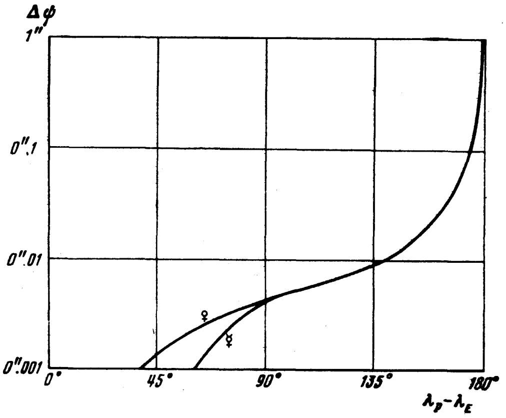

FIG. 5. Relativistic effect in the angle distance ψ of inner planet from the sun. Initial measurements: radar data at the instants $t=0$ (inferior conjunction) and $t=T_s/4$ (T_s is the synodic period).

of the motion of the celestial bodies, calculate the observational characteristics of the motion using formulas such as (11), (42), and (47) (with the same value of the function $\alpha(r)$ as used in the dynamic theory), determine the parameters of the motion by comparison with the observations, and then assess the reliability of the employed gravitation theory from the degree of agreement with subsequent observation.

The relativistic expressions obtained here for the measured angle distances φ and ψ add to the possibility of new tests of relativistic effects. These effects are of particular interest because the relativistic corrections to φ and ψ are necessitated by three factors: the dynamic theory of the motion of the bodies, the laws of light propagation, and the bending of space in the vicinity of the sun.

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Quantum mechanical analysis of the sensitivity of a gravitational antenna

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The quantum mechanical problem of the optimal estimate of the amplitude of an external force acting on a gravitational antenna of a given structure is solved. The optimal spectral operation for processing the output signal of the antenna which forms the observed variable is found. It is shown that there is no quantum sensitivity limit when the optimal procedures are followed. A practical possibility of attaining the resolution needed for second-generation antennas is illustrated.

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§1. INTRODUCTION

The realistic estimate for the intensity of bursts of gravitational radiation arriving at the Earth from outer-space covers the range $W\sim10^4-1$ erg/cm^2 with a duration $\hat{\tau}\sim10^{-3}-10^{-4}$ sec (see, for example, the reviews Ref. 1). For a gravitational detector of Weber type (measuring $l=10^2$ cm), such a pulse is equivalent to the action of an acceleration field $F/m\sim10^{-9}-10^{-11}$ cm/sec^2. Is it possible to detect such a weak disturbance? The answer to this question is crucial for modern gravitational-wave experiments.

Taking a quantum oscillator as a model of a gravitational detector in the limiting case of zero temperature, and assuming that it is in a coherent state (as the state nearest to a classical state), we can formulate a rule for detecting a force acting on the oscillator. It is natural to regard the force as detectable if it shifts the wave packet (or rather, its center) by an amount of the order of its width. In the coordinate representation, this shift is $\Delta k_{\rm qu}\approx(\hbar/2m\omega)^{1/2}$. Hence, for the quantum sensitivity limit we have

$$(F/m)_{\rm qu}\approx\hat{\tau}^{-1}(\hbar\omega/m)^{1/2}, \tag{1}$$

which for the typical parameters $m\approx10^4$, $\omega\sim10^4$, and $\hat{\tau}\sim2\times10^{-4}$ of a gravitational detector[1] gives $(F/m)_{\rm qu}\sim10^{-10}$ cm/sec^2, which is in the middle of the range in which we are interested.

This and similar considerations[2] forces us to approach the problem of detecting gravitational bursts with more care. In reality, the temperature of an antenna is not zero but, in fact, corresponds to a high excitation level and, it would seem, there is no need to invoke quantum arguments. A classical analysis of the sensitivity of a

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gravitational antenna based on the theory of optimal filtration of the signal from the noise is given in Ref. 3. However, it has been emphasized by Braginskiĭ[2,4] more than once that a vibrational system of frequency ω_μ having a high quality factor Q_μ (as is the case for a gravitational detector) can exhibit quantum properties at high temperatures $\varkappa T \gg \hbar\omega_\mu$ if the observation time is significantly shorter than the relaxation time of the system: $\hat{\tau} \ll \tau_\mu$. It is therefore of interest to consider the quantum detection problem, following the theoretical ideas of Refs. 5–7 as applied to a gravitational-wave experiment.

