2}M\xi^2; \qquad (12b)$$

$$\Delta n=(\langle n^2\rangle-\langle n\rangle^2)^{1/2} \sim 1/\alpha. \qquad (12c)$$

From the formula (9) we calculate the phase uncertainty:

$$\langle\Delta\varphi^2\rangle=2\left[1-\int_0^\infty w(\xi)\,w(\xi+\alpha)\,d\xi\right],$$

$$\alpha(k+1)=\xi+\alpha. \tag{13}$$

We assume that the function $w(\xi+\alpha)$ can be expanded in powers of α:

$$w(\xi+\alpha)=w(\xi)+\alpha w'(\xi)+{}^1/_2\alpha^2 w''(\xi)+{}^1/_6\alpha^3 w'''(\xi+\theta\alpha),\quad 0\leqslant\theta\leqslant 1.$$

This yields

$$\langle\Delta\varphi^2\rangle=2\left[-\alpha\int_0^\infty w(\xi)\,w'(\xi)\,d\xi-\frac{\alpha^2}{2}\int_0^\infty w(\xi)\,w''(\xi)\,d\xi-\frac{\alpha^3}{6}R\right]. \tag{14}$$

It is easily seen that

$$-\int_0^\infty w(\xi)\,w'(\xi)\,d\xi=w^2(0)/2,$$

$$\int_0^\infty w(\xi)\,w''(\xi)\,d\xi=-w(0)\,w'(0)-\int_0^\infty [w'(\xi)]^2 d\xi,$$

and formula (14) yields

$$\langle\Delta\varphi^2\rangle=\alpha w^2(0)+\alpha^2\left\{w(0)\,w'(0)+\int_0^\infty [w'(\xi)]^2 d\xi\right\}-\frac{\alpha^3}{3}R. \tag{15}$$

According to (12a) we have $n\sim 1/\alpha$, so that if (2) is to be satisfied Eq. (15) should have no term linear in α. This leads to the following condition for the sought distributions:

$$w^2(0)= \tag{16a}$$

If the conditions

$$w(0)\,w'(0)=0, \tag{16b}$$

$$\int_0^\infty [w'(\xi)]^2 d\xi<\infty, \tag{16c}$$

$$R=\int_0^\infty w(\xi)\,w'''(\xi+\alpha\theta)\,d\xi<\infty,\quad \alpha\to 0, \tag{16d}$$

are satisfied, we obtain ultimately

$$\Delta\varphi=\alpha\left(\int_0^\infty [w'(\xi)]^2 d\xi\right)^{1/2}\sim 1/n. \tag{17}$$

The conditions (16) determine the sought class of distributions. The conditions (16a) and (16b) mean that the distribution does not contain weakly excited states, while (16c) and (16d) mean that the sought distribution is smooth enough.

We note that the length uncertainty Δl for the states (2) depends only on the total energy $\mathscr{E}$ of the state and does not depend on the frequency ω:

$$\Delta l=\frac{c}{\omega}\Delta\varphi=\frac{c}{\omega n}=\frac{\hbar c}{\mathscr{E}},\quad n=\frac{\mathscr{E}}{\hbar\omega}. \tag{18}$$

We consider the simplest case of a distribution from this class:

$$w(\xi)=2^{-1/2}\xi e^{-\xi/2},\quad M\xi=3,\quad M\xi^2=12,$$
$$n=3/\alpha,\quad \Delta n=\sqrt{3}/\alpha=n/\sqrt{3}. \tag{19}$$

For the derivative we have

$$w'(\xi)=2^{-1/2}(1-\xi/2)\,e^{-\xi/2},\quad \int_0^\infty [w'(\xi)]^2 d\xi={}^1/_4,$$

which yields for the phase uncertainty

$$\Delta\varphi=\alpha/2=3/2n,\quad \Delta n\Delta\varphi=\sqrt{3}/2>{}^1/_2. \tag{20}$$

For the considered example, the product $\Delta n\Delta\varphi$ is only $\sqrt{3}$ times larger than the minimal value. We recall that for the coherent state we have

$$\Delta\varphi=1/2\sqrt{n} \tag{21}$$

and consequently the gain in the accuracy of the measurements of the phase and of the linear distances amounts to $\sqrt{n}/3$. The use of quantum states with minimum phase uncertainty permits therefore a more accurate measurement of distances than the use of coherent signals; this makes it possible in principle either to work with smaller photon fluxes or with longer wavelengths. Both circumstances weaken the disturbing action of the sounding radiation on the investigated object. Therefore the most enticing application of these still exotic states of optical photons is the analysis of the structure of complex biological molecules, but this calls for development of both generators and receivers for such states.

[1] W. Heitler, The Quantum Theory of Radiation, Oxford, 1954.
[2] A. I. Baz', Ya. B. Zel'dovich, and A. M. Perelomov, Rasseyanie, reaktsii i raspady v nerelyativistskoĭ kvantovoĭ mekhanike (Scattering, Reactions and Decays in Quantum Mechanicas), Nauka, 1966.
[3] V. M. Faĭn, Fotony i nelineĭnye sredy. Kvantovaya radiofizika (Photons and Nonlinear Media. Quantum Radiophysics), vol. 1, Sovetskoe Radio, 1972.
[4] A. E. Siegman, Microwave Solid Lasers, McGraw-Hill.
[5] R. Serber and C. H. Townes, Quantum Electronics, New York, 1960, p. 233.
[6] V. B. Braginskiĭ and Yu. I. Vorontsov, Usp. Fiz. Nauk 114, 41 (1974) [Sov. Phys. Usp. 17, 644 (1975)].
[7] V. B. Braginskiĭ, Yu. I. Vorontsov, and F. Ya. Khalili, Pis'ma Zh. Eksp. Teor. Fiz. 27, 296 (1978) [JETP Lett. 27, 276 (1978)].
[8] Kip S. Torne, Ronald W. P. Drevar, Carlton M. Caves, Mark Zimmerman, and Venon D. Sandberg, Phys. Rev. Lett. 40, 667 (1978).
[9] D. E. Vakman and L. A. Vaĭnshteĭn, Usp. Fiz. Nauk 123, 657 (1977) [Sov. Phys. Usp. 20, 1002 (1977)].
[10] P. Carruthers and M. M. Nietto, Rev. Mod. Phys. 40, 411 (1968).
[11] V. I. Tikhonov and M. A. Mironov, Markovskie protsessy (Markov Processes), Sovetskoe Radio, 1977.

Translated by J. G. Adashko