Superradiance of an extended system

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A comparison is presented of the results of the quantum and semiclassical theories of superradiance. It follows from the comparison that the one-dimensional semiclassical model of superradiance is valid at large Fresnel numbers. It is shown that the applicability of the spatially homogeneous solution is restricted to short or closed systems, and comes into play for extended systems only during the time of one passage of the light through the system. Numerical calculations are made of the superradiance pulses, of the population kinetics, and of the polarization under spatially homogeneous initial conditions. The properties of these quantities, which are due to invariance of the equations and of the initial and boundary conditions to scaling transformations, are discussed.

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1. INTRODUCTION

Even though Dicke's paper[1] in which he predicted superradiance theory was published more than 20 years ago, discussions of the models and approximations that describe this phenomenon adequately still continue.[2-4] Superradiance is collective spontaneous emission of a polyatomic system under conditions when the relaxation of the off-diagonal density-matrix elements of the atom is slower than the de-excitation process. One characteristic feature of superradiance, the proportionality of its intensity to the inversion-density squared, is due to the onset of a macroscopic dipole moment. The phasing of the atomic dipoles, which accounts for this effect, takes place spontaneously in the course of the emission, regardless of the character of the excitation (coherent or incoherent). This distinguishes superradiance from optical induction, where the quadratic effect is connected with coherent pumping.[1)]

Intense superradiance was observed in optically pumped HF gas on the transition between the rotational sublevels of an excited vibrational state.[5] The observed increase of the radiation intensity (by ten orders of magnitude compared with spontaneous emission) and the quadratic dependence of the intensity on the inversion density areed with Dicke's theory. To explain the waveform of the superradiance pulse, however, and to determine the influence exerted on the collective effect by the homogeneous broadening of the spectrum, a more detailed theory was necessary. The details were obtained in two approximations, quantum (quantum description of the electromagnetic field and of the atomic system)[6-10] and semiclassical (classical description of the electromagnetic field and quantum description of the atomic system).[4,11-18]

The purpose of the present paper is to compare the results of the quantum and semiclassical theories, to determine the regions of applicability of various semiclassical models of superradiance, and to investigate the solutions obtained for them.

2. QUANTUM THEORY OF SUPERRADIANCE

The quantum theory of superradiance is primarily of heuristic significance, since a consistent explanation of this phenomenon (as well as of ordinary spontaneous emission) is possible only with the aid of quantum electrodynamics. At the same time, for sufficiently simple models of superradiant systems (a system in a volume with linear dimensions smaller than the wavelength, in a system in the single-mode approximation without retardation and Coulomb interaction) the quantum theory yields the dynamics of the de-excitation most effectively.

It is known that in a two-level approximation all the operators pertaining to a polyatomic system can be expressed in terms of Pauli matrices (or energy spin operators). In particular, the energy operator of a polyatomic system interacting with an electromagnetic field is given by

$$\hat{H}=\hat{H}_0+\hat{H}_{int}, \tag{1}$$

where

$$\hat{H}_0=\hbar\omega_0\sum_j \hat{R}_z^{(j)}+\sum_{k\lambda}\hbar ck\hat{a}_{k\lambda}^{+}\hat{a}_{k\lambda},$$

$$\hat{H}_{int}=g_0'\frac{\hbar}{\sqrt{N}}\sum_{j,k,\lambda}(\mathbf{e}_\lambda\mu)\left[\hat{R}_+^{(j)}\hat{a}_{k\lambda}\exp(i\mathbf{kr}_j)+\hat{R}_-^{(j)}\hat{a}_{k\lambda}^{+}\exp(-i\mathbf{kr}_j)\right]. \tag{2}$$

Here $\hat{R}_z^{(j)}$ and $\hat{R}_\pm^{(j)}$ are the operators of the energy spin of the j-th atoms, $\hat{a}_{k\lambda}$ and $\hat{a}_{k\lambda}^{+}$ are the operators of annihilation and production of a photon with momentum $\mathbf{k}$ and polarization $\mathbf{e}_\lambda$, ω_0 is the resonant frequency of the atom, μ is the matrix element of the dipole moment of the transition, $\mathbf{r}_j$ is the radius vector of the j-th atom, N is the number of atoms, and the interaction constant g_0' is given by

$$g_0'=\left(\frac{N}{V}\frac{2\pi\omega_0\mu^2}{\hbar}\right)^{1/2}, \tag{3}$$

where V is the quantization volume.

