Stationary weakly turbulent distributions in a spin-wave system

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Stationary power-law solutions, corresponding to conditions of weak turbulence, are found for the kinetic equation of spin waves in a ferromagnet. Methods of producing and observing weakly turbulent distributions are considered.

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For systems whose dynamics is described by a kinetic equation, when the range of energy injection and the dissipative range are appreciably separated in wave-vector space, there can be produced nontrivial stationary distributions corresponding to conditions of weak turbulence. In such states, as a result of a fast relay transfer mechanism, there are constant flows of energy or of number of particles over the spectrum. Weakly turbulent solutions were first constructed by Zakharov[1] for the case of acoustic waves with a decomposition-type dispersion law. At present the general theory of finding weakly turbulent solutions is quite well developed, and there are many examples of such distributions.[2,3]

The present paper considers the question of weakly turbulent distributions in a spin-wave system. We shall be interested in non-equilibrium stationary distributions in the absence of parametric excitation, in the classical limit, when the occupation numbers are large: $n_k \gg 1$. The question of a stationary state of parametrically excited spin waves has already been studied in detail.[4]

The paper consists of two parts. In the first section, exact stationary, weakly turbulent solutions are found for the kinetic equation of spin waves. Then the possibility of realizing such solutions is considered, and methods of experimental observation of distributions corresponding to these solutions are discussed.

1. STATIONARY NONEQUILIBRIUM DISTRIBUTIONS OF SPIN WAVES

The Hamiltonian of the spin-wave system of a ferromagnet has the form

$$\mathscr{H} = \sum_k \varepsilon_k c_k^+ c_k + \frac{1}{\sqrt{N}} \sum_{123} (\Psi_{12,3} c_1^+ c_2^+ c_3 + \text{H.c.}) + \frac{1}{N} \sum_{1234} \Phi_{12,34} c_1^+ c_2^+ c_3 c_4 + V(t), \quad (1)$$

where $\Psi_{12,3}$ is the amplitude of three-magnon dipole processes, $\Phi_{12,34}$ is the amplitude of four-magnon exchange scattering processes, and $V(t)$ describes the external perturbation that insures influx of energy into the spin-wave system. Hereafter only the weak-anisotropy case will be considered. By means of the Hamiltonian (1) one can immediately write the kinetic equation for magnons; in the limit of large occupation numbers, it has the form

$$\begin{aligned} \frac{\partial n_1}{\partial t} = & \frac{1}{2} \int d\tau_2 d\tau_3 W(\mathbf{k}_2, \mathbf{k}_3; \mathbf{k}_1) [n_2 n_3 - n_1 n_2 - n_1 n_3] \\ & + \int d\tau_2 d\tau_3 W(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3) [n_1 n_3 + n_2 n_3 - n_1 n_2] \\ & + \frac{1}{2} \int d\tau_2 d\tau_3 d\tau_4 W(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4) \\ & \times [n_1 n_3 n_4 + n_2 n_3 n_4 - n_1 n_2 n_3 - n_1 n_2 n_4], \end{aligned} \quad (2)$$

where

$$d\tau_i = v_0 d\mathbf{k}_i / (2\pi)^3,$$

$$W(\mathbf{k}_2, \mathbf{k}_3; \mathbf{k}_1) = 2\pi |\Psi_{23,1} + \Psi_{32,1}|^2 \delta(\varepsilon_1 - \varepsilon_2 - \varepsilon_3) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3), \quad (3)$$

$$W(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4) = 2\pi |\Phi_{12,34} + \Phi_{21,34} + \Phi_{12,43} + \Phi_{21,43}|^2 \times \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4). \quad (4)$$

An estimate of the various terms in (2) leads to the isolation of two wave-vector ranges (see below):

$$\frac{(\mu M_0)^{3/2}}{\Theta_C (\hbar \dot{E})^{1/4}} \gg (ak)^2 \gg \max\left\{ \frac{\mu H}{\Theta_C}, \frac{\mu M_0}{\Theta_C} \right\} \quad (5)$$

$$(ak)^2 \gg \max\left\{ \frac{(\mu M_0)^{3/2}}{\Theta_C (\hbar \dot{E})^{1/4}}, \frac{\mu M_0}{\Theta_C} \right\}, \quad (6)$$

where μ is the Bohr magneton, Θ_C is the Curie temperature, $M_0 = \mu S / v_0$ is the saturation magnetization, v_0 is the volume of the elementary cell, S is the spin of an ion, a is a quantity of the order of an atomic dimension, $\dot{E}$ is the energy being introduced in unit time per elementary cell, and H is the external biasing field.