The quantum generalization of problems of optimal filtration is constructed by replacing the numerical variables by operators with definite commutation relations. The sequence of operators

$$\bar{Y}^T=(\hat{y}_1,\ldots,\hat{y}_n), \quad \hat{y}_i=\hat{y}(t_i), \quad t_i\in[0,T] \qquad (2)$$

to be observed (the quantum analog of the classical process $y(t)$) is the point of departure for the obtaining of an estimate $\bar{u}^T=D(\bar{Y})$ of the required vector $\bar{X}^T=(x_1,\ldots,x_n)$; D is the decision function, or the filtration algorithm, determined with allowance for the sequence of action of the operators from the condition of a minimum of the mean penalties. The form of the algorithm was found by Grishanin and Stratonovich[6] for the case of filtration with minimization of the rms error. Below, we shall not attempt to solve the complete problem of synthesis of an optimal detector for a gravitational signal. As in Ref. 3, we shall assume that the structure of the gravitational antenna, which consists of the gravitational detector and the detecting device, is given or almost given. The aim of the present paper is to find, in the framework of the quantum mechanical description, an operation for optimal processing of the output signal of the antenna (2) that realizes maximal sensitivity to the detection of an external force acting on the gravitational detector.

The formalism of the quantum theory of optimal filtration[5-7] also makes it possible to answer a number of experimentally important questions that arise in connection with Refs. 8–10. For example, does there exist a fundamental quantum limit to the accuracy of detection of a classical force (of the type (1)) acting on a quantum oscillator? When do the classical estimates for the sensitivity of a gravitational antenna break down? Is the Callen-Welton fluctuation-dissipation theorem sufficient for the determination of the fluctuation limitations on the sensitivity or is it necessary to make a further special analysis of the measuring instrument (as in Refs. 8-10) and so forth?

These questions are of interest for the general class of high-precision experiments with macroscopic test bodies as a field of application for the conclusions of the modern quantum theory of measurements.[11] For the gravitational-wave experiments, the answers to these questions are of practical importance, since they determine the basic possibility of constructing second-generation gravitational antennas for receiving gravitational radiation from cosmic objects.

§2. QUANTUM OSCILLATOR AND FILTRATION ALGORITHM FOR QUANTUM GAUSSIAN SYSTEMS

The main reason why a high-Q oscillator, even in a highly excited state with $\varkappa T_\mu \sim n_0\hbar\omega_\mu$, $n_0 \gg 1$, can manifest quantum properties is as follows. First, the inherent Brownian motion (due to interaction with the thermal bath) must have quantum features; for example, the energy-time diagram must approximately have the form of steps with length $\hat{\tau} \lesssim n_0^{-2}\tau_\mu$ (Refs. 2 and 4). Second, quantum features arise when a force is detected by means of the response of the oscillator. For a short disturbance, $\hat{\tau} \ll \tau_\mu$, the response does not depend on Q, and the thermodynamic spreading of the wave packet of the initial state is small, and therefore the results obtained for an ideal without damping hold. As we noted above, for a weak force F the change in the expectation value of the canonical variable (coordinate or momentum) may be less than the characteristic width of the coherent state: $F\hat{\tau}/m\omega_\mu < (\hbar/m\omega_\mu)^{1/2}$. The corresponding change in the mean energy of the oscillator is less than the Poisson standard deviation $\sim n_0^{1/2}\hbar\omega_\mu$. As a result, the limit (1) arises.

The analysis of Refs. 8 and 9 shows that the limitation (1) is not fundamental. It can be avoided by, for example, using the so-called nondisturbative method of measuring the energy,[8a,10a] which permits a resolution $F_{\min} \gtrsim F_{\text{qu}}/\sqrt{n_0}$. It follows that with increasing n_0 an arbitrarily weak force can be detected. (For what follows, it is important to note that the dependence of the increment of the oscillator energy on n_0 makes it possible to regard the "signal" contained in the response of the oscillator to an external force as classical (i.e., many-quantum) for an arbitrary weak force F if n_0 is sufficiently large.)

Another way to circumvent the limitation (1) is associated with the stroboscopic method of measurement of the oscillator coordinate at intervals $\sim\omega_\mu^{-1}$. In this case, the limiting resolution is of order $F_{\min} \gtrsim F \sqrt{\omega_\mu \Delta t}$ and improves without limit if the stroboscopic interval Δt is shortened.[3b]

Taken together, Refs. 8-10 emphasize that the result of detection of an external force (classical) on a quantum oscillator depends strongly on the measurement procedure that is adopted.

We consider the problem of detecting a classical force acting on a complex quantum system consisting of a quantum oscillator and a sensor from the traditional points of view of optimal processing of the signal of a system with given properties. The aim of the processing is to achieve the best signal-to-noise ratio. The concept of "noise" must here also include quantum limitations such as the noncommutativity of observables and the reduction of the density matrix as a result of measurement.