If the linear dimensions of the system are less than the wavelength, the exponentials in (2) can be omitted. In this case the energy operator commutes with the operator of the square of the total energy spin:

$$[\hat{H},\hat{R}^2]=0,$$
$$\hat{R}^2=\hat{R}_x^2+\hat{R}_y^2+\hat{R}_z^2,\quad \hat{R}_\alpha=\sum_j R_\alpha^{(j)}. \tag{4}$$

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Assume that at the initial instant of time all the atoms were excited. Then the corresponding state is an eigenstate of the operator $\hat{R}^2$ with eigenvalue $R(R+1) = \frac{1}{2}N(\frac{1}{2}N+1)$. Since $\hat{R}^2$ is an integral of the motion, the quantum number $R = N/2$ is conserved while the system evolves. The de-excitation is accompanied by a cascade transition between states with quantum numbers $R_z = \frac{1}{2}N, \frac{1}{2}N-1, \ldots, -\frac{1}{2}N$. The probability per unit time of a transition between the states $|\frac{1}{2}N, R_z\rangle$ and $|\frac{1}{2}N, R_z-1\rangle$ can be calculated by perturbation theory:

$$\gamma_{R_z} = \frac{2\pi}{\hbar^2}\int \frac{V\omega^2}{(2\pi)^3c^3}\left|\left\langle \frac{1}{2}N,\ R_z\left|\hat{H}_{int}\right|\frac{1}{2}N, R_z-1\right\rangle\right|^2 \times\delta(\omega-\omega_0)\,d\omega\sin\theta\,d\theta\,d\varphi = \gamma\left(\frac{1}{2}N+R_z\right)\left(\frac{1}{2}N-R_z+1\right), \tag{5}$$

where $\gamma = 4\mu^2\omega_0^3/3\hbar c^3$ is the radiation constant of one atom.

The realization of the cascade transition in time with probabilities (5) can be obtained by the Monte Carlo method.[19] The intensity fluctuations that appear in this case were investigated in Refs. 7, 20, and 21.

The superradiance intensity averaged over different realizations can be obtained by summing the average transition times $T_{Rz} = \gamma_{Rz}^{-1}$. In fact, the average time in which n photons will be emitted (under the conditions $n \gg 1, N \gg 1$) is

$$t = \sum_{R_z=N/2}^{N/2-n} T_{R_z} = \gamma^{-1}\int_{N/2}^{N/2-n} \frac{dR_z}{(N/2+R_z)(N/2-R_z+1)} = \frac{2\gamma^{-1}}{N+1}\operatorname{arcth}\frac{2R_z-1}{N+1}\Bigg|_{N/2}^{N/2-n}. \tag{6}$$

From this we get for n

$$n = \frac{1}{2}N + \frac{1}{2}N\operatorname{th}\left[\frac{1}{2}\gamma N(t-t_0)\right], \tag{7}$$

where

$$t_0 = \gamma^{-1}\operatorname{arcth}\frac{N-1}{N+1} = \frac{\gamma^{-1}}{N}\ln N. \tag{8}$$

With the aid of (7) we obtain the average number of photons emitted per unit time:

$$\dot{n} = \frac{N}{4}\frac{1}{\tau_R}\operatorname{sech}^2\frac{t-t_0}{2\tau_R}, \quad \tau_R = (\gamma N)^{-1}. \tag{9}$$

The operator $\hat{R}^2$ of an extended system does not commute with the interaction Hamiltonian (2). However, as shown by Dicke,[1] at a fixed value of the wave vector $\mathbf{k}$ the operators

$$\hat{R}_{k\pm} = \sum_j \hat{R}_\pm^{(j)} \exp(\pm i\mathbf{k}\mathbf{r}_j), \quad \hat{R}_z, \tag{10}$$