In the range (5), the important terms in the collision integral are those corresponding to three-magnon interaction. In this case the amplitude $\Psi_{12,3}$ and the spectrum ε_k have the form[5]

$$\Psi_{12,3} = \pi \left(\frac{2}{S} \right)^{1/2} \mu M_0 (e^{i\varphi_1} \sin 2\theta_1 + e^{i\varphi_2} \sin 2\theta_2), \quad \varepsilon_k = \Theta_C (ak)^2, \quad (7)$$

where θ and φ are the polar and azimuthal angles that the vector $\mathbf{k}$ makes with the direction of $\mathbf{M}$.

Since the general methods of finding weakly turbulent solutions are inapplicable in this case because of the anisotropy of the transition probability $W(\mathbf{k}_2, \mathbf{k}_3; \mathbf{k}_1)$, we shall use direct calculation. After integration over angles and over the wave vector $\mathbf{k}_2$, on the supposition that the distribution function is isotropic, the dipole-dipole collision integral takes the form

$$\begin{aligned} \frac{\partial n_\varepsilon}{\partial t} = & \frac{\pi}{16 S \hbar} [1 + 2\cos^2\theta - 3\cos^4\theta] \frac{(\mu M_0)^2}{\varepsilon^{1/2} \Theta_C^{3/2}} \\ & \times \left\{ \int_0^\varepsilon d\varepsilon' [n_{\varepsilon'} n_{\varepsilon-\varepsilon'} - n_\varepsilon n_{\varepsilon'} - n_{\varepsilon-\varepsilon'} n_\varepsilon] - 2 \int_0^\infty d\varepsilon' [n_\varepsilon n_{\varepsilon'} - n_{\varepsilon+\varepsilon'} n_\varepsilon - n_{\varepsilon+\varepsilon'} n_{\varepsilon'}] \right\}. \end{aligned} \quad (8)$$

We shall seek a stationary solution of the collision integral (8) in the form of an isotropic power-law function, $n \propto \varepsilon^s$. Here one can use the method of linear fractional

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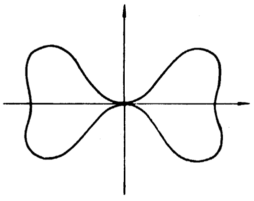

FIG. 1.

transformations.[6]

There are two solutions that make the collision integral vanish:

$$n \propto \varepsilon^{-1}, \quad n \propto \varepsilon^{-3/2}.$$

The first solution corresponds to the equilibrium Rayleigh-Jeans distribution, the second to a constant flow of energy through the spectrum. To within a numerical coefficient, it has the form

$$n_k \propto (\hbar\dot{E})^{1/2}/\mu M_0 (ak)^3. \tag{9}$$

This distribution is local; that is, the main contribution to the interaction, over the whole inertial range, comes from wave vectors of the same order of magnitude. Although, just as in the isotropic case, only the radial component of the energy flow is nonzero, the anisotropy of the three-magnon collision integral leads to a dependence of it on angles:

$$I_\varepsilon \propto [1+2\cos^2\theta-3\cos^4\theta]. \tag{10}$$

The angular dependence of the energy flow is shown in Fig. 1; I_s reaches its maximum value at $\cos\theta = \pm 1/\sqrt{3}$.

Furthermore, in the first order of perturbation theory there is a nontrivial correction to the equilibrium distribution, $\delta n \propto \varepsilon^{-2}$ (Ref. 7), which corresponds to a small deviation from the equilibrium distribution in the presence of a slight flow of energy through the spectrum. This distribution is also found to be local (in estimating how local, one must take into account that the logarithmic divergences that occur at small ε_k in each of the two terms in (8) have opposite signs and compensate each other). It must be noted, however, that extension of perturbation theory to higher orders encounters considerable difficulties even in the second order of perturbation theory. The angular dependence of the radial component of energy flow is also described by formula (10).

