Interaction between hydrodynamic modes in superfluid He^3

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A Hamiltonian technique based on the Poisson brackets is developed to describe the hydrodynamics of superfluid He^3. The technique is used to obtain the nonlinear-hydrodynamics equations for the A and B phases. The normal coordinates corresponding to first and second sound, and also the spin waves in both phases, are found. The interaction between these hydrodynamic modes is considered. The conditions for the excitation of various parametric processes are described. The interaction vertices for first and second sound, and also for sound interacting with spin waves, are obtained by expanding the Hamiltonian in normal coordinates.

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INTRODUCTION

The basis of the present work is a Hamiltonian formalism that turns out to be convenient for the consideration of nonlinear problems of hydrodynamics. The Hamiltonian formalism was developed by Pokrovskiĭ and Khalatnikov[1,2] as applied to superfluid He^4. In Ref. 2 and also in the work of the author[3] this method was used for the investigation of the process of parametric excitation of second sound by first sound in He II. In the works of Khalatnikov and the author,[4-6] a Hamiltonian formalism was developed for the description of the hydrodynamics of the anisotropic superfluid liquid He^3-A. In the present paper, the Hamiltonian method is used for the consideration of processes of interaction between different hydrodynamic modes in superfluid He^3.

A review of the properties of superfluid He^3 can be found in the papers of Wheatley[7] and Leggett.[8] This liquid is characterized by an order parameter $\hat{\Delta}$ the physical meaning of which is that of a gap in the excitation spectrum:

$$\hat{\Delta}(\mathbf{k})=id_{ji}k_j\hat{\sigma}_i\hat{\sigma}_y/k. \tag{1}$$

Here $\mathbf{k}$ is the wave vector of the excitation and $\hat{\sigma}$ are the Pauli matrices. The structure of the tensor d_{ji} is different in the different phases of the superfluid He^3.[8]

The tensor d_{ji} is factored in the A phase:

$$d_{ji}=\Delta_0\Phi_j n_i. \tag{2}$$

Here Δ_0 is the maximal value of the gap, the unit vector $\mathbf{n}$ has the sense of the spin anisotropy vector, and the vector

$$\mathbf{\Phi}=\mathbf{\Phi}'+i\mathbf{\Phi}'' \tag{3}$$

characterizes the orbital part of the order parameter. The vectors $\mathbf{\Phi}'$ and $\mathbf{\Phi}''$ are mutually orthogonal and have unit length, and the unit vector

$$\mathbf{l}=[\mathbf{\Phi}'\times\mathbf{\Phi}''] \tag{4}$$

has the sense of the orbital anisotropy vector.

The order parameter in the B phase of superfluid He^3 is determined by the angle of rotation of the spin part of the order parameter relative to the orbital part:

$$d_{ji}=\Delta_0\left(\delta_{ji}\cos\theta+\frac{\theta_i\theta_j}{\theta^2}(1-\cos\theta)+e_{jik}\frac{\theta_k}{\theta}\sin\theta\right). \tag{5}$$

At equilibrium, $\cos\theta=-\frac{1}{4}$.

THE HAMILTONIAN FORMALISM

We assume that reversible hydrodynamic processes can be described by means of a local variational principle. Let $q(\mathbf{r})$ be the set of generalized coordinates, the specification of which, together with their time derivatives, uniquely determines the state of the system. Then the Lagrangian density can be written down in the form

$$\Lambda=p\frac{\partial q}{\partial t}-H(p,q,\nabla q), \tag{6}$$

where p are the generalized momenta, variables that are canonically conjugate to q, and H is the Hamiltonian density.

The equations corresponding to (6) are written in the form

$$\frac{\partial q}{\partial t}=\frac{\partial H}{\partial p}\equiv\{\mathscr{H},q\},\quad \frac{\partial p}{\partial t}=-\frac{\partial H}{\partial q}+\nabla\frac{\partial H}{\partial\nabla q}\equiv\{\mathscr{H},p\}, \tag{7}$$

where the Hamiltonian is

$$\mathscr{H}=\int d^3r\,H(p,q,\nabla q),$$

and $\{\ ,\}$ are the Poisson brackets. For the canonical variables,

$$\{p(t,\mathbf{r}),p(t,\mathbf{r}')\}=0,\quad \{q(t,\mathbf{r}),q(t,\mathbf{r}')\}=0,$$
$$\{p(t,\mathbf{r}),q(t,\mathbf{r}')\}=\delta(\mathbf{r}-\mathbf{r}'). \tag{8}$$

Let $\hat{G}$ be the generator of the group whose representations are p and q. If the Lagrangian density (5) is invariant relative to this group then, with account of the equation of motion (7), the conservation laws

$$\frac{\partial}{\partial t}(p\hat{G}q)-\nabla\left(\frac{\partial H}{\partial\nabla q}\hat{G}q\right)=0 \tag{9}$$

hold as usual. Thus,

$$G(\mathbf{r})=-ip(\mathbf{r})\hat{G}q(\mathbf{r}) \tag{10}$$

has the meaning of the density of a quantity corresponding to the generator $\hat{G}$. The coefficient in (10) is so chosen that standard expressions for the quantum-mechanical operators can be used; thus, for the momentum density we have

$$\mathbf{g}(\mathbf{r})=-p(\mathbf{r})\nabla q(\mathbf{r}). \tag{11}$$

Let $\hat{G}_a$ be the set of generators of the intrinsic symmetry group of a system, for which the relations

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$$\hat{G}_a\hat{G}_b-\hat{G}_b\hat{G}_a=t_{ab}{}^c\hat{G}_c \tag{12}$$

hold. Here t_{ab}^c are the structural constants of the group. With account of the definitions (10), (11) and also (8), we find the rules for the Poisson brackets:

$$\{G_a(\mathbf{r}_1),\ G_b(\mathbf{r}_2)\}=it_{ab}{}^cG_c\delta(\mathbf{r}_1-\mathbf{r}_2); \tag{13}$$

$$\{G_a(\mathbf{r}_1),\ q(\mathbf{r}_2)\}=-i\hat{G}_aq\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{14}$$

$$\{G_a(\mathbf{r}_1),\ p(\mathbf{r}_2)\}=-i\hat{G}_ap\delta(\mathbf{r}_1-\mathbf{r}_2);$$
$$\{\mathbf{g}(\mathbf{r}_1),\ q(\mathbf{r}_2)\}=-\nabla q\delta(\mathbf{r}_1-\mathbf{r}_2); \tag{15}$$

$$\{g_k(\mathbf{r}_1),\ g_i(\mathbf{r}_2)\}=g_i(\mathbf{r}_1)\nabla_k\delta(\mathbf{r}_1-\mathbf{r}_2)+\nabla_i(g_k(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2)); \tag{16}$$

$$\{\mathbf{g}(\mathbf{r}_1),\ G_a(\mathbf{r}_2)\}=G_a(\mathbf{r}_1)\nabla\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{17}$$

In the description of real systems, we are not dealing with the Hamiltonian density H but with the density of thermodynamic energy E, which is represented by the function

$$E(G_a,\ \mathbf{g},\ q,\ \nabla q). \tag{18}$$

Various combinations of the order parameter play the role of q; quantities associated with the symmetry group of the order parameter as well as quantities associated with the "normal" degrees of freedom enter into the set of generator densities G_a. The rules (13)–(17) enable us to formulate the hydrodynamic equations in this case also; for the Hamiltonian we use here (7) with the substitution $H\to E$.