We use the results of Grishanin and Stratonovich,[6,7] who found a procedure for optimal filtration of a classical signal $\bar{S}(t,\theta)$ on the background of quantum noise $\hat{\xi}(t)$. Namely, we consider the mixture

$$\hat{Y}(t)=\bar{S}(t,\theta)+\hat{\xi}(t). \qquad (3)$$

The quantum operator variable $\hat{Y}(t)$ is assumed capable of observation (measurement, processing), and we assume that the signal profile is known. We aim at an optimal estimation of the parameter θ. In the simplest case, θ is the signal amplitude, i.e. $\overline{S}(t,\theta)=\theta S(t)$. In accordance with Refs. 6 and 7, the quantum generalization of the algorithm for optimal filtration leads to the following unbiased and locally effective estimate of the parameter θ:

$$\hat{u}(\theta)=\theta+\hat{L}(\theta)/I(\theta), \tag{4}$$

which minimizes the residual mean penalties (for all g)

$$R(u)=\mathrm{Tr}[(\hat{u}-\theta)^T g(\hat{u}-\theta)\hat{\rho}(\theta)]. \tag{5}$$

Here, $\hat{\rho}(\theta)$ is the density matrix describing the statistical properties of the quantum variables $\hat{y}_i=\hat{y}(t_i)$, $t_i \in[0,T]$. (We shall use a replacement of the continuous variable $\hat{y}(t)$ by the vector row $\overline{Y}=(\hat{y}_1,\hat{y}_2,\ldots,\hat{y}_N)$; then the transposed vector column will be denoted by $\overline{Y}^T$.)

In Eqs. (4) and (5), $\hat{L}(\theta)$ is the symmetrized logarithmic derivative with respect to θ^T of $\hat{\rho}(\theta)$,[5] and I is the information matrix; these are given by the expressions

$$\hat{L}\hat{\rho}+\hat{\rho}\hat{L}=2\,\partial\hat{\rho}/\partial\theta^T, \tag{6}$$

$$I(\theta)=\mathrm{Tr}\tfrac{1}{2}[\hat{L}\hat{L}^T+(\hat{L}\hat{L}^T)^T]\hat{\rho}(\theta). \tag{7}$$

The operator (4) is the quantum analog of the classical estimate of maximal likelihood, and the condition of a minimum of (5) is a generalization of the requirement of minimization of the mean square error of the estimate (4). For Gaussian quantum noise $\hat{\xi}(t)$ (we shall, as usual, assume in what follows that the Gaussian approximation reflects the physical conditions with a sufficient accuracy), Eqs. (4)–(7) can be particularized. The density matrix of the variables y_i has the form

$$\hat{\rho}(\theta)=\exp[\Gamma-(\overline{Y}-S(\theta))^T Q(\overline{Y}-S(\theta))]. \tag{8}$$

Given quantities are the numerical matrix $Q=\|Q(t,t')\|$ and the commutator $C=[\hat{Y},\hat{Y}^T]=[\hat{\xi},\hat{\xi}^T]$, which is assumed to be a nonoperator quantity (also a c-number matrix). Calculation of the operator $\hat{L}$ and the matrix I in accordance with (6) and (7) leads to the expressions

$$\hat{L}=\frac{\partial S^T(\theta)}{\partial\theta^T}K^{-1}[\hat{Y}-S(\theta)], \tag{9}$$

$$K=\tfrac{1}{2}C\,\mathrm{cth}\,QC, \tag{10}$$

$$I=\frac{\partial S^T(\theta)}{\partial\theta^T}K^{-1}\frac{\partial S(\theta)}{\partial\theta}. \tag{11}$$

Taking $\overline{S}(t,\theta)=\theta S(t)$, we find in accordance with (4) the optimal estimate of the amplitude θ in the matrix form

$$\hat{u}=(S^T K^{-1}\hat{Y})/(S^T K^{-1}S). \tag{12}$$

It is easy to show that the matrix K in Eqs. (9)–(12) coincides with the correlation matrix of the quantum fluctuations $\hat{\xi}(t)$, i.e.

$$K=\mathrm{Tr}\left[\frac{\hat{\xi}\hat{\xi}^T+(\hat{\xi}\hat{\xi}^T)^T}{2}\hat{\rho}\right]. \tag{13}$$

We recall that the correlation matrix of a quantum system can be found by means of the Callen-Welton fluctuation–dissipation theorem.[12,13] Thus, this theorem takes into account fully the noncommutativity of the observables y_i. (In the language of Ref. 8 of Braginskiĭ *et al.*, one should say that allowance is made for the perturbation of the quantities y_i at times t_i subsequent to the time t_i of measurement of y_1.) In accordance with the postulates of quantum mechanics, there are no fundamental restrictions on exact measurement of a single Hermitian observable.[11] This is accompanied by a distrubance of the canonically conjugate observable, which can be constructed from (12) from, for example, the given commutator $[\hat{u},\hat{v}]=\hbar$. In principle, to measure (estimate) the amplitude of a signal of known profile it is sufficient to make a single observation of the variable $\hat{u}$ after its synthesis over the interval of action of the signal. From this point of view, it is not important what happens to the canonical pair $(\hat{u},\hat{v})$ after the estimate has been obtained. As in the classical case, the estimate is obtained by comparison with the threshold, which depends on the *a priori* mean characteristics of $\hat{u}$. Going over to a continuous representation of the vector $\overline{Y}$ as a function of the time, we can readily show that the algorithm (12) is analogous to a classical matched linear filter, so that the variable $\hat{u}$ is the result of the integral operation

$$\hat{u}=\int_0^T B(t)\hat{y}(t)\,dt\Big/\int_0^T B(t)S(t)\,dt; \tag{14}$$

in which $B(t)$ is the solution of the integral equation

$$\int_0^T K(t,\tau)B(\tau)\,d\tau=S(t), \tag{15}$$

where $K(t_1,t_2)=1/2\langle\hat{\xi}(t_1)\hat{\xi}(t_2)+\hat{\xi}(t_2)\hat{\xi}(t_1)\rangle$ is the correlation function of the fluctuations $\hat{\xi}(t)$.