In the range (6), the determining role is played by four-magnon interaction. For this case, the spin-wave interaction amplitude $\Phi_{12,34}$ has the form[5]

$$\Phi_{12,34} = -{}^1/_4\Theta_C a^2(\mathbf{k}_1\mathbf{k}_2+\mathbf{k}_3\mathbf{k}_4). \tag{11}$$

At strong fields, $H \gg M$, in the spectrum ε_k a term μH is added in (7); but for the case of four-magnon interaction, the presence of an additive correction in the spectrum does not destroy the self-modeling character of the collision integral. The scattering probability (11) is an isotropic power-law function. In this case, the methods of Refs. 6 and 7 are applicable for finding the weakly turbulent solutions. Besides the Rayleigh-Jeans distribution, there are two solutions, $n^{(0)} \propto k^{-11/3}$ and $n^{(1)} \propto k^{-13/3}$, that correspond to constant flow of particles and of energy through the spectrum.[1)] The investigation of the local character in this case is practically the same as that carried out earlier by Zakharov[6] (where only the second solution was found). Both solutions are found to be local. In a comparison with the estimate of local character given in Ref. 6, one must bear in mind that at large $\mathbf{k}_2$ the degree of $\mathbf{k}_2$ in the denominator of the integrand is increased by unity after integration over the angles, thus ensuring the convergence at large $\mathbf{k}_2$ (see also the general case of investigation of locality in Ref. 7). To within a constant multiplier of order of magnitude unity, the distributions have the form

$$n_k^{(0)} \propto (\hbar\dot{N})^{1/3}/\Theta_C^{1/3}(ak)^{11/3}, \tag{12}$$

where $\dot{N}$ is the change of the number of magnons in unit time per unit cell, and

$$n_k^{(1)} \propto (\hbar\dot{E})^{1/3}/\Theta_C^{2/3}(ak)^{13/3}. \tag{13}$$

An estimate on the ranges of the wave vectors (5) and (6) will be obtained on substitution of the distributions (9) and (13) in the collision integral (2). In the case of the distribution (12), in order that the four-magnon interaction may play the determining role following conditions must be satisfied:

$$(ak)^2 \gg \max\left\{\frac{\mu M_0}{\Theta_C}, \left(\frac{\mu M_0}{\Theta_C}\right)^{4/5}\left(\frac{\Theta_C}{\hbar\dot{N}}\right)^{1/5}\right\}. \tag{14}$$

In the first order of perturbation theory, there are nontrivial isotropic and anisotropic corrections to the stationary solutions obtained.[7] We shall consider only the isotropic case. For the Rayleigh-Jeans distribution, under the condition $(ak)^2 \gg \mu H/\Theta_C$ we have $\delta n \propto k^{-7}$ and $\delta n \propto k^{-9}$, which corresponds to small flows of the number of particles and of energy. In the opposite limiting case $\mu M_0/\Theta_C \ll (ak)^2 \ll \mu H/\Theta_C$, we have $\delta n \propto k^{-11}$ and $\delta n \propto k^{-13}$. For the solution (12) there is a correction $\delta n \propto k^{-17/3}$. All these solutions are found to be nonlocal. Only for the solution (13) is there a local correction $\delta n \propto k^{-7/3}$, which corresponds to a small flow of the number of particles.

In the investigation of stationary solutions of the collision integral (2), we have considered interaction of waves having only large occupation numbers and have disregarded interaction of waves with large and with small occupation numbers. Only the local distributions are mathematically correct and of physical interest. On the other hand, the local property corresponds to the fact that the only important interaction is that of waves with occupation numbers of a single order of magnitude, which insures self-consistency of the approach.

In the collision integral (2), umklapp processes have also been disregarded.[2)] This is certainly correct if the characteristic scales of turbulence of the distributions (9), (12), and (13),

$$\mathscr{L}_1 = a\frac{(\mu M_0)^{1/6}}{(\hbar\dot{E})^{1/6}}, \quad \mathscr{L}_2 = a\left(\frac{\Theta_C}{\hbar\dot{N}}\right)^{1/11}, \quad \mathscr{L}_3 = a\left(\frac{\Theta_C^2}{\hbar\dot{E}}\right)^{1/13} \tag{15}$$

are much larger than an atomic dimension. Further estimates show that this requirement can be satisfied.