have the same commutation properties as $\hat{R}_\pm$ and $\hat{R}_z$, and the operator $\hat{R}_k^2$ commutes with the corresponding terms of the Hamiltonian (having the same wave vector $\mathbf{k}$). It is shown later that the eigenstates $|R_k, R_z\rangle$ of the operators $\hat{R}_k^2$ and $\hat{R}_z^2$ approximately diagonalize the matrix of the self-energy part.[8] An investigation of the cascade transition between these states yields an estimate of the influence of the extended character of the system on the dynamics of the superradiance. The calculation of the probability of the transition between the state $|R_k, R_z\rangle$ and $|R_k, R_z-1\rangle$ is similar to (5), except that the integration of the sphere of radius $k_0 = \omega_0/c$ in k space must be replaced by integration over certain regions of this sphere with angular dimensions Ω about the point $\mathbf{k}_0$ (Ref. 8):

$$\gamma_{k,R_z} = \gamma'(N/2+R_z)(N/2-R_z+1). \tag{11}$$

If (for the sake of argument) the direction of the radiation is perpendicular to the vector μ, then

$$\gamma' = \frac{\omega_0^3\mu^2}{2\pi\hbar c^3}\Omega = \frac{3}{8\pi}\gamma\Omega. \tag{12}$$

The value of the solid angle Ω is connected with the uncertainty of the quasimomentum $\mathbf{k}$ of the state $|R_k, R_z\rangle$ and depends on the finite character of the dimensions of the system and on the inhomogeneity of the disposition of the atoms. Assume that the distribution of the atoms is homogeneous within a rectangular parallelepiped with dimensions L, D, D, and let $\mathbf{k}$ be directed along the edge L. It is obvious that in this case Ω is the solid angle subtending the intersection a sphere of radius $k_0 = \omega_0/c$, with center at the point k_0 and a parallelepiped with dimensions $1/L, 1/D, 1/D$.

$$\Omega = D^{-2}/k_0^2 = \lambda^2/D^2, \quad F \geqslant 1, \qquad \Omega = \lambda^{-1}L^{-1}/k_0^2 = \lambda/L, \quad F \leqslant 1, \tag{13}$$

where $F = D^2/\lambda L$ is the Fresnel number. Thus, the fact that the system is extended changes the time scale of the superradiance by a factor λ^2/D^2 or λ/L, respectively for large and small values of the Fresnel number.

According to (12) and (13) we have

$$\tau_R^{-1} = \gamma' N = \begin{cases} 3\gamma N\lambda^2/8\pi D^2 = 2\pi\omega_0\mu^2 nL/\hbar c, & F \geqslant 1, \quad (14a) \\ 3\gamma N\lambda/8\pi L = \omega_0^2\mu^2 nD^2/\hbar c^2, & F \leqslant 1, \quad (14b) \end{cases}$$

where $n = N/LD^2$ is the inversion density.

The quantity Ω determines the angle divergence of one mode.[2)] Comparing it with the angular dimension D^2/L^2 of the system, we find that for large values of Fresnel numbers there can exist several modes, and for small values of Fresnel numbers a single-mode regime exists. One can expect in the latter case (if the retardation can still be neglected) the kinetics of the superradiance pulse to have the form (10) with τ_R equal to (14b). For the multimode regime, the mixing of the different modes in the course of the radiation leads to loss of coherence of the atomic system, and consequently to a partial suppression of the superradiance effect. At $F > 1$ the quantity τ_R does not depend on the transverse dimension D of the system. Since large values of the Fresnel number are reached with increasing D, this result can be attributed to the homogeneity of the field in the transverse direction.

3. SEMICLASSICAL APPROXIMATION

In the semiclassical approximation, the problem of superradiance of two-level atoms can be described by the system of equations for the single-atom density matrix $\|\rho_{ab}\|$ and the equation for the electric field intensity:

$$i\hbar\dot{\rho}_{aa} = V_{ab}\rho_{ba} - \rho_{ab}V_{ba}, \qquad i\hbar\dot{\rho}_{ba} = \hbar\omega_0\rho_{ba} + V_{ba}\rho_{aa} - \rho_{bb}V_{ba}, \qquad \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \varkappa'\frac{\partial}{\partial t}\right)\mathbf{E} = \frac{4\pi}{c^2}\ddot{\mathbf{P}}, \tag{15}$$

where the subscripts a and b number the ground and ex-

cited states of the atom, and $V_{ba}-\mu\cdot\mathbf{E}$ is the matrix element of the interaction. The polarization vector P is expressed in terms of the density matrix:

$$\mathbf{P}(\mathbf{r},t)=n\mu(\rho_{ab}+\rho_{ba}). \tag{16}$$

Here n is the density of the radiating atom, which we assume to be equal to the initial inversion density. The term with the first derivative in the equation for the electric field describes the value of losses.