HYDRODYNAMICS OF He^3-A

Since the order parameter of He^3-A has independent orbital and spin degrees of freedom, we must take into account the dependence of the energy density E both on the density of the spin angular momentum $\mathbf{S}$ and on the density of the orbital angular momentum $\mathbf{L}$. The dependence on the mass density ρ and on the superfluid momentum density $\mathbf{j}$ associated with the order parameter are also important. The entropy density s and the relative normal momentum density $\mathbf{p}$ describe the degrees of freedom connected with the normal motion.

Carrying out a Galilean transformation from the set of coordinates in which $\mathbf{j}=0$, we find

$$E=\frac{j^2}{2\rho}+\frac{\mathbf{jp}}{\rho}+\varepsilon(\mathbf{p},s,\mathbf{L},\mathbf{S},\mathbf{l},\nabla_i\mathbf{l},\mathbf{n},\nabla_i\mathbf{n}). \tag{19}$$

The differential of the energy has the form

$$dE=\mu\,d\rho+T\,ds+\mathbf{v}\,d\mathbf{p}+\frac{\mathbf{g}}{\rho}\,d\mathbf{j}+\omega^L\,d\mathbf{L}+\omega^s\,d\mathbf{S}+\theta^l\,d\mathbf{l}+\theta^n\,d\mathbf{n}$$
$$+\nabla_i\left(\frac{\partial\varepsilon}{\partial\nabla_i\mathbf{l}}d\mathbf{l}\right)+\nabla_i\left(\frac{\partial\varepsilon}{\partial\nabla_i\mathbf{n}}d\mathbf{n}\right). \tag{20}$$

Here μ is the chemical potential, T is the temperature, $\mathbf{v}$ is the normal velocity, and ω are the angular velocities connected with the orbital and spin angular momenta,

$$\theta^l=\frac{\partial\varepsilon}{\partial\mathbf{l}}-\nabla_i\frac{\partial\varepsilon}{\partial\nabla_i\mathbf{l}},\quad \theta^n=\frac{\partial\varepsilon}{\partial\mathbf{n}}-\nabla_i\frac{\partial\varepsilon}{\partial\nabla_i\mathbf{n}}. \tag{21}$$

The momentum density $\mathbf{j}$ is connected with the order parameter; therefore the rules for the mutual Poisson brackets of $\mathbf{j}$ with combinations made up of the order parameter are given by the expression (15) with the substitutions $\mathbf{g}\to\mathbf{j}$ and $q\to\mathbf{l}$, $\mathbf{n}$, while the rules for the mutual Poisson brackets with the generator densities—by the expression (17) with the substitutions $\mathbf{g}\to\mathbf{j}$ and $G_a\to\mathbf{L}$, $\mathbf{S}$, ρ. The enumerated variables have null Poisson brackets with those describing the normal motion s, $\mathbf{p}$, the rules for the mutual Poisson brackets of which are given by the expression (17) with the substitutions $\mathbf{g}\to\mathbf{p}$ and $G_a\to s$, since the entropy in the non-dissipative regime is a conserved quantity. The rules for the Poisson brackets of $\mathbf{j}$ and $\mathbf{p}$ with themselves have the form (16). Taking into account the commutation relations for the spin operators and the rules for their action on the vector indices (see, for example, Ref. 9), we find

$$\{L_i(\mathbf{r}_1),\ L_j(\mathbf{r}_2)\}=-e_{ijn}L_n\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{22}$$

$$\{L_i(\mathbf{r}_1),\ l_j(\mathbf{r}_2)\}=-e_{ijn}l_n\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{23}$$

$$\{S_i(\mathbf{r}_1),\ S_j(\mathbf{r}_2)\}=-e_{ijn}S_n\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{24}$$

$$\{S_i(\mathbf{r}_1),\ n_j(\mathbf{r}_2)\}=-e_{ijn}n_n\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{25}$$

The complete set of Poisson brackets has thus been constructed for the quantities on which E depends, which allows us to formulate the equations of motion:

$$ds/dt=\{\mathscr{H},\ c\}=-\nabla(sv), \tag{26}$$

$$\partial\mathbf{p}/\partial t=\{\mathscr{H},\ \mathbf{p}\}=-\nabla_i(v_i\mathbf{p})-(p_i\nabla_i)\mathbf{v}-s\nabla T, \tag{27}$$

$$\partial\rho/\partial t=\{\mathscr{H},\ \rho\}=-\nabla\mathbf{g}, \tag{28}$$

$$\frac{\partial\mathbf{L}}{\partial t}=\{\mathscr{H},\mathbf{L}\}=-\nabla_i\left(\frac{g_i}{\rho}\mathbf{L}\right)+[\omega^L\times\mathbf{L}]+[\theta^l\times\mathbf{l}], \tag{29}$$

$$\frac{\partial\mathbf{S}}{\partial t}=\{\mathscr{H},\mathbf{S}\}=-\nabla_i\left(\frac{g_i}{\rho}\mathbf{S}\right)+[\omega^s\times\mathbf{S}]+[\theta^n\times\mathbf{n}], \tag{30}$$

$$\frac{\partial\mathbf{l}}{\partial t}=\{\mathscr{H},\mathbf{l}\}=-\left(\frac{\mathbf{g}}{\rho}\nabla\right)\mathbf{l}+[\omega^L\times\mathbf{l}], \tag{31}$$

$$\frac{\partial\mathbf{n}}{\partial t}=\{\mathscr{H},\mathbf{n}\}=-\left(\frac{\mathbf{g}}{\rho}\nabla\right)\mathbf{n}+[\omega^s\times\mathbf{n}], \tag{32}$$

$$\partial g_i/\partial t=\{\mathscr{H},\ j_i+p_i\}=-\nabla_k\Pi_{ik}, \tag{33}$$

where the stress tensor is

$$\Pi_{ik}=\delta_{ik}P+j_i\frac{g_k}{\rho}+v_ip_k+\frac{\partial\varepsilon}{\partial\nabla_i\mathbf{l}}\nabla_k\mathbf{l}+\frac{\partial\varepsilon}{\partial\nabla_i\mathbf{n}}\nabla_k\mathbf{n} \tag{34}$$

and the pressure is

$$P=sT+\rho\mu+\frac{\mathbf{g}}{\rho}\mathbf{j}+\mathbf{vp}+\omega^L\mathbf{L}+\omega^s\mathbf{S}-E$$
$$=sT+\rho\frac{\partial\varepsilon}{\partial\rho}+\left(\mathbf{v}-\frac{\mathbf{j}}{\rho}\right)\mathbf{p}+\omega^L\mathbf{L}+\omega^s\mathbf{S}-\varepsilon. \tag{35}$$

We note that the transfer of all quantities connected with the order parameter takes place with the mean mass velocity. As is seen from (31) and (32), the conditions $\mathbf{l}^2=1$ and $\mathbf{n}^2=1$ are first integrals and therefore are compatible with the derived system. The system that has been obtained corresponds to the equations of Volovik.[10]

The orbital angular momentum $\mathbf{L}$ fails to satisfy even an approximate conservation law. The equation for $\mathbf{L}$ has therefore a dissipative term [added to the right side of (29)] proportional to ω^L (see, for example, the work of Leggett and Takagi[11]). This dissipative term leads to a relaxation of $\mathbf{L}$ in a finite time to its equilibrium value, which leads in turn to the effective removal of the degrees of freedom connected with $\mathbf{L}$; here the equation for $\mathbf{l}$ (in (31) there are also kinetic terms) is a purely diffusion one, even with account taken of the spontaneous orbital angular momentum.[12] Therefore we shall assume $\mathbf{l}$ to be constant when the hydromatic modes are considered.