The condition for applicability of the approximation of weak turbulence requires that the turbulent perturbation of the momentum shall be much smaller than the equilibrium value of the momentum; for the distributions (9), (12), and (13) respectively, this leads to the following conditions:

$$\frac{(\hbar\dot{E})^{1/2}}{\mu M_0} \ll 1, \quad \frac{(\hbar\dot{N})^{1/3}}{\Theta_C^{1/3}(ak_{min})^{2/3}} \ll 1, \quad \frac{(\hbar\dot{E})^{1/2}}{\Theta_C^{2/3}(ak_{min})^{4/3}} \ll 1, \tag{16}$$

where the value of $k_{\min}$ is determined by the condition (6). The characteristic values of the quantities that occur in all these conditions are discussed in the following section.

2. CONDITIONS FOR REALIZATION OF WEAKLY TURBULENT DISTRIBUTIONS

In order that a transition may occur to conditions of weak turbulence, it is necessary, on the one hand, that the external action shall be strong enough so that the condition of perturbation theory ceases to be applicable,[3)]

$$\delta n/n_0 \gtrsim 1, \tag{17}$$

and on the other hand, there must be satisfaction of the criterion (16) for weak turbulence, of the conditions (15) $\mathcal{L}_i \gg a$, and of the requirement that the occupation numbers must be large, $n_k \gg 1$, in the inertial interval. As is evident from the corresponding expressions, simultaneous satisfaction of these requirements is in general contradictory; therefore there is substantial interest in the question of the possibility of realization of weakly turbulent distributions with any concrete method of introduction of energy into the spin-wave system. Below, we shall consider only the most typical case of introduction of energy, by microwave pumping.

We shall calculate the value of the energy flow $\dot{E}$ through the spectrum; this determines the degree of deviation of the spin-wave system from equilibrium. The value of the energy absorbed by a ferromagnet has been calculated previously in two cases: when splitting of a microwave-field quantum into two magnons can occur ($\hbar\omega > 2\varepsilon_0$),[8] and in the limiting low-frequency case $\omega\tau^{(0)} \ll 1$, when the motion of the magnetic moment is adiabatic.[8,9] We are interested in the frequency range $\tau_0^{-1} \ll \omega \lesssim 2\varepsilon_0/\hbar$, in which parametric resonance is absent (the half-frequency of pumping lies below the gap in the spin-wave spectrum) and the external perturbation is nonadiabatic. We shall consider the case in which the alternating field $h(t)$ is applied along the biasing field H. Then the last term of the expression (1) has the form

$$V(t) = \mu h(t) \sum_{\mathbf{k}} \left[\frac{1}{2} (u_{\mathbf{k}}^2 + |v_{\mathbf{k}}|^2 c_{\mathbf{k}}^+ c_{\mathbf{k}} + u_{\mathbf{k}} v_{\mathbf{k}} c_{\mathbf{k}}^+ c_{-\mathbf{k}}^+ + \text{H.c.} \right], \tag{18}$$

where

$$u_{\mathbf{k}} = \left(\frac{A_{\mathbf{k}} + \tilde{\varepsilon}_{\mathbf{k}}}{2\tilde{\varepsilon}_{\mathbf{k}}} \right)^{1/2}, \quad v_{\mathbf{k}} = e^{-2i\varphi_{\mathbf{k}}} \left(\frac{A_{\mathbf{k}} - \tilde{\varepsilon}_{\mathbf{k}}}{2\tilde{\varepsilon}_{\mathbf{k}}} \right)^{1/2}, \quad \tilde{\varepsilon}_{\mathbf{k}} = (A_{\mathbf{k}}^2 - |B_{\mathbf{k}}|^2)^{1/2};$$