In practice the system (15) can be solved only in the one-dimensional approximation (i.e., assuming the field to be homogeneous in the transverse direction); this approximation, as already noted, corresponds to a large number of the Fresnel number. In this case the field **E** and the density matrix depend only on one spatial coordinate z directed along the radiation. The direction of the dipole moments of the transition of all the atoms will be assumed to be the same and perpendicular to the z axis. The vectors **E** and **P** have the same direction, and we shall therefore omit the vector designations from now on.

Neglecting the volume losses ($\varkappa'=0$), Eq. (15) for the electromagnetic field has the following explicit solution:

$$E(z,t)=-\frac{2\pi}{c}\int_0^L \dot{P}\left(t-\frac{|z-z'|}{c}\right)dz'. \tag{17}$$

Using (16), we get

$$E(z,t)=-\frac{2\pi}{c}\mu n\int_0^L\left\{\dot{\rho}_{ba}\left(z',t-\frac{|z-z'|}{c}\right)+\dot{\rho}_{ab}\left(z',t-\frac{|z-z'|}{c}\right)\right\}. \tag{18}$$

We separate in the density-matrix elements the essential dependence on the time:

$$\rho_{ba}=R^-(z,t)\exp(-i\omega_0 t),\quad \rho_{ab}=R^+(z,t)\exp(i\omega_0 t),\quad (R^{-*}=R^+), \tag{19}$$

and reduce the Bloch vector with components X, Y, and Z:

$$X+iY=R^+,\quad Z=(\rho_{bb}-\rho_{aa})/2. \tag{20}$$

We then obtain with the aid of (18)

$$E=-i\frac{2\pi}{c}\omega_0\mu n\int_0^L\left\{R^+\left(z',t-\frac{|z-z'|}{c}\right)\exp\left[i\omega_0\left(t-\frac{|z-z'|}{c}\right)\right]-R^-\left(z',t-\frac{|z-z'|}{c}\right)\exp\left[-i\omega_0\left(t-\frac{|z-z'|}{c}\right)\right]\right\}dz'. \tag{21}$$

(a) *Short system* ($L\ll\lambda$).[3] If $L\ll\lambda$, we can neglect the retardation and the dependence of the polarization on z. Then

$$E(z,t)=-i\frac{2\pi}{c}\omega_0\mu nL\left\{R^+(t)\exp\left[i\omega_0\left(t-\frac{|z|}{c}\right)\right]-R^-(t)\exp\left[-i\omega_0\left(t-\frac{|z|}{c}\right)\right]\right\}. \tag{22}$$

Using (22) and neglecting the rapidly oscillating terms, we obtain

$$\dot{Z}=-\frac{2L}{c}g_0^2R^+R^-,\quad \dot{R}^\pm=\frac{2L}{c}g_0^2R^\pm, \tag{23}$$

where

$$g_0^2=2\pi n\mu^2\omega_0/\hbar.$$

It is easy to verify with the aid of (23) that the Bloch vector preserves its length. Under the initial condition of total inversion we have $(X^2+Y^2+Z^2)^{1/2}=1/2$. We introduce the polar angle θ and the azimuthal angle φ. Then

$$X=\tfrac{1}{2}\sin\theta\cos\varphi,\quad Y=\tfrac{1}{2}\sin\theta\sin\varphi,\quad Z=\tfrac{1}{2}\cos\theta \tag{24}$$

and the system (23) takes the form

$$\dot{\theta}-\frac{1}{2\tau_R}\sin\theta=0,\quad \dot{\varphi}=0, \tag{25}$$

where

$$\tau_R=\frac{1}{g_0^2}\frac{c}{2L}=\left(\frac{4\pi\mu^2\omega_0 L}{\hbar c}\right)^{-1}. \tag{26}$$

This is half the value of τ_R obtained in quantum theory [see (14a)].