SOUND WAVES IN He^3-A

The variables describing the sound waves are the mass density and the entropy density, and also the densities of the normal and superfluid momenta. The corresponding part of the energy, with accuracy to third order in the vector quantities, has the form

$$\varepsilon=(\mathbf{pl})^2/2\rho_\parallel+[\mathbf{p}^2-(\mathbf{pl})^2]/2\rho_\perp+u(\rho,s), \tag{36}$$

where $\rho_\parallel$ and $\rho_\perp$ have the meaning of longitudinal and orthogonal (to $\mathbf{l}$) components of the normal density. Substitution of this expression in (26), (27), (28) and (33) yields equations describing the propagation of first and second sound.

We assume the parameter

$$\beta=-\frac{\sigma}{\rho}\left(\frac{\partial\rho}{\partial\sigma}\right)_p, \tag{37}$$

which characterizes the relative value of the dependence of the thermodynamic quantities on the temperature ($\sigma=s/\rho$ is the specific entropy), to be small. By virtue of the smallness of this quantity, the estimate $(c_2/c_1)^2\sim\beta$ holds, where c_1 and c_2 are the phase velocities of first and second sound. The expressions for them in the variables ρ and σ have the form[13]

$$c_1^2=\frac{\partial P}{\partial\rho},\quad c_2^2=\Gamma\frac{\sigma^2}{\rho}\frac{\partial^2u}{\partial\sigma^2}, \tag{38}$$

where (k is the wave vector of the wave)

$$\Gamma=\frac{\rho-\rho_\parallel}{\rho_\parallel}\left(\frac{\mathbf{lk}}{k}\right)^2+\frac{\rho-\rho_\perp}{\rho_\perp}\left(1-\left(\frac{\mathbf{lk}}{k}\right)^2\right). \tag{39}$$

Now let a_1 and a_2 be the normal coordinates corresponding to first and second sound. They satisfy the relations (in the Fourier components)

$$\{a(\mathbf{k}_1),\ a^*(\mathbf{k}_2)\}=i(2\pi)^3\delta(\mathbf{k}_1-\mathbf{k}_2). \tag{40}$$

We require diagonalization, in second order in a, of the Hamiltonian corresponding to (36):

$$\mathscr{H}^{(2)}=\int\frac{d^3k}{(2\pi)^3}(c_1ka_1^*(\mathbf{k})a_1(\mathbf{k})+c_2ka_2^*(\mathbf{k})a_2(\mathbf{k})). \tag{41}$$

With account of the rules (14) and (40) for the Poisson brackets and keeping in mind the smallness of the parameter β, we find (departures from equilibrium values are denoted by δ)

$$\begin{gathered}\delta\rho(\mathbf{k})=\rho A(a_1(\mathbf{k})+a_1^*(-\mathbf{k}))+\beta\rho(a_2(\mathbf{k})+a_2^*(-\mathbf{k}))B,\\ \delta s(\mathbf{k})=-sB(a_2(\mathbf{k})+a_2^*(-\mathbf{k}))+\rho^{-1}s\delta\rho(\mathbf{k}),\\ -ik(\alpha_1(\mathbf{k})+\alpha_2(\mathbf{k}))=c_1A(a_1(\mathbf{k})-a_1^*(-\mathbf{k})),\\ -ik\alpha_2(\mathbf{k})=c_1\beta A(a_1(\mathbf{k})-a_1^*(-\mathbf{k}))-\frac{c_2}{\Gamma}B(a_2(\mathbf{k})-a_2^*(-\mathbf{k})),\\ \mathbf{j}=-(\rho+\delta\rho)\nabla\alpha_1,\quad \mathbf{g}=-\rho\nabla(\alpha_1+\alpha_2)-\sigma^{-1}\delta s\nabla\alpha_2-\delta\rho\nabla\alpha_1,\end{gathered} \tag{42}$$

where

$$A=(\omega_1/2\rho c_1^2)^{1/2},\quad B=(\Gamma\omega_2/2\rho c_2^2)^{1/2}.$$

The expressions for $\mathbf{g}$ and $\mathbf{j}$ contain the terms, of second order in a, which are necessary for the calculation of the third-order interaction Hamiltonian:

$$\begin{aligned}\mathscr{H}^{(3)}=\int d^3r(&{}^1/_2\delta\rho(\nabla\alpha_1)^2+\sigma^{-1}\delta s(\nabla\alpha_1\nabla\alpha_2+\rho\nabla_i\alpha_2(\rho^n)_{ij}^{-1}\nabla_j\alpha_2)\\ &+{}^1/_2\rho^2\nabla_i\alpha_2D(\rho^n)_{ij}^{-1}\nabla_j\alpha_2+{}^1/_6D^3\varepsilon),\end{aligned} \tag{43}$$

where

$$D=\delta\rho\frac{\partial}{\partial\rho}+\delta s\frac{\partial}{\partial s},\quad \rho_{ik}^n=\rho_\parallel l_il_k+\rho_\perp(\delta_{ik}-l_il_k).$$

Separating in (43) the coefficients of the corresponding products of normal coordinates,[2,3] we can find the vertices corresponding to the interaction of first and second sound. The Hamiltonian (43) describes a three-wave interaction. It follows from the condition $c_1>c_2$ and from the conservation laws that a process is possible with decay of a wave of first sound into two waves of second sound, as well as a Cerenkov process in which the wave of first sound is transformed into another wave of first sound with emission of a wave of second sound.

The decay process is characterized by the frequency ω of the wave of first sound, the direction of its wave vector $\mathbf{k}_1$, and the direction of the wave vector of one of the waves of second sound $\mathbf{k}_2$ (with account taken of $c_1\gg c_2$, the wave vector of the other wave of first sound is $-\mathbf{k}_2$, and the energy is distributed equally). The corresponding vertex has the form

$$U=\frac{\omega^{3/2}}{2^{5/2}\rho^{1/2}c_1}\left(-1-\frac{2(\mathbf{k}_1\mathbf{k}_2)^2}{k_1^2k_2^2}-\frac{\rho^2}{\Gamma}\frac{\partial}{\partial\rho}\frac{\Gamma}{\rho}+\frac{\partial^2P/\partial\sigma^2}{\partial^2\varepsilon/\partial\sigma^2}\right). \tag{44}$$

In the Cerenkov process, with account taken of $c_1\gg c_2$, the frequencies of the waves of first sound are almost equal (and are denoted by ω),

$$\omega_2=\frac{2c_2}{c_1}\omega\sin\frac{\varphi}{2},$$

where φ is the angle between the directions of propagation of the waves of first sound. The Cerenkov vertex has the form

$$V=\frac{\omega^{3/2}\Gamma^{1/2}\beta}{2\rho^{1/2}c_2}\left(\frac{c_2}{c_1}\sin\frac{\varphi}{2}\right)^{1/2}\left(-\rho\frac{\partial}{\partial\rho}\ln\beta-\cos\varphi\right). \tag{45}$$

These formulas are a generalization of those given in Ref. 2 for the case of an anisotropic superfluid. We note that the Cerenkov vertex V has smallness of the order of $\beta^{3/4}$ in comparison with the decay vertex U; moreover, the Cerenkov process is suppressed at small angles φ owing to the conservation laws.