$\varepsilon_{\mathbf{k}}$ is the spin-wave spectrum with allowance for dipole-dipole interaction.[5] By virtue of the condition $\hbar\omega < 2\varepsilon_0$, in the lowest order of perturbation theory the processes described by the Hamiltonian (18) are forbidden. In the calculation of $\dot{E}$, it is necessary, in the next order of perturbation theory, to put together processes with a larger number of magnons. With allowance for various terms (for fusion of a microwave quantum with a magnon and two magnons, see Fig. 2), the Hamiltonian of interaction of microwave-field quanta with magnons takes the form

$$V_{int} = \sum_{123} [V_1(12,3;\varepsilon_i) c_\nu^+ c_1^+ c_2^+ c_3 + V_2(12,3;\varepsilon_i) c_\nu^+ c_3^+ c_2 c_1 + \text{H.c.}]; \tag{19}$$

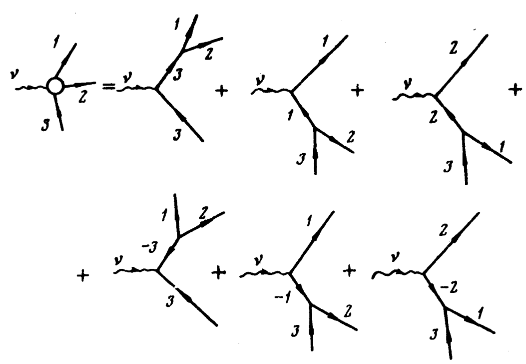

FIG. 2.

$$V_1(12,3;\varepsilon_i) = 2\frac{\tilde{\Psi}_{12,3}A}{\hbar\omega}[2|v_3|^2 - |v_1|^2 - |v_2|^2] + \frac{\tilde{\Psi}^{\bullet}_{13,2}Au_1v_1}{\hbar\omega - 2\varepsilon_1} + \frac{\tilde{\Psi}^{\bullet}_{23,1}Au_2v_2}{\hbar\omega - 2\varepsilon_2} - 2\frac{\tilde{\Psi}'_{123}Au_3v_3}{\hbar\omega + 2\varepsilon_3},$$
$$V_2(12,3;\varepsilon_i) = -V_1(12,3;-\varepsilon_i), \tag{20}$$

$\tilde{\Psi}_{12,3}$ is the amplitude of splitting of a magnon into two, which differs from (7) by allowance for the u, v transformation; $\tilde{\Psi}'_{123}$ is the amplitude of "explosive" formation of three magnons from the vacuum; A is a coefficient of proportionality that appears upon quantization of the microwave field.[10]

It must be emphasized that at relatively large wave vectors, determined by the condition $(ak)^2 \gg \mu M_0/\Theta_C$, when $u \approx 1$ and $v \ll 1$, the interaction amplitudes (20) of microwave-field quanta with magnons vanish. This means that the main injection of energy occurs in the small-wave-vector range $(ak)^2 \lesssim \mu M_0/\Theta_C$ outside the inertial range. (The situation is similar with the process of interaction of a microwave-field quantum with four magnons; but allowance for this process would lead to an insignificant additional term $(T/\Theta_C)(\mu H/\Theta_C)^{1/2}$ in the kinetic equation.) For the change of the occupation numbers of the microwave-field quanta we have

$$\frac{dN_\nu}{dt} = \frac{4\pi}{\hbar} \int d\tau_1 \, d\tau_2 \, d\tau_3 \, \delta(\mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2) \times \{ |V_1|^2 N_\nu [(n_1+n_2)n_3 - n_1 n_2] \delta(\hbar\omega + \varepsilon_1 + \varepsilon_2 - \varepsilon_3) + |V_2|^2 N_\nu [n_1 n_2 - n_3(n_1+n_2)] \delta(\hbar\omega + \varepsilon_3 - \varepsilon_1 - \varepsilon_2) \} \tag{21}$$

The calculations are carried out most simply for $H \gg M_0$, when $u_k \approx 1$ and $v_k \sim \mu M_0/\varepsilon_k$. Since for estimation of the applicability of perturbation theory, $\delta n/n_0 \lesssim 1$, the occupation numbers of the magnons may be considered in equilibrium, we get from (20) and (21)

$$\dot{E} = \frac{d}{dt}(\hbar\omega N_\nu) \approx \begin{cases} \dfrac{(\mu h)^2}{\hbar} \dfrac{(\mu M_0)^4 T^2}{(\mu H)^3 \Theta_C^3}, & \hbar\omega \ll \mu H \\[2ex] \dfrac{(\mu h)^2}{\hbar} \dfrac{(\hbar\omega)^2 (\mu M_0)^3 T^2}{(\mu H)^4 \Theta_C^3}, & \hbar\omega \lesssim 2\mu H \end{cases} \tag{22}$$