The solution of the system (25) is

$$\cos\theta=-\operatorname{th}\frac{t-t_0}{2\tau_R},\quad t_0=2\tau_R\ln\frac{2}{\theta(0)},\quad \varphi=\text{const}. \tag{27}$$

It follows therefore that the number of photons emitted per unit time is [cf.(9)]

$$I=-N\dot{Z}=\frac{1}{2}N\frac{1}{2}\tau_R^{-1}\operatorname{sech}^2\frac{t-t_0}{2\tau_R}. \tag{28}$$

The delay time t_0 is determined from the initial value of $\theta(0)$, which in turn can be determined from the condition that at the start of the process the de-excitation intensity $\bar{J}$ averaged over the period and corresponding to the initial polarization $P(0)$ is equal to the spontaneous-emission intensity J_{sp}:

$$J=\frac{2}{3}c^{-3}\ddot{P}^2(0)\cdot\frac{1}{2}=\frac{1}{3}\omega_0^4c^{-3}\mu^2(2X(0))^2N^2=\hbar\omega\gamma X^2(0)N^2,$$
$$J_{sp}=\hbar\omega\gamma N. \tag{29}$$

Equating these two quantities we get

$$X(0)=\left(\frac{1}{N}\right)^{1/2},\quad \theta(0)=X(0)\Big/\frac{1}{2}=2\left(\frac{1}{N}\right)^{1/2},\quad t_0=\tau_R\ln N. \tag{30}$$

It is interesting to note that the expressions for the intensity (28) and for the delay time (30) coincide with the corresponding results obtained in the quantum analysis of single-mode superradiance (see formulas (9) and (14a)). The difference, by a factor of two, of the values of τ_R (cf. (14a) and (26)) can be attributed to the fact that in semiclassical description account is taken of emission in two opposite directions, whereas the quantum results pertain to a single mode, i.e., emission in a single direction.

(b) *Extended system (spatially homogeneous solution).* We consider an extended system with $L\gg\lambda$. We seek the solution of Eqs. (15) in the form of plane waves whose amplitudes do not depend on the spatial coordinates:

$$E=\mathscr{E}(t)\exp(-i\omega_0 t+ikz)+\text{c.c.},$$
$$\rho_{ba}=R^-(t)\exp(-i\omega_0 t+ikz),\quad \rho_{ab}=R^+(t)\exp(i\omega_0 t-ikz). \tag{31}$$

In the approximation of slowly varying amplitudes, we get from (15)

$$\dot{A}+\varkappa A=g_0R^-,\quad \dot{R}^-=2g_0AZ,\quad \dot{Z}=-g_0(AR^++A^*R^-), \tag{32}$$

where A is the dimensionless amplitude of the electric field intensity, $|A|^2$ is the number of photons in the volume per atom:

$$A=-i(2\pi\hbar\omega_0 n)^{-1/2}\mathscr{E},\quad \varkappa=\frac{1}{2}\varkappa' c^2.$$

Eq. (32) reduces to the equation of a mathematical pendulum for a polar angle θ:

$$\ddot{\theta}+\varkappa\dot{\theta}-g_0^2\sin\theta=0. \tag{33}$$

The solution of this equation was used in Refs. 3, 7, 9, 17, and 18 to determine the waveform of the superradiance pulse, wherein the volume losses described effectively the radiation emitted from the volume. To this end, the coefficient $\varkappa$ was assumed equal to $c/2L$. Although the pulse obtained in this manner has the same superradiance properties, we must agree with the authors of Refs. 2 and 4 that these solutions cannot be used for a quantitative description in those cases (particularly, in the experiment of Ref. 5) when the spatial-inhomogeneity condition is not satisfied.

Obviously, the spatial-homogeneity condition is satisfied under cyclic boundary conditions, i.e., for a closed system. The dynamics of the radiation at a fixed inversion density does not depend in this case on the length of the system and is determined by the ratio of the constants $\varkappa$ and g_0. The intensity of radiation (the number of photons passing through the cross section per unit time) is given by

$$I=\frac{A^*A}{L}c=\frac{Nc}{4g_0^2L}\dot{\theta}^2, \tag{34}$$

i.e., it is proportional to the kinetic energy of the pendulum. In the case of large damping ($\varkappa \gg g_0$) it follows from (33) and (34) that