SPIN WAVES IN He^3-A

The spin variables are $\mathbf{n}$ and $\mathbf{S}$ and the spin waves are respectively described by Eqs. (30) and (32). These equations, in particular lead to the conservation law

$$\frac{\partial}{\partial t}(\mathbf{nS})+\nabla_i\left(\frac{g_i}{\rho}\mathbf{nS}\right)=0. \tag{46}$$

Thus the oscillations of the longitudinal part $\mathbf{n}$ of $\mathbf{S}$ correspond to a harmonic with zero frequency and will henceforth not be taken into account by us.

Accurate to third order, the spin part of the energy is given by

$$u_s=\frac{(\mathbf{Sn})^2}{2\chi_\parallel}+\frac{\mathbf{S}^2-(\mathbf{Sn})^2}{2\chi_\perp}+\frac{1}{2}\nabla_i\mathbf{n}M_{ij}\nabla_j\mathbf{n}-\gamma H\mathbf{vS}-\xi\mathbf{nl}, \tag{47}$$

where

$$M_{ij}=M_\parallel l_il_j+M_\perp(\delta_{ij}-l_il_j).$$

The energy (47) includes the interaction with the homogeneous external field H (ν is a unit vector in its direction), and γ is the gyromagnetic ratio. The last term of (47) takes into account the ferromagnetic ordering as well as the spin-order interaction.

At equilibrium, $\mathbf{n}$ is directed along $\mathbf{l}$,[14] so that $\xi>0$; in addition, $\chi_\perp>\chi_\parallel$,[15] so that at equilibrium $\mathbf{n}\perp\boldsymbol{\nu}$. In first-order approximation, Eqs. (30), (32), and (47) lead to equations that correspond to the linear hydrodynamics of Graham and Pleiner[16]:

$$\frac{\partial \mathbf{S}}{\partial t}=-\chi_\perp\gamma H\nu\left(\nabla\frac{\mathbf{g}}{\rho}\right)-\xi[\mathbf{l}\times\mathbf{n}]-\gamma H[\nu\times\mathbf{S}]+[\mathbf{l}M_{ij}\times\nabla_i\nabla_j\mathbf{n}], \quad (48)$$

$$\frac{\partial \mathbf{n}}{\partial t}=-\left[\mathbf{l}\times\frac{\delta\mathbf{S}}{\chi_\perp}\right]. \quad (49)$$

We label with $\parallel$ and $\perp$ the oscillations for which $\delta\mathbf{S}$ is respectively parallel and perpendicular to the magnetic-field direction ν. The transverse oscillations correspond to the dispersion law

$$\omega_\perp^2=c_s^2k^2+(\gamma H)^2+\Omega^2, \quad (50)$$

where the gap is

$$\Omega^2=\xi/\chi_\perp \quad (51)$$

and the phase velocity of the spin waves is

$$c_s=\left(\frac{M_\parallel-M_\perp}{\chi_\perp}\left(\frac{\mathbf{lk}}{k}\right)^2+\frac{M_\perp}{\chi_\perp}\right)^{1/2}. \quad (52)$$

It is seen from (48) that the longitudinal oscillations are mixed with the first sound, but this mixing is small to the extent that $(\chi_\perp/\rho)^{1/2}\gamma H/c$ is small and will be disregarded by us. The dispersion law for the longitudinal oscillations takes the form

$$\omega_\parallel^2=c_s^2k^2+\Omega^2.$$

Let b be the normal coordinates corresponding to the spin waves; they satisfy relations of the type (40). Then, recognizing that the rules (24) and (25) must be satisfied in the zeroth approximation, and stipulating diagonalization of the Hamiltonian corresponding to (47), we obtain in second order in b

$$\delta\mathbf{n}(\mathbf{k})=\frac{i[\nu\times\mathbf{l}]}{(2\chi_\perp\omega_\parallel(\mathbf{k}))^{1/2}}(b_\parallel(\mathbf{k})-b_\parallel^*(-\mathbf{k}))+\frac{i\nu}{(2\chi_\perp\omega_\perp(\mathbf{k}))^{1/2}}(b_\perp(\mathbf{k})-b_\perp^*(-\mathbf{k})), \quad (53)$$

$$\delta\mathbf{S}(\mathbf{k})=\nu\left(\frac{\chi_\perp\omega_\parallel(\mathbf{k})}{2}\right)^{1/2}(b_\parallel(k)+b_\parallel^*(-k))-[\nu\times\mathbf{l}]\left(\frac{\chi_\perp\omega_\perp(k)}{2}\right)^{1/2}(b_\perp(\mathbf{k})+b^*(-\mathbf{k}))-i\gamma H\mathbf{l}\left(\frac{\chi_\perp}{2\omega_\perp(\mathbf{k})}\right)^{1/2}(b_\perp(\mathbf{k})-b_\perp^*(-\mathbf{k})). \quad (54)$$

We consider now the interaction of the spin waves with the sound. To write down the interaction Hamiltonian of third order in b it is necessary to substitute for the quantities on which E depends expressions of second order in the normal coordinates. The second-order corrections can be found by requiring that the rules for the mutual Poisson brackets of the quantities on which E depends be satisfied accurately to first order in b. For the description of the interaction of the spin waves with sound, an important second order correction is the contribution of spin waves to the momentum density:

$$g_i'=-\delta\mathbf{S}[\mathbf{l}\times\nabla_i\mathbf{n}]. \quad (55)$$

When this is taken into account, the Hamiltonian that corresponds to interaction of sound and spin waves has the form

$$\mathscr{H}^{(3)}=\int d^3r\,({}^1/_2\nabla_i\delta\mathbf{n}DM_{ij}\nabla_j\delta\mathbf{n}+{}^1/_2\delta\mathbf{S}D^\perp\chi_\perp\delta\mathbf{S}+\nabla_i(\alpha_1+\alpha_2)\delta\mathbf{S}[\mathbf{l}\times\nabla_i\delta\mathbf{n}]+{}^1/_2D\xi(\delta\mathbf{n})^2). \quad (56)$$

The Hamiltonian (56) corresponds to processes in which a single sound wave and two spin waves participate. For the A phase, which exists only in the region of temperatures near the transition curve, it is natural to expect that $c_\perp>c_s$.[8] Therefore, decay into two spin waves having a frequency threshold due to gaps in the spin wave spectrum is possible with the participation of first sound. The Hamiltonian (56) gives nonzero vertices for the decay into spin waves of like polarization ($\mathbf{k}_1$ is the wave vector of the sound wave, $\mathbf{k}_2$ and $\mathbf{k}_3$ are the wave vectors of the spin waves)

$$U_\parallel=\frac{k_1^{1/2}}{2^{3/2}c_1^{1/2}\rho^{1/2}}\left[c_1k_1+\frac{k_{2i}k_{3j}\rho\,\partial M_{ij}/\partial\rho-\rho\,\partial\xi/\partial\rho}{\chi_\perp(\omega_\parallel(\mathbf{k}_2)\omega_\parallel(\mathbf{k}_3))^{1/2}}-(\omega_\parallel(\mathbf{k}_2)\omega_\parallel(\mathbf{k}_3))^{1/2}\rho\frac{\partial}{\partial\rho}\ln\chi_\perp\right], \quad (57)$$

$$U_\perp=\frac{k_1^{1/2}}{2^{3/2}c_1^{1/2}\rho^{1/2}}\left[c_1k_1+\frac{k_{2i}k_{3j}\rho\,\partial M_{ij}/\partial\rho-\rho\,\partial\xi/\partial\rho}{\chi_\perp(\omega_\parallel(\mathbf{k}_2)\omega_\parallel(\mathbf{k}_3))^{1/2}}-\left(1-\frac{\gamma^2H^2}{\omega_\perp(\mathbf{k}_2)\omega_\perp(\mathbf{k}_3)}\right)(\omega_\perp(\mathbf{k}_2)\omega_\perp(\mathbf{k}_3))^{1/2}\rho\frac{\partial}{\partial\rho}\ln\chi_\perp\right]. \quad (58)$$