In spectral language, introducing the quantum energy spectrum

$$N_\xi(\omega)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\langle\hat{\xi}(t)\hat{\xi}(t+\tau)+\hat{\xi}(t+\tau)\hat{\xi}(t)\rangle e^{j\omega\tau}\,d\tau, \tag{16}$$

we can represent the signal-to-noise ratio at the filter output (14) as

$$\theta^{-2}\geqslant\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|S(j\omega)|^2}{N_\xi(\omega)}\,d\omega. \tag{17}$$

Thus, the quantum results of the filtration problem differ from the classical results only in the quantum generalization of the correlation matrix K and the energy spectrum $N_\xi(\omega)$, as was noted in Refs. 6 and 7.

§3. QUANTUM CALCULATION OF THE FLUCTUATIONS OF A GRAVITATIONAL ANTENNA

To apply the recommendations of the preceding section to an estimation of the sensitivity of a gravitational antenna, we must, on the basis of its given structure, calculate the correlation function or the energy spectrum of the output noise. The dynamical equations of antennas are given, for example, in Ref. 3; as there, we consider successively the case of a detector with passive transducer and then with a transducer of parametric type.

a) A gravitational detector with passive transducer (without external energy source) is described by a system of linear equations with constant coefficients that relate the coordinate of the detector ξ and the sensor η to the external force f_c acting on the detector:

$$\ddot{\xi}+2\delta_\mu\dot{\xi}+\Omega^2\xi-\omega_\mu^2\Gamma_1\eta=\Omega^2 f_c(t),$$
$$-\omega_e^2\Gamma_2\xi+\ddot{\eta}+2\delta_e\dot{\eta}+\Omega^2\eta=0 \qquad (18)$$

(the normalization is the same as in Ref. 3: $\xi=x/l$, $\eta=v/V_0$, $f_c=F/m\omega_\mu^2 l$). Here, the macroscopic coordinates are quantum-mechanical expectation values: $\xi=\langle\hat{\xi}\rangle$, $\eta=\langle\hat{\eta}\rangle$. For virtually all known antennas of this type, the partial frequencies of both degrees of freedom are approximately equal to the resonance frequency of the detector, $\Omega\approx\omega_e\approx\omega_\mu$, and therefore we shall ignore their difference below.

To calculate the spectral density of the quantum variable $\hat{\eta}$, we use the fluctuation-dissipation theorem, assuming an equilibrium state of the system (18) for $f_c=0$. In accordance with this theorem, the spectral density of the fluctuations of $\hat{\eta}$ can be expressed in terms of the anti-Hermitian part of the generalized susceptibility $\alpha(j\omega)$ of the dynamical system (18):

$$N(j\omega)=j\mathscr{E}(\omega,T)\omega^{-1}[\alpha(j\omega)-\alpha^*(j\omega)]. \qquad (19)$$
$$\mathscr{E}(\omega,T)=E(\omega,T)/E_0,\quad E(\omega,T)={}^1/_2\hbar\omega\,\mathrm{cth}(\hbar\omega/2\varkappa T),$$
$$\alpha(j\omega)\approx\omega_\mu^2(\omega_\mu^2-\omega^2+2j\delta_\mu\omega)/\mathrm{Det}(j\omega),$$
$$\mathrm{Det}(j\omega)=(\omega_\mu^2-\omega^2+2j\delta_\mu\omega)(\omega_\mu^2-\omega^2+2j\delta_e\omega)-\lambda^2\omega_\mu^4.$$

Here, we have introduced the transformation coefficient $\lambda^2=\Gamma_1\Gamma_2$, which is equal to the ratio of the electrical energy of the sensor to the mechanical energy of the antenna, and E is a normalization coefficient ($E_0 = cV_0^2$, in which c is the capacitance of the sensor[3]). From Eqs. (19), we find

$$N(\omega)\approx4\omega_\mu^2|\mathrm{Det}(j\omega)|^{-2}\{[(\omega_\mu^2-\omega^2)^2+4\delta_\mu^2\omega^2]\delta_e+\lambda^2\omega_\mu^4\delta_\mu\}\mathscr{E}(\omega,T). \qquad (20)$$

In the limit $\varkappa T\gg\hbar\omega$, Eq. (20) coincides with the result of the calculation in accordance with the classical correlation theory.[3] In what follows, we shall use (20), taking $\delta_\mu=0$, which corresponds to a high-Q detector. In the absence of coupling to the thermal bath also through the electrical degree of freedom, in the limit $\delta_e\to0$, the spectrum (20) goes over into

$$N(\omega)\approx\frac{\pi}{2}\sum_k\left(\frac{\omega_\mu}{\omega_k}\right)^2\mathscr{E}(\omega_k,T)\delta(\omega-\omega_k), \qquad (21)$$

where $\omega_k^2=\omega_\mu^2\ (1\pm\lambda)$ are the eigenfrequencies of the system (18), i.e., the energy spectrum of the fluctuations takes the form of δ-function spikes at the eigenfrequencies of the antenna.