$$I=\frac{cN}{2L\varkappa}\frac{1}{\tau_R'}\operatorname{sech}^2\frac{t-t_0}{2\tau_R'}, \tag{35}$$

where

$$\tau_R'=\frac{\varkappa}{g_0^2},\quad t_0=\tau_R'\ln\frac{2}{\theta(0)}.$$

In the general case ($\varkappa \sim g_0$) damped oscillations of the intensity, connected with energy exchange between the atoms in the field will take place in the system. In the limiting case $\varkappa = 0$ the period of these oscillations is equal to $2g^{-1}\ln[8/\theta(0)]$—the half-period of the oscillations of the mathematical pendulum, and the intensity at the maxima (corresponding to the maxima of the kinetic energy of the pendulum) reaches the value $I_{\max} = Nc/L$.

It should be noted that the solution described above (of the mathematical-pendulum type) will determine the intensity of the radiation in the single-mode regime during a time L/c under all boundary conditions. Since the maximum of the intensity in this solution is inversely proportional to the length of the system, while the number of pulsations is proportional to this length, the integrated intensity of these pulsations is practically independent of L and amounts to about to 20% of the stored energy (see Fig. 1c).

(c) *Extended system (zero boundary conditions).* In the general case the superradiance pulse of an extended system can be obtained by numerically solving Eqs. (15). We seek the solution in the form of plane waves propagating in one direction, with amplitudes that vary slowly in space and in time:

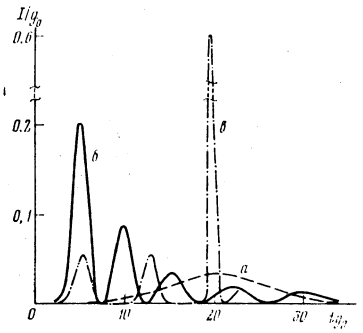

FIG. 1. Superradiance pulses (intensity per atom) for systems of various lengths: $\theta(0) = 4\cdot10^{-2}$; a) $L = 0.5\,c/g_0$; b) $L = 4c/g_0$; c) $L = 18\,c/g_0$.

$$\begin{aligned}E&=\mathscr{E}^-(t,z)\exp(-i\omega_0t+ikz)\\&\quad+\mathscr{E}^+(t,z)\exp(i\omega_0t-ikz),\\ \rho_{ba}&=R^-(t,z)\exp(-i\omega_0t+ikz),\\ \rho_{ab}&=R^+(t,z)\exp(i\omega_0t-ikz).\end{aligned} \tag{36}$$

Substituting (36) in (15) and neglecting the relatively oscillating terms, we obtain

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}Z&=-\mu(\mathscr{E}^-R^+-\mathscr{E}^+R^-),\\ i\hbar\frac{\partial}{\partial t}R^\mp&=\pm2\mu\mathscr{E}^\mp Z,\\ \frac{\partial}{\partial z}\mathscr{E}^\mp+\frac{1}{c}\frac{\partial}{\partial t}\mathscr{E}^\mp&=\pm i\frac{2\pi\omega_0\mu n}{c}R^\mp.\end{aligned} \tag{37}$$

If we introduce in place of t the retarded time $T = t - z/c$, then we can rewrite (37) in the form

$$\begin{aligned}&i\hbar\frac{\partial}{\partial T}Z=-\mu(\mathscr{E}^-R^+-\mathscr{E}^+R^-),\quad i\hbar\frac{\partial}{\partial T}R^\mp=\pm2\mu\mathscr{E}^\mp Z,\\ &\frac{\partial}{\partial z}\mathscr{E}^\mp=\pm i\frac{2\pi\omega_0\mu n}{c}R^\mp.\end{aligned} \tag{38}$$

In the dimensionless variables $\zeta = z/L$ and $\tau = T/\tau_R$ we obtain

$$\begin{aligned}&\frac{\partial Z}{\partial\tau}=-\frac{1}{2}(\varepsilon^-R^++\varepsilon^+R^-),\quad \frac{\partial R^\mp}{\partial\tau}=\varepsilon^\mp Z,\quad \frac{\partial\varepsilon^\mp}{\partial\zeta}=R^\mp,\\ &\varepsilon^\mp=\mp\frac{ic}{2\pi\omega_0n\mu L}\mathscr{E}^\mp.\end{aligned} \tag{39}$$