In the high frequency limit $\omega\gg\Omega$, both these formulas give identical expressions:

$$U=\frac{k_1^{1/2}}{2^{3/2}c_1^{1/2}\rho^{1/2}}\left[c_1k_1+\frac{k_{2i}k_{3j}\rho\,\partial M_{ij}/\partial\rho}{\chi_\perp(c_s(\mathbf{k}_2)c_s(\mathbf{k}_3)k_2k_3)^{1/2}}-(c_s(\mathbf{k}_2)c_s(\mathbf{k}_3)k_2k_3)^{1/2}\rho\frac{\partial}{\partial\rho}\ln\chi_\perp\right]. \quad (59)$$

Because of the smallness of β, the estimate $(c_2/c_s)^2\sim\beta\ll1$ holds.[8] Second sound exists only in the region of very low frequencies; we therefore assume the frequency of the wave of second sound to be much smaller than the gap Ω. This condition excludes the possibility of consideration of the decay process, and we shall deal with the Cerenkov emission of a wave of second sound by a spin wave, since the conservation laws do not impose any limitations on the frequency in this case. By virtue of the assumed condition $\Omega\gg c_2k_1$ ($\mathbf{k}_1$ is the wave vector of the second sound) the frequency of the spin wave changes little; we shall denote it simply by ω.

As before, let $\mathbf{k}_2$ and $\mathbf{k}_3$ be the wave vectors of the spin waves. Just as in the case of interaction with first sound, only processes without change in the polarization of the spin wave are possible. The corresponding Cerenkov vertices, obtained from the interaction Hamiltonian (56) have the form

$$V_\parallel=\frac{\Gamma^{1/2}k_1^{1/2}}{2^{3/2}\rho^{1/2}c_2^{1/2}}\left[\omega\sigma\frac{d}{d\sigma}\ln\chi_\perp-\frac{1}{\chi_\perp\omega}\sigma\frac{d}{d\sigma}(\xi+M_{ij}k_{2i}k_{3j})\right], \quad (60)$$

$$V_\perp=\frac{\Gamma^{1/2}k_1^{1/2}}{2^{3/2}\rho^{1/2}c_2^{1/2}}\left[\left(1+\frac{(\gamma H)^2}{\omega^2}\right)\omega\sigma\frac{d}{d\sigma}\ln\chi_\perp-\frac{1}{\chi_\perp\omega}\sigma\frac{d}{d\sigma}(\xi+M_{ij}k_{2i}k_{3j})\right], \quad (61)$$

where the derivative $d/d\sigma$ denotes differentiation at constant pressure. In the high-frequency limit $\omega\gg\Omega$ we can assume $\mathbf{k}_2\approx\mathbf{k}_3=\mathbf{k}$. Here the expressions for both vertices simplify and are identical:

$$V=-\left(\frac{\Gamma k_1}{2\rho c_2}\right)^{1/2}k\sigma\frac{d}{d\sigma}c_s. \quad (62)$$

The expressions (57) and (58) give the long-wave limit of the vertices of phonon–two magnon interaction and can be used for the study of kinetic phenomena in He^3-A. We also note that the linkage of the spin waves with second sound is much weaker than with first sound.

HYDRODYNAMICS OF He^3-B

In connection with the structure of the order parameter of the B phase (5), we must take into account, in addition to ρ, s and p, the dependence of the energy density ε on the density of the spin S which generates the rotation of the spin part of the order parameter relative to the orbital part, and also the dependence on the order parameter itself and its derivatives:

$$\varepsilon=\varepsilon(\rho,\, s,\, \mathbf{p},\, \mathbf{S},\, d_{ji},\, \nabla d_{ji}),$$
$$dE=(\mathbf{g}/\rho)\,d\mathbf{j}+\mathbf{v}\,d\mathbf{p}+\mu d\rho+T\,ds+\boldsymbol{\omega}\,d\mathbf{S}+\Xi_{ji}dd_{ji}+\nabla\left(\frac{\partial\varepsilon}{\partial\nabla d_{ji}}dd_{ji}\right), \quad (63)$$

where

$$\Xi_{ji}=\frac{\partial\varepsilon}{\partial d_{ji}}-\nabla\frac{\partial\varepsilon}{\partial\nabla d_{ji}}.$$

In connection with (14) we have

$$\{S_k,\, d_{ji}\}=-e_{kin}d_{jn}; \quad (64)$$

taking also (24) into account, we find the equations of motion

$$\partial\mathbf{S}/\partial t=\{\mathscr{H},\mathbf{S}\}=-\nabla_i\left(\frac{g_i}{\rho}\mathbf{S}\right)+[\boldsymbol{\omega}\times\mathbf{S}]+[\Xi_j\times\mathbf{d}_j], \quad (65)$$

$$\frac{\partial\mathbf{d}_j}{\partial t}=\{\mathscr{H},\mathbf{d}_j\}=-\frac{g_i}{\rho}\nabla_i\mathbf{d}_j+[\boldsymbol{\omega}\times\mathbf{d}_j]. \quad (66)$$

The equations of motion for ρ, s, and p have the same form as (28), (26), (27).

We complete the set of hydrodynamic equations with the law of momentum conservation, which has the form (33) with the stress tensor

$$\Pi_{ik}=\delta_{ik}P+j_i\frac{g_k}{\rho}+v_ip_k+\frac{\partial\varepsilon}{\partial\nabla_i\mathbf{d}_j}\nabla_k\mathbf{d}_j, \quad (67)$$

where the pressure is given by

$$P=sT+\rho\mu+(\mathbf{g}/\rho)\mathbf{j}+\mathbf{v}\mathbf{p}+\boldsymbol{\omega}\mathbf{S}-E=sT+\rho\frac{\partial\varepsilon}{\partial\rho}+\left(\mathbf{v}-\frac{\mathbf{j}}{\rho}\right)\mathbf{p}+\boldsymbol{\omega}\mathbf{S}-\varepsilon. \quad (68)$$

At equilibrium, the angle of rotation is determined by the condition $\cos\theta=-\frac{1}{4}$.[8] At small departures from the equilibrium angle, we shall use the symbol $\tilde{\boldsymbol{\theta}}$:

$$d_{ji}=\exp(-i\tilde{\theta}\hat{S})\,d_{ji}^{(0)}.$$

As is known, in the presence of a magnetic field the angle $\boldsymbol{\theta}$ tends to become oriented along the field.[17] We assume therefore that at equilibrium $\boldsymbol{\theta}$ is directed along the uniform external magnetic field $\mathbf{H}=H\boldsymbol{\nu}$ (i.e., we neglect boundary effects). However, the corresponding contribution to the energy is of fourth order in $\tilde{\boldsymbol{\theta}}$ and we shall not take it into account.