b) For a gravitational detector with a parametric transducer, it is convenient to use the Langevin method of calculation associated with the introduction of locally uncorrelated quantum sources of fluctuations for each of the degrees of freedom of the antenna ($\hat{f}_\mu(t), \hat{e}(t)$). (In the absence of fluctuations "at the input" of the dynamical system, the operators in the limit $t\to\infty$ will converge to zero, which entails vanishing of the commutators over a long time, in contradiction with one of the fundamental principles of quantum mechanics.) Then the dynamical equations of the antenna under the condition of sufficiently strong pumping,

$$\eta_p=C_p\cos(\omega_p t+\varphi_p),\quad |\eta_p|\gg\langle\eta\rangle,$$

can be written in the form

$$\ddot{\hat{\xi}}+2\delta_\mu\dot{\hat{\xi}}+\omega_\mu^2\hat{\xi}-2\lambda\omega_p\eta_p\hat{\eta}\approx\omega_\mu^2[f_c+\hat{f}_\mu],$$
$$-\omega_e^2\eta_p\hat{\xi}+\ddot{\hat{\eta}}+2\delta_e\dot{\hat{\eta}}+\omega_e^2\hat{\eta}\cong\omega_e^2\hat{e} \qquad (22)$$

(the normalization is as follows: $\hat{\xi}=\hat{x}/d$, $\hat{\eta}=\hat{v}/V_0$, $f_c=F_c/m\omega_\mu^2 d$, $\lambda=cV_0^2/m(\omega_\mu d)^2$, $c=S/4\pi d$, $C_p=V_p/V_0$ (Ref. 3)). As above, we assume that the inherent fluctuations of the detector are small, $\delta_\mu=0$, and accordingly $\hat{f}_\mu=0$. The consistent quantum calculationof the source spectral density $\hat{e}(t)$ gives $N_e=\mathscr{E}(\omega,T)\cdot4\delta_e\omega_e^{-2}$.

The solution of (22) is analogous to the calculation in the classical case.[3] The smallness of the deformation of the detector, $\langle\hat{\xi}\rangle\ll1$, and the strong difference between the partial frequencies, $\omega_\mu\ll\omega_e\sim\omega_p$, make it possible to go over to truncated equations; the solution of (22) is sought in the form

$$\hat{\xi}(t),\ \hat{\eta}(t)=\hat{a}(t)\cos\omega_p t+\hat{b}(t)\sin\omega_p t,$$

where $\hat{a}$ and $\hat{b}$ are slow functions. It is they that carry the information about the external force, and therefore in what follows demodulation of the variable $\hat{\eta}$ is necessary. As is shown in Ref. 3, the demodulation can be performed quasioptimally by means of synchronous detection with reference signal proportional to the pumping $\eta_p(t)$. Constructing the quantum analog of such processing, we go over in (22) to new operator variables:

$$\hat{\xi},\hat{\eta}\to\hat{\xi},\hat{\eta}_{sd}=\hat{a}\cos\varphi_p+\hat{b}\sin\varphi_p,$$

which satisfy the equations

$$\ddot{\hat{\xi}}+\omega_\mu^2\hat{\xi}-\lambda\omega_\mu^2 C_p\hat{\eta}_{sd}=\omega_\mu^2 f_c(t),$$
$$-{}^1/_2\Delta\omega_e C_p\hat{\xi}+\ddot{\hat{\eta}}_{sd}+2\delta_e\dot{\hat{\eta}}_{sd}+(\delta_e^2+\Delta^2)\hat{\eta}_{sd}=\omega_e\hat{\Phi}_e(t). \qquad (23)$$

Here, $\Delta=(\omega_e-\omega_p)\sim\delta_e$; the spectral density of the fluctuations $\hat{\Phi}_e(t)$ is given by

$$N_f(\omega)\approx(\omega^2+\delta_e^2+\Delta^2)\omega_e^{-2}N_e(\omega). \qquad (24)$$

The solution (23) gives the desired fluctuation spectrum of the variable $\hat{\eta}_{sd}$:

$$N(\omega)\approx[\omega^2+\delta_e^2+\Delta^2](\omega_\mu^2-\omega^2)|\mathrm{Det}(j\omega)|^{-2}N_e(\omega),$$
$$\mathrm{Det}(j\omega)=(\omega_\mu^2-\omega^2)(\delta_e^2+\Delta^2-\omega^2+2j\delta_e\omega)-{}^1/_2\Delta\lambda C_p^2\omega_e\omega_\mu^2. \qquad (25)$$

Like (21), $N(\omega)$ in (25) has δ-function spikes at the eigenfrequencies of the system (23) in the limit $\delta_e\to0$; at the same time, (25) coincides with the classical result[3] at high temperatures. The expressions (20) and (25) will be used below to estimate the antenna sensitivity.