(the expression (22) is correct in order of magnitude even when $H \lesssim M_0$). At large energy flows $\dot{E}$, when a break occurs to conditions of weak turbulence, an exact solution of the problem requires simultaneous investigation of the system of kinetic equations for microwave-field quanta and for spin waves. But for the range $\delta n/n_0 \sim 1$, we may still use the expression (22) as an estimate of the value of $\dot{E}$. In the estimate of $\hbar\dot{E}$, we take as characteristic values

$$T/\Theta_C\sim0.1,\quad \mu M_0\sim10^{-2}-10^{-1}\ \text{K},\quad \Theta_C\sim10^2-10^3\ \text{K},\quad H\sim1\ \text{kOe}\ .$$

As follows from (16) and (22), the criteria for weak turbulence and the condition $n\gg1$ are satisfied in the inertial region with a wide margin. The nonlinear shift of frequency may also be neglected under these conditions (cf. Ref. 6). Condition (16) in turn is satisfied only on the boundary of applicability of the solutions found, when $(ak)^2\sim\mu M_0/\Theta_C$, but in the inertial interval $\delta n/n_0\sim0.1$. Under these conditions it is impossible to make a definite choice between a stationary weakly turbulent distribution and a weakly turbulent correction to the Rayleigh-Jeans distribution. With the parameter values assumed, with allowance for (5) and (6), the distribution (9) should be realized. Then in the wave-vector range

$$\max\left\{\frac{\mu M_0}{\Theta_C},\ \frac{\mu H}{\Theta_C}\right\}\ll(ak)^2\ll\left(\frac{\mu M_0}{\Theta_C}\right)^{2/5}\left(\frac{\mu M_0}{T}\right)^{2/5}\tag{23}$$

the weakly turbulent correction δn is also determined by three-magnon dipole-dipole interaction, and the total distribution function has the form

$$n_k=\frac{T}{\Theta_C(ak)^2}+\text{const}\,\frac{\hbar\dot{E}}{(\mu M_0)^2(ak)^4}\,\frac{\Theta_C}{T}.\tag{24}$$

Apparently the most direct method of measurement of the spin-wave distribution function is experiments on light scattering,[11] which could answer the question which distribution is realized.

This paper has considered the case in which parametric excitation of spin waves is absent. It is to be expected that under conditions of parametric resonance, the interaction of spin waves will still be determined by the collision integral (2) far from the resonance surface $\varepsilon_k=\hbar\omega/2$. Energy is first injected into the system of resonant spin waves, which are in a stationary state, and is then transferred to the nonresonant spin waves. Thus in the system of nonresonant spin waves there is an energy flow, and for this reason one may expect a transition to a weakly turbulent distribution. The value of the energy flow in this case is determined by the phase limitation mechanism[4] (for the case of transverse pumping, see also Ref. 12). Since the absorption under resonance conditions is large, the value of the energy flow is also large and may exceed by two or three orders of magnitude the corresponding value (21) calculated under nonresonance conditions. Consequently, there may again occur a transition to weakly turbulent distributions determined by four-magnon interaction. A weakly turbulent distribution with the flow of particles (12) may play an important role in relaxation of the magnetic moment.

A transition to conditions of weak turbulence in the presence of injection of energy into the system must, in all probability, occur in many problems of solid-state physics. Such distributions can easily be found also for magnon-magnon and magnon-phonon interactions in antiferromagnets, and also phonon-phonon interaction. Study of turbulent distributions in various specific situations would be of considerable interest.

In conclusion, the authors express their thanks to A. L. Chernyakov for discussion of questions related to weak turbulence.

[1)]In the presence of a constant term in the spectrum ε_k, a flow of particles is accompanied also by a flow of energy.

[2)]We note that umklapp processes are known to guarantee an energy sink in the large-wave-vector range.

[3)]The condition (17) assumes, as a rule, satisfaction of the condition $\omega\tau^{(0)}\gg1$ (ω is the frequency of the external perturbation, $\tau^{(0)}$ is the equilibrium relaxation time); that is, nonadiabaticity of the external perturbation.

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