Finally, the spin part of the energy density, with account taken of the gradient terms and the spin-orbit interaction, has the form[18]

$$\varepsilon_s=\frac{\mathbf{S}^2}{2\chi}-\gamma\mathbf{H}\mathbf{S}+\frac{1}{5}\rho^s\left(\frac{\hbar}{2m}\right)^2(2(\nabla_j\tilde{\theta}_i)^2-\nabla_j\tilde{\theta}_i\nabla_i\tilde{\theta}_j)+\frac{\zeta}{2}(\tilde{\theta}\mathbf{v})^2, \quad (69)$$

with accuracy to third order. Here $\rho^s=\rho-\rho^n$ is the superfluid density and m is the He^3 mass.

It is more convenient to write out the equations of first order for $\tilde{\boldsymbol{\theta}}$ and S with account of the fact that

$$\{S_j(\mathbf{r}_1),\ \theta_i(\mathbf{r}_2)\}=\delta_{ji}\delta(\mathbf{r}_2-\mathbf{r}_1) \quad (70)$$

follows from (64) in zeroth order in $\tilde{\boldsymbol{\theta}}$. These equations have the form[19]

$$\frac{\partial\mathbf{S}}{\partial t}=\gamma[\mathbf{S}\times\mathbf{H}]-\zeta(\tilde{\theta}\mathbf{v})\mathbf{v}-\frac{\rho^s}{5}\frac{\hbar^2}{2m^2}(\nabla_j\nabla\tilde{\theta}_j-2\nabla^2\tilde{\theta}), \quad (71)$$

$$\partial\tilde{\theta}/\partial t=\delta\mathbf{S}/\chi. \quad (72)$$

Just as in the case of He^3-A, we neglect the transport term that leads to the mixing of spin waves with first sound and is small in terms of the parameter $(\chi/\rho)^{1/2}\gamma H/c_s\approx H/10^6$G (the velocity of the spin waves is taken from Ref. 20).

Equations (71) and (72) describe the spin waves for which the characteristic parameters are the phase velocity

$$c_s=(\hbar^2\rho^s/10m^2\chi)^{1/2} \quad (73)$$

and the gap Ω_L:

$$\Omega_L^2=\zeta/\chi. \quad (74)$$

In correspondence with the number of degrees of freedom, there are three spin wave branches with rather complicated dispersion laws in the general case. We consider two limiting cases depending on the relation between γH and $\Omega_L^2+c_s^2k^2$.

We consider the case $(\gamma H)^2\ll\Omega_L^2+c_s^2k^2$. In this limit, the dispersion laws for the three branches of the spin waves have the form

$$\omega_1^2=2c_s^2k^2,$$
$$\omega_2^2={}^1/_2(\Omega_L^2+3c_s^2k^2)+{}^1/_2((\Omega_L^2+c_s^2(k_\perp^2-k_\parallel^2))^2+4c_s^2k_\parallel^2k_\perp^2)^{1/2},$$
$$\omega_3^2={}^1/_2(\Omega_L^2+3c_s^2k^2)-{}^1/_2((\Omega_L^2+c_s^2(k_\perp^2-k_\parallel^2))^2+4c_s^2k_\parallel^2k_\perp^2)^{1/2}, \quad (75)$$

where $k_\parallel=\mathbf{k}\boldsymbol{\nu}$, $k_\perp^2=k^2-k_\parallel^2$. Let b_1, b_2 and b_3 be the normal coordinates corresponding to these waves. Then, keeping in mind the fact that (24) and (70) should be satisfied in zeroth approximation, recognizing that the b satisfy the rules for Poisson brackets of the type (40), and requiring diagonalization of the Hamiltonian corresponding to (69), we find

$$\delta\mathbf{S}(\mathbf{k})=({}^1/_2\chi\omega_1(\mathbf{k}))^{1/2}(b_1(\mathbf{k})+b_1^*(-\mathbf{k}))\mathbf{n}_1$$
$$+({}^1/_2\chi\omega_2(\mathbf{k}))^{1/2}(b_2(\mathbf{k})+b_2^*(-\mathbf{k}))\mathbf{n}_2+({}^1/_2\chi\omega_3(\mathbf{k}))^{1/2}(b_3(\mathbf{k})+b_3^*(-\mathbf{k}))\mathbf{n}_3, \quad (76)$$

$$\tilde{\theta}(\mathbf{k})=\frac{i}{(2\chi\omega_1(\mathbf{k}))^{1/2}}(b_1(\mathbf{k})-b_1^*(-\mathbf{k}))\mathbf{n}_1$$
$$+\frac{i}{(2\chi\omega_2(\mathbf{k}))^{1/2}}(b_2(\mathbf{k})-b_2^*(-\mathbf{k}))\mathbf{n}_2+\frac{i}{(2\chi\omega_3(\mathbf{k}))^{1/2}}(b_3(\mathbf{k})-b_3^*(-\mathbf{k}))\mathbf{n}_3. \quad (77)$$

Here $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ are a triad of orthonormalized vectors specified by their components

$$\mathbf{n}_1\mathbf{k}=\mathbf{n}_1\mathbf{v}=0,$$
$$\mathbf{n}_2\mathbf{v}=A_2(2k^2c_s^2-\omega_2^2-c_s^2k_\perp^2), \qquad \mathbf{n}_3\mathbf{v}=A_3(2k^2c_s^2-\omega_s^2-c_s^2k_\perp^2),$$
$$\mathbf{n}_2\mathbf{k}=A_2(2k^2c_s^2-\omega_2^2)k_\parallel, \qquad \mathbf{n}_3\mathbf{k}=A_3(2k^2c_s^2-\omega_3^2)k_\parallel, \quad (78)$$

where

$$A_2=(4k^4c_s^4-3c_s^4k^2k_\perp^2-2c_s^2(2k^2-k_\perp^2)\omega_2^2+\omega_2^4)^{-1/2},$$
$$A_3=(4k^4c_s^4-3c_s^4k^2k_\perp^2-2c_s^2(2k^2-k_\perp^2)\omega_3^2+\omega_3^4)^{-1/2}.$$

In the high-frequency limit $\omega\gg\Omega_L$ we have $\omega_1=\omega_2=2^{1/2}c_sk$ and $\omega_3=c_sk$. The first two dispersion laws correspond to waves of transverse polarization, $\mathbf{n}_1$ and $\mathbf{n}_2$ orthogonal to k, and the third corresponds to longitudinal polarization, $\mathbf{n}_3=\mathbf{k}/k$.

We now consider the case $(\gamma H)^2\gg\Omega_L^2+c_s^2k^2$. In this limit, the dispersion laws for the three branches of the spin waves have the form

$$\omega_4=\gamma H+\frac{c_s^2}{\gamma H}\left(2k^2-\frac{1}{2}k_\perp^2\right),$$
$$\omega_5^2=\left(\frac{2c_s^2k^2}{\gamma H}\right)^2\frac{\Omega_L^2-k_\perp^2\Omega_L^2/2k^2+c_s^2k^2}{\Omega_L^2+c_s^2(2k^2-k_\parallel^2)}, \quad (79)$$
$$\omega_6^2=\Omega_L^2+c_s^2(2k^2-k_\parallel^2).$$