§4. ANALYSIS OF THE SENSITIVITY AND QUANTUM LIMITATIONS

In accordance with the prescription (17), the minimal detectable force acting on the gravitational detector can be found from the condition

$$(2\pi)^{-1}\int_{-\infty}^{+\infty}|\eta_c(\omega)|^2/N(\omega)\,d\omega=1, \qquad (26)$$

where $\eta_c(\omega)$ is the spectral intensity of the signal response, which is determined by the solution of the classical equations (18) and (23). We shall assume that the spectrum $\eta_c(\omega)$ (like the spectrum $f_c(\omega)$) is in a certain neighborhood of ω_μ around the central frequency of the detector; the widths of the spectrum satisfies the condition $\Delta\omega\hat{\tau}\gtrsim1$ (this description corresponds to astrophysical models of a gravitational burst; see the bibliography in Ref. 1).

a) Gravitational detector with passive sensor. Using

(20), we estimate the integral (26). In dimensional form, the antenna resolution is

$$F_{min}\approx 2\hat{\tau}^{-1}[2mE(\omega_\mu, T)]^{1/2}(\lambda_0/\lambda)(\Delta\omega\hat{\tau})^{1/2}. \quad (27)$$

In (27), we have introduced the coupling coefficient in the matched regime: $\lambda_0=(Q_e\omega_\mu\hat{\tau})^{-1/2}$ (Ref. 3); the classical and quantum limits correspond to replacement of $E(\omega_\mu, T)$ by $\varkappa T$ or $\hbar\omega_\mu/2$; by the temperature T we understand the temperature T_e of the sensor, though for an antenna with passive transducer it is natural to take $T_e\approx T_\mu=T$.

b) Gravitational detector with parametric sensor. The analogous procedure using (25) and (26) leads to the expression

$$F_{min}\approx 2\hat{\tau}^{-1}[2mE(\omega_e, T_e)(\omega_\mu/\omega_e)]^{1/2}(C_0/C_p)(\Delta\omega\hat{\tau})^{1/2}, \quad (28)$$

where C_0 is the pumping amplitude in the matched regime, which depends on the actual parameters of the antenna $((V_0C_0)^2\approx 2\sqrt{2}m\omega_\mu d^2\cdot(c\hat{Q}_e\tau)^{-1}$; for $\delta_e=0$, Q_e is replaced by $\omega_e\hat{\tau}/2$; for details see Ref. 3).

Equations (28) and (27) are valid when $C_p\geq C_0$ and $\lambda\geq\lambda_0$; if the opposite inequalities hold, the corresponding coupling ratio must also be inverted (i.e., λ_0/λ and C_0/C_p replaced by λ/λ_0 and C_p/C_0). The factor ω_μ/ω_e in (28) characterizes the increase in the resolution due to the "parametric cooling", $\omega_\mu\ll\omega_e$. With increasing pumping frequency and the passage through the boundary $\varkappa T_e=\hbar\omega_e$, the combination $E(\omega_e, T_e)(\omega_\mu/\omega_e)$ is transformed into $\hbar\omega_\mu$ and the effect of parametric cooling ceases to be important (the quantum fluctuations increase at the same rate as the signal).

The main result contained in Eqs. (28) and (27) is that they predict an increase in the sensitivity ($F_{min}\to 0$) with increasing coupling between the detector and the sensor. In the passive variant of the antenna, the possibilities for increasing λ are few: theoretically $\lambda\leq 0.5$, while in practice $\lambda\approx 0.1$ has been achieved.[14] In the case of a parametric antenna, the coupling increases with the pumping. The maximal admissible C_p is limited by the requirement of dynamical stability of the system (22). In the working regime, when $\delta_e\sim\Delta\gg\omega_\mu$, the stability condition is $C_p^2\leq C_0^2\,\omega_\mu\hat{\tau}$. Thus, increasing the resolution by increasing the pumping (coupling) is possible (without additional stabilization) only for long trains: $\omega_\mu\hat{\tau}\gg 1$. For short bursts, $\omega_\mu\hat{\tau}\sim 1$, it is expedient to make the filtration band $\Delta\omega$ narrower relative to the width of the signal spectrum, which is $\sim\hat{\tau}^{-1}$; For $\Delta\omega\hat{\tau}\ll 1$, one can in prinicple achieve high resolution, though is is paid for by a reduction in the signal.

The physical reason for the increase in the sensitivity with increasing coupling is to be sought in the effect of dynamical damping[15] generalized to the quantum case. The fluctuation spectrum of the system of coupled oscillators (18), (23) for $\delta_\mu=0$ has a characteristic dip at the frequency ω_μ of the detector, as can be seen from Eqs. (20) and (25). In contrast, the signal response at this frequency is nonzero. A gain in sensitivity is possible only in the case of measurements in a small neighborhood of ω_μ, whose width must be at least less than the interval between the eigenfrequencies.