The radiation intensity (the number of photons passing through the cross section per unit time) can be represented in the form

$$I=\frac{\mathscr{E}^+\mathscr{E}^-}{2\pi}c\frac{N}{\hbar\omega_0nL}=\frac{1}{2\tau_R}\varepsilon^+\varepsilon^-. \tag{40}$$

Fig. 1 shows the superradiance pulses obtained under the homogeneous initial conditions

$$Z(0,z)=\frac{1}{2},\quad \theta(0)=\frac{\operatorname{Re}R^\mp(0,z)}{Z(0,z)}=4\cdot10^{-2}$$

for different lengths L. At $L > ct_0$, pulsations appear, corresponding to the "mathematical pendulum" regime. One can expect for a system with sufficiently large di-

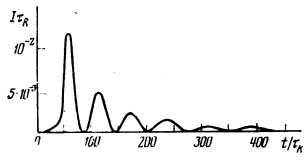

FIG. 2. Pulse of superradiance of an extended system (without retardation): $\theta(0) = 4\times10^{-4}$.

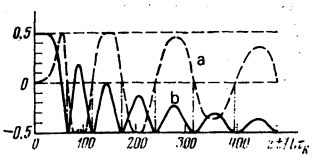

FIG. 3. Scaling properties of superradiance of an extended system (without retardation): $\theta(0)=4\times10^{-4}$; a) polarization; b) population.

mensions the emission of radiation in the form of pulsations to be suppressed by the competing process of radiation in the direction with the shorter length (i.e., with smaller grain).

4. SCALING PROPERTIES OF SUPERRADIATION

A solution that describes superradiance stimulated by a like pulse of small area θ was obtained for the system (15) in Ref. 4.[4)] The authors called attention to the fact that the shape of the pulse is determined only by the quantities θ and τ_R. This property was called the scaling property of superradiance, since it is governed by the scale invariance of Eqs. (39) and of the initial conditions to a scale transformation (see also Ref. 11):

$$\zeta\to\alpha\zeta,\quad T\to\alpha^{-1}T,\quad \varepsilon\to\alpha\varepsilon, \tag{41}$$

The homogeneous initial conditions used in our paper are in the general case not invariant to the scale transformation (41). Therefore the shape of the superradiance pulse depends also on the length of the system (see Fig. 1). Under homogeneous initial conditions, the scaling properties manifest themselves in the case when the retardation can be neglected, i.e., if $\tau_R > L/c$. In this case the variable τ in the equations has the meaning of time (and not retarded time); since the initial condition is homogeneous over the length $\theta(0,\zeta)=\text{const}$ and Eqs. (39) do not depend explicitly on L, the distribution of all the quantities Z, $R^{\pm}$, and $\varepsilon^{\pm}$ for systems of different but at equal instants of time (in units of τ_R) are similar in form. The consequence is a universal form of the superradiance pulse under identical initial conditions for systems with different lengths (Fig. 2).

The invariance of Eq. (39) and of the initial and boundary conditions $R^{\pm}(0,\zeta)=R^{\pm}(\tau,0)$ to the scaling transformation (1) leads to the following properties of the solution. The dependence of $R^{\pm}$ and of Z on τ at a fixed value of ζ coincides with the dependence of these quantities on ζ at a fixed value of τ, i.e., $R^{\pm}$ and Z are functions of the product $\zeta\tau$ (Fig. 3). The distribution of the quantities $R^{\pm}$, Z, and $\varepsilon^{\pm}$ over the length at the instant of time τ_1 is obtained by stretching the distribution of these quantities at the instant of time τ_2 with a stretching coefficient $\alpha=\tau_2/\tau_1$; the intensity $\varepsilon^{\pm}$ is multiplied in this case by the stretching coefficient (Figs. 4, 5, 6).

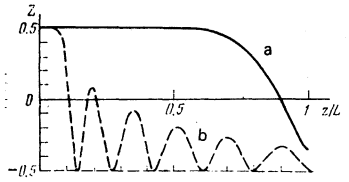

FIG. 4. Distribution of the population in superradiance of an extended system (without retardation) at various instants of time: $\theta(0)=4\cdot10^{-4}$; a) $\tau_1=60\tau_R$; b) $\tau_2=400\tau_R$.