Let b_4, b_5, and b_6 be the normal coordinates corresponding to these waves. Then, keeping it in mind that (24) and (70) should be satisfied in zeroth approximation, taking it into account that the b satisfy the rules for Poisson brackets of type (40), and requiring diagonalization of the Hamiltonian corresponding to (69), we find

$$\delta\mathbf{S}(\mathbf{k})=\left(\frac{1}{2}\chi\omega_6(\mathbf{k})\right)^{1/2}(b_6(\mathbf{k})+b_6^*(-\mathbf{k}))\mathbf{v}+\left(\frac{\gamma\chi H}{2}\right)^{1/2}(b_4(\mathbf{k})$$
$$+b_4^*(-\mathbf{k}))\mathbf{v}'-i\left(\frac{\gamma\chi H}{2}\right)^{1/2}(b_4(\mathbf{k})-b_4^*(-\mathbf{k}))\mathbf{v}'', \tag{80}$$

$$\tilde{\boldsymbol{\theta}}(\mathbf{k})=\frac{i}{(2\gamma\chi H)^{1/2}}(b_4(\mathbf{k})-b_4^*(-\mathbf{k}))\mathbf{v}'$$
$$+\frac{c_s k}{\gamma H(\omega_5(\mathbf{k})\chi)^{1/2}}(b_5(\mathbf{k})+b_5^*(-\mathbf{k}))\mathbf{v}'$$
$$+\frac{1}{(2\gamma\chi H)^{1/2}}(b_4(\mathbf{k})+b_4^*(-\mathbf{k}))\mathbf{v}''+\left(\frac{\omega_5(\mathbf{k})}{\chi}\right)^{1/2}\frac{i}{2c_s k}(b_5(\mathbf{k})-b_5^*(-\mathbf{k}))\mathbf{v}''$$
$$+\frac{i}{(2\chi\omega_6(\mathbf{k}))^{1/2}}(b_6(\mathbf{k})-b_6^*(-\mathbf{k}))\mathbf{v}$$
$$+\frac{c_s^3 k k_\perp k_\parallel}{\gamma H(\omega_5(\mathbf{k})\chi)^{1/2}\omega_6^2(\mathbf{k})}(b_5(\mathbf{k})-b_5^*(-\mathbf{k}))\mathbf{v}, \tag{81}$$

where $\boldsymbol{\nu}'$, $\boldsymbol{\nu}''$, $\boldsymbol{\nu}$ is a triad of orthonormalized vectors, $\boldsymbol{\nu}''\mathbf{k}=0$.

INTERACTION OF SOUND WAVES AND SPIN WAVES IN He^3-B

The superfluid He^3-B is characterized by an isotropic superfluid density, therefore the consideration of the interaction of first and second sound is completely analogous to the case of superfluid He^4.[2] The expressions for the normal coordinates and the scattering vertices can also be obtained from the corresponding formulas for He^3-A, except that now $\Gamma=\rho^s/\rho^n$ is isotropic. In what follows, we shall use the same notation for the sound waves as in He^3-A.

It is necessary to know the interaction Hamiltonian in order to consider the interaction of sound and spin waves. From among the second order corrections in b to the quantities on which E depends, the contribution to the momentum density

$$g_i^s=-\delta\mathbf{S}\nabla_i\theta$$

is the most important. The third-order Hamiltonian then takes the form

$$\mathscr{H}^{(3)}=\int d^3r\left(\nabla_i(\alpha_1+\alpha_2)\delta\mathbf{S}\nabla_i\tilde{\theta}+\frac{1}{2}(\delta\mathbf{S})^2D\frac{1}{\chi}\right.$$
$$\left.+\frac{\hbar^2}{20m^2}D\rho s(2(\nabla_j\tilde{\theta}_i)^2-\nabla_j\tilde{\theta}_i\nabla_i\tilde{\theta}_j)+\frac{1}{2}D\zeta(\tilde{\theta}\mathbf{v})^2\right). \tag{82}$$

This Hamiltonian corresponds to processes in which a single acoustic and two spin waves participate.

We consider the interaction of first sound with spin waves. The experimental data are such[2,20] that the condition $c_1\gg c_2$ is satisfied. Thus, with the participation of a wave of first sound, only a decay process is possible which has a number of frequency thresholds connected with the presence of gaps in the spectrum of the different branches of the spin waves. This condition leads also to the result that for all frequencies, except those very close to threshold, the wave vectors of the spin waves are much greater in magnitude than the wave vector of first sound, and we can assume them to be equal in magnitude and opposite in direction.

We now consider the case $(\gamma H)^2\ll\Omega_L^2+c_s^2k^2$. In this limit the expressions for the quantities entering into (82) are given by the expressions (42), (76), (77). We shall assume that the wave of first sound, which has the frequency ω, decays into two spin waves with frequencies ω' and ω'', with wave vectors $\mathbf{k}'$ and $\mathbf{k}''$, and with polarizations $\mathbf{n}'$ and $\mathbf{n}''$. The Hamiltonian (82) gives the decay vertex in the general case:

$$U=\frac{\omega^{3/2}}{2^{3/2}\rho^{1/2}c_1}\left[\mathbf{n}'\mathbf{n}''-\frac{(\omega'\omega'')^{1/2}}{\omega}\rho\frac{\partial}{\partial\rho}\ln\chi\mathbf{n}'\mathbf{n}''\right.$$
$$+\frac{\hbar^2}{10m^2}\rho\frac{\partial}{\partial\rho}\rho^s\frac{1}{\chi\omega(\omega'\omega'')^{1/2}}(2(\mathbf{k}'\mathbf{k}'')\mathbf{n}'\mathbf{n}''$$
$$\left.-(\mathbf{k}'\mathbf{n}'')(\mathbf{k}''\mathbf{n}'))-\rho\frac{\partial}{\partial\rho}(\chi\Omega_L^2)\frac{1}{\omega\chi(\omega'\omega'')^{1/2}}(\mathbf{n}'\mathbf{v})(\mathbf{n}''\mathbf{v})\right]. \tag{83}$$

Recognizing that $\mathbf{k}'=-\mathbf{k}''=\mathbf{k}$, and that in decay into waves of identical polarization the frequency is equally divided, we obtain from (83) for the different polarizations (the expressions corresponding to the expression in the square brackets in (83))

$$[U_{11}]=1-\frac{3}{2}\rho\frac{\partial}{\partial\rho}\ln\chi-2\rho\frac{\partial}{\partial\rho}\ln c_s,\qquad U_{12}=U_{13}=0,$$
$$[U_{22}]=1-\frac{1}{2}\rho\frac{\partial}{\partial\rho}\ln\chi-2\rho\frac{\partial}{\partial\rho}(\chi\Omega_L^2)\frac{(\mathbf{n}_2\mathbf{v})^2}{\chi\omega^2}$$
$$-2(2k^2-(\mathbf{k}\mathbf{n}_2)^2)\frac{1}{\chi\omega^2}\rho\frac{\partial}{\partial\rho}(\chi c_s^2), \tag{84}$$
$$[U_{23}]=\frac{1}{\chi\omega(\omega_2\omega_3)^{1/2}}\left((\mathbf{k}\mathbf{n}_2)(\mathbf{k}\mathbf{n}_3)\rho\frac{\partial}{\partial\rho}(\chi c_s^2)-(\mathbf{n}_2\mathbf{v})(\mathbf{n}_3\mathbf{v})\rho\frac{\partial}{\partial\rho}(\chi\Omega_L^2)\right).$$

The expression for U_{33} is obtained from the expression for U_{22} by the substitution $\mathbf{n}_2\to\mathbf{n}_3$.