For a real antenna $\delta_\mu=\tau_\mu^{-1}\neq 0$, and the damping formulas (27) and (28) will be valid only until the resolution reaches the potential level

$$F_{pot}\approx 2\hat{\tau}^{-1}[mE(\omega_\mu, T_\mu)\hat{\tau}/\tau_\mu]^{1/2}. \quad (29)$$

The value of the pumping $C_0/C_p=(\hat{\tau}/\tau_\mu)^{1/2}$ needed for this may be difficult to achieve; it is then necessary to make the filtration band narrower. This narrowing of the band is accompanied by multiplication of (29) by the factor $(\Delta\omega\hat{\tau})^{-1/2}$; making a comparison with (28), we readily obtain an estimate for the optimal reduction. Thus, in the quantum case, $E(\omega_\mu, T_\mu)\approx E(\omega_e, T_e)\omega_\mu/\omega_e\approx\hbar\omega_\mu$, this band is $\Delta\omega_{opt}\approx(C_p/C_0)(\hat{\tau}\tau_\mu)^{-1/2}$, and the detectable force is

$$F_{min}\approx 2\hat{\tau}^{-1}(m\hbar\omega_\mu)^{1/2}(C_0/C_p)^{1/2}(\hat{\tau}/\tau_\mu)^{1/4}, \quad (30)$$

for

$$\omega_\mu\hat{\tau}\sim 1,\quad C_p=C_0\to F_{min}\approx 2\hat{\tau}^{-1}(m\hbar\omega_\mu)^{1/2}Q_\mu^{-1/4}.$$

Equation (30) demonstrates an interesting feature, which is a consequence of the dynamical damping: even for $C_p=C_0$, an increase in the Q of the detector, $Q_\mu\to\infty$, leads in prinicple to unbounded growth of the sensitivity, i.e., to the absence of any quantum limit.

Note specially that in the quantum case, i.e., in the limit $T\to 0$, an observer who uses the algorithm (26) cannot, any more than any other observer, arrive at any conclusions contradicting the uncertainty principle. It is easy to show by direct calculation that in the algorithm (26) or in its quasioptimal replacement by narrow-band filtration with $\Delta\omega\sim\hat{\tau}^{-1}$ around ω_μ reconstruction of the "trajectory" of the detector is possible only with accuracy

$$\Delta\xi\Delta\dot{\xi}\sim\Delta x\Delta p\approx\hbar(\omega_e/\omega_\mu)(C_p/C_0)\gg\hbar\quad(C_p\gg C_0);$$

if one gives up band filtration, which is accompanied by a loss of sensitivity, one achieves a reconstruction with $\Delta x\Delta p\approx\hbar\omega_e/\omega_\mu$. With regard to the coordinate of the sensor, observation of the complete spectrum of the variable $\hat{\eta}_{sd}$ ensures the accuracy

$$\Delta\eta\Delta\dot{\eta}\sim\Delta qL\Delta i=(C_p/C_0)\hbar>\hbar\quad(C_p\gg C_0);$$

working in a band $\Delta\omega\sim\hat{\tau}^{-1}$, the observer always remains at the level of uncertainty of the coherent state, i.e., $L\Delta i\Delta q\approx\hbar$ (here, L is the inductance of the resonance circuit of the sensor).

We show, finally, that the proposed detection algorithm (26) belongs to the class of asymptotically nondisturbative methods of detection in the sense of Ref. 8a. For this, it is sufficient to calculate the commutator of the variable $\hat{u}$ and the quantum process $\hat{\eta}(t)$ or $\hat{\eta}_{sd}$ (23). Omitting the details of the calculation, we write down the result:

$$\left.\begin{matrix}[\hat{u},\hat{\eta}]\\ [\hat{u},\hat{\eta}_{sd}]\end{matrix}\right\}=\frac{\hbar\omega_\mu}{E_0}\frac{1}{\delta_e\hat{\tau}}\cdot\left\{\begin{matrix}(\lambda_0/\lambda)^4\\ (C_0/C_p)^4\end{matrix}\right. \quad (31)$$

$$(\Delta\sim\delta_e,\ \Delta\omega\sim\hat{\tau}^{-1},\ \lambda\gg\lambda_0,\ C_p\gg C_0).$$

Thus, an increase in the coupling between the detector and the sensor is accompanied by a decrease in the disturbance of the quantum variables $\hat{\eta}$ and $\hat{\eta}_{sd}$ accompanying a measurement of $\hat{u}$ and an increase in the resolution of the antenna.

§5. DISCUSSION OF THE RESULTS

In our discussion, we shall first consider some fundamental aspects of the theory of measurements touched on in the introduction; we shall then analyze the possibilities for practical realization of our algorithms, and, finally, on this basis estimate the sensitivity of a gravitational antenna made possible by the present-day technology.