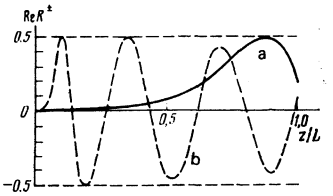

FIG. 5. Scaling properties of the distribution of the polarization of superradiance of an extended system (without retardation: $\theta(0)=4\cdot10^{-4}$; a) $\tau_1=60\tau_R$; b) $\tau_2=400\tau_R$.

5. CONCLUSION

A comparison of the results of the quantum and semiclassical (one-dimensional) theory of superradiation shows that at Fresnel numbers greater than unity, the time scale of the superradiance pulse, determined by the quantity τ_R, is the same. The complexity of the quantum analysis is due in this case to the need for taking into account several modes with close values of the momentum, and makes therefore the semiclassical description preferable. For a quantitative investigation of the superradiance pulse, even for spatially homogeneous initial conditions, the assumption that the amplitudes of the solution are homogeneous is in the general case not justified. The solutions are homogeneous either for systems shorter than the wavelength, or for closed systems with cyclical boundary conditions. In the latter case the value of τ_R does not depend on the length of the system and is determined by the coefficient of the volume losses. A homogeneous solution for the superradiance pulse of an extended system is obtained only during a time $t < L/c$. The invariance of the initial conditions and of the equations that describe the evolution of the superradiance to scaling transformation leads to scaling properties of the solution. These properties have a particular simple and illustrative form for sufficiently short systems, when the retardation can be neglected ($\tau_R > L/c$). A consequence of the scaling invariance is the universality of the shape of the superradiance pulse and the similarity properties of the distribution of the field, of the polarization, and of the population along the system at different instants of time.

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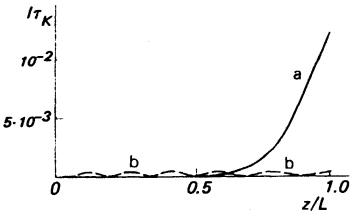

FIG. 6. Scaling properties of the distribution of the intensity of superradiance of an extended system (without retardation): $\theta(0)=4\cdot10^{-4}$; a) $\tau_1=60\tau_R$; b) $\tau_2=400\tau_R$.

[1)] In some papers, superradiance produced by incoherent pumping is called superfluorescence or intense superradiance, while optical induction and optical echo are called limited superradiance.
[2)] A more exact estimate of the angular dimension of one mode is given in Ref. 24.
[3)] This particular case of the solution of the system (21) was considered in Ref. 16.
[4)] The area of the pulse coincided with the polar angle of the Bloch vector.

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Calculations of inelastic electron scattering from atoms in the second Born approximation

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The high-energy asymptotic behavior of the second Born approximation is examined. The formulas obtained are used to calculate differential and total cross sections for inelastic scattering of electrons from hydrogen and helium atoms. The results of the calculations are compared with experiment.

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1. INTRODUCTION

Until recently the second Born approximation, unlike the first one, had not been much used in the theory of collisions of electrons with atoms. This is evidently due to the simplicity and efficacy of the first approximation and the complexity of the second, together with the fact that the early studies seemed to indicate that the second approximation was not very efficacious.

Of course problems involving two-electron transitions cannot be treated consistently without going beyond the limitations of first-order perturbation theory. But such problems are complex and have not been worked out well so far. In most of the studies total cross sections for single-electron transitions were calculated and compared with experiment, and various attempts were made to improve the results of the first Born approximation in the region of low collision energies where it usually gives values some 50–100% higher than the experimental values. In particular, attempts were made to improve these results by taking the higher-order (mainly the second-order) perturbation-theory contributions into account.

Difficulties involved in calculating the integrals that arose hindered the use of the second Born approximation for a long time. But when Dalitz,[1] Lewis,[2] and others had developed a technique for calculating these integrals, the calculation of the contributions from individual intermediate states ceased to present any difficulties in principle, although the calculations remained rather laborious.

The proposed technique was used in a series of studies by Moiseevich and collaborators[3-6] and by Wollings and McDowell[7] to calculate the total elastic scattering cross sections of hydrogen and helium as well as the total cross sections for excitation of the $2s$ and $2p$ levles of hydrogen and the 2^1S and 2^1P levels of helium.

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