In the high-frequency limit $\omega\gg\Omega_L$, U_{23} vanishes, the expression for U_{22} is identical with the expression for U_{11}, and as should be the case for the mode with transverse polarization, the limiting expression for U_{33} has the form

$$[U_{33}]=1-\rho\frac{\partial}{\partial\rho}\ln\chi-\rho\frac{\partial}{\partial\rho}\ln c_s. \tag{85}$$

We now proceed to consideration of the interaction of spin waves with second sound. The decay process is possible if $c_2>c_s$ and the Cerenkov process, in the opposite case. The experimental relation between these two quantities is unknown; however, we can obtain from the Hamiltonian (82) an expression that describes both the Cerenkov and the decay processes. In this same limit $(\gamma H)^2\ll\Omega_L^2+c_s^2k^2$, it has the form (the notation is the same as in (83) except that ω now refers to second sound)

$$\frac{\Gamma^{1/2}\omega^{1/2}}{2^{1/2}\rho^{1/2}c_2\chi(\omega'\omega'')^{1/2}}\sigma\frac{d}{d\sigma}[\chi\omega'\omega''(\mathbf{n}'\mathbf{n}'')-\chi c_s^2(2(\mathbf{k}'\mathbf{k}'')(\mathbf{n}'\mathbf{n}'')$$
$$-(\mathbf{k}'\mathbf{n}'')(\mathbf{k}''\mathbf{n}'))\pm(\chi\Omega_L^2)(\mathbf{n}'\mathbf{v})(\mathbf{n}''\mathbf{v})]. \tag{86}$$

Here only χ, c_s and Ω_L are differentiated; the plus sign refers to the decay vertex, the minus sign to the Cerenkov vertex.

We now consider the case $(\gamma H)^2\gg\Omega_L^2+c_s^2k^2$. The possible polarizations of the spin waves are enumerated for this limit in (80) and (81). In the consideration of the decay of an acoustic wave into two spin waves we assume, as before, that the frequency of the wave of first sound is not too close to threshold, so that the spin waves have the wave vectors **k** and $-\mathbf{k}$.

First we consider the decay processes with participation of a Larmor spin wave, to which the index 4 corresponds. By virtue of the conservation law the frequency of first sound is here $\sim\gamma H$, and the condition on the wave vector of the decay waves has the form $1/c_s \gg k/\gamma H \gg 1/c_1$. In the decay into two Larmor waves, the frequency of the wave of first sound is $\approx 2\gamma H$; from the interaction Hamiltonian (82) we find the corresponding vertex

$$U_{44}=\frac{\rho^{1/2}k_\perp^2}{2\chi(\gamma H)^{1/2}c_1}\frac{\partial}{\partial\rho}(\chi c_s^2). \tag{87}$$

In the decay into a Larmor wave and a spin wave of another type, the frequency of the wave of first sound is $\approx \gamma H$. With account taken of the inequalities for the wave vector **k**, we find the expressions for the decay vertices:

$$U_{45}=\frac{i\gamma H\cos\varphi}{2\rho^{1/2}c_s\omega_5^{1/2}}\left(\frac{\omega_5}{2}+\frac{c_s^2k^2}{\gamma H}\right),\quad U_{46}=\frac{\rho^{1/2}}{2^{3/2}\chi c_1\omega_6^{1/2}}\frac{\partial}{\partial\rho}(\chi c_s^2), \tag{88}$$

where φ is the angle between the directions of the wave vectors of the wave of first sound and the Larmor wave.

We now consider the decay of a wave of first sound into spin waves that are not of the Larmor type. It must be kept in mind that the frequency ω of first sound is divided in two upon decay into waves of the same type, and is almost completely transferred to the wave of type 6 upon decay into waves of type 5 and 6. Taking this into account, we find the expressions for the decay vertices:

$$U_{55}=\frac{2^{1/2}\rho^{1/2}c_s^2k^2}{c_1(\gamma H)^2\omega^{1/2}\chi}\left[\frac{c_s^4k_\perp^2k_\parallel^2}{\omega_6^4}\frac{\partial}{\partial\rho}(\chi\Omega_L^2)\right.$$
$$\left.+\left(\frac{k^2(\omega_6^4+c_s^4k_\parallel^2k_\perp^2)+k_\parallel^2(\Omega_L^2+c_s^2k^2)^2}{\omega_6^4}-\frac{(\gamma H\omega)^2}{8c_s^4k^2}\right)\frac{\partial}{\partial\rho}(\chi c_s^2)\right], \tag{89}$$
$$U_{56}=\frac{c_s^3kk_\parallel k_\perp\rho^{1/2}}{2\gamma Hc_1\omega_5^{1/2}}\frac{\partial}{\partial\rho}\ln(\chi\omega_6^2),\qquad U_{66}=\frac{\omega^{1/2}}{2^{3/2}\rho^{1/2}c_1}\left(1-\rho\frac{\partial}{\partial\rho}\ln(\chi\omega_6)\right),$$

where we have $\omega_6=\omega_6(\mathbf{k})$. In the high-frequency limit $\gamma H\gg\omega\gg\Omega_L$, these expressions yield

$$U_{55}=\frac{2^{1/2}\rho^{1/2}c_s^4k^2k_\parallel^2}{c_1(\gamma H)^2\omega^{1/2}\chi\omega_6^4}\left[c_s^2k_\perp^2\frac{\partial}{\partial\rho}(\chi\Omega_L^2)+\Omega_L^2k_\parallel^2\frac{\partial}{\partial\rho}(\chi c_s^2)\right],$$
$$U_{56}=\frac{k_\parallel k_\perp}{k^2}\frac{\rho^{1/2}}{2^{3/2}c_1}\left(\frac{c_s^3k^3\omega_6}{\gamma H}\right)^{1/2}\frac{\partial}{\partial\rho}\ln(\chi c_s^2), \tag{90}$$
$$U_{66}=\frac{\omega^{3/2}}{2^{3/2}\rho^{1/2}c_1}\left(1-\rho\frac{\partial}{\partial\rho}\ln(\chi c_s)\right).$$

In the long-wave limit, which for U_{55} means $\omega\ll\Omega_L$, and for U_{56} and U_{66} means $\Omega_L/c_1\ll k\ll\Omega_L/c_s$, we find

$$U_{55}=-\frac{\omega^{3/2}\rho^{1/2}}{2^{5/2}c_1(1-k_\perp^2/2k^2)}\frac{\partial}{\partial\rho}\ln(\chi c_s^2),$$
$$U_{56}=\frac{k_\parallel c_s\rho^{1/2}\omega_5^{1/2}}{4c_1(k^2/k_\perp^2-1/2)^{1/2}}\frac{\partial}{\partial\rho}\ln(\chi\Omega_L^2), \tag{91}$$
$$U_{66}=\frac{\omega^{3/2}}{2^{3/2}\rho^{1/2}c_1}\left(1-\rho\frac{\partial}{\partial\rho}\ln(\chi\Omega_L)\right).$$

In the consideration of the interaction of second sound with spin waves, it is necessary to keep in mind the smallness of the frequency of second sound ω; we assume $\omega\ll\Omega_L$. Here the conservation laws allow only decay of second sound into two spin waves of type 5. By virtue of the quadratic dispersion law for these waves, we can assume that their wave vectors are equal to **k** and $-\mathbf{k}$, the frequency of the second sound is equally divided in this case. The expression for the corresponding decay vertex, obtained from the interaction Hamiltonian (82), has the form

$$\tilde U=\frac{\omega^{3/2}\Gamma^{1/2}}{2^{5/2}c_2\rho^{1/2}(1-k_\perp^2/2k^2)}\sigma\frac{d}{d\sigma}\ln(\chi c_s^2). \tag{92}$$

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