1. The algorithm for constructing the variable $\hat{u}$ (§2) reduces the estimate of the antenna sensitivity to an ordinary procedure of spectral filtration (17), but with allowance for vacuum noise; this allowance is made by calculating the fluctuations in accordance with the fluctuation-dissipation theorem. For comparison with the language used in Ref. 8, we emphasize that allowance for noncommutativity is allowance for the quantum disturbance of one of the observables when the other is measured, i.e., the signal-to-noise ratio calculated in accordance with the theorem includes the quantum mechanical uncertainty introduced by the measuring device. At the same time, the theorem does not contain a procedure for constructing the estimate. When the scheme for choosing the estimate is not optimal, a limited sensitivity is predicted. For example, if one detects the force F acting on the detector without band filtration around ω_μ and measures the total variance $\Delta\eta^2_{sd}$ (23), the fluctuation-dissipation theorem gives a/quantum limit for the minimal amplitude of the force (1). In the optimal procedure adopted in the text, there is no fundamental quantum limit.

2. The process of detecting F in our algorithm reduces to a measurement of $\hat{u}$, i.e., to an observation after the output of the band filter. A procedure corresponding exactly to the algorithm (12) consists of a single measurement of $\hat{u}$ at the time at which the signal ends and comparison of this value with the threshold, which is established *a priori* from the known statistics of the quantum (and thermal) fluctuations. The comparison yields a judgement on the magnitude of F. The threshold can also be found empirically by observing the fluctuations at the filter output for a sufficient time. Such detection in the case of a single measurement of $\hat{u}$ naturally eliminated problems associated with the disturbance (or nondisturbance) of the system by the act of measurement. However, it is easy to show that continuous following of $\hat{u}$ also preserves the conclusions (27) and (28) about the attainable sensitivity. Indeed, calculating the commutator $\hat{u}$ at different times, we find ($C_p \gtrsim C_0$, $\Delta\omega = \hat{\tau}^{-1}$)

$$[\hat{u}(t), \hat{u}(t+\tau)] = \frac{\hbar\omega_\mu}{E_0} \frac{1}{\delta_e \hat{\tau}} \left(\frac{C_0}{C_p}\right)^4 \frac{\sin(\tau/\hat{\tau})}{(\tau/\tau)} \sin\omega_\mu\tau. \tag{32}$$

It can be seen that the commutator $[\hat{u}, \hat{u}_\tau]$ vanishes with increasing pumping, making possible an asymptotic increase in the accuracy with which the trajectory of $\hat{u}$ can be measured. This demonstrates once more that the discussed procedure is an "asymptotically nondisturbative method of detection" in accordance with Ref. 8. Finally, it follows from (32) that at times $\omega_\mu\tau \sim 2\pi k (k = 0, 1, 2, \ldots)$ the measurement becomes "nondisturbative", in accordance with the stroboscopic rule.[8b]

3. Essentially, detection in accordance with the algorithm (12), (17) amounts to separating one statistical set (in the given case, the deterministic signal function) on the background of another set (quantum and thermal fluctuations). It makes use of subtle distinction criteria in the spectral color of the disturbances. These ideas, which are typical for the classical theory of filtration, remain valid for quantum systems. The analysis made in §2 shows that the process of forming the estimate $\hat{u}$ contains no operations forbidden by quantum theory.

It is remarkable that this approach does not require one to consider practically inconvenient categories and operations such as "reduction of the statistical state," "preparation of the state," "following of the relaxation of the state," and so forth, which is characteristic of Refs. 2, 8, and 9. The result of filtration in accordance with the algorithm (12), (17) is invariant with respect to the initial state of the observed system. These circumstances obviously free us from the need to apply the concept of "quantumness of the system" (i.e., the obligatory recourse to quantum theory for its investigations) to parameters such as the relaxation time and measurement time. The usual consideration remains valid: quantum effects are important only under the condition $\hbar\omega_\mu > \varkappa T$.

It is obvious that a gravitational antenna can be described by classical (or quasiclassical) theory even at very low temperatures $T \sim 1$°K. At frequencies $\omega_\mu \sim 10^4$, the mean energy $\langle E\rangle = n_0\hbar\omega_\mu$ of the detector corresponds to the level $n_0 = 10^7$; in a stationary state, $\langle\Delta E^2\rangle^{1/2} = n_0\hbar\omega_\mu \gg \hbar\omega_\mu$. The force F changes the energy of the detector by $\Delta E_c \approx (\langle E\rangle(F_0\hat{\tau})^2/m)^{1/2}$; for the amplitude of the force at the sensitivity threshold (28) $\Delta E_c \approx n_0^{1/2}\hbar\omega_\mu C_0/C_p$. Even under the assumption that n_0 is unchanged, we have a sufficient reserve of damping $C_p/C_0 \sim 10^3$ until ΔE_c is comparable with the energy of a quantum. However, with increasing coupling n_0 increases as $\sim(C_p/C_0)^2$ due to the "heating" of the detector from the active sensor.[16] The upshot is that for $C_p > C_0$ we always have $(\Delta E_c)_{min} \sim n_0^{1/2}\hbar\omega_\mu$. The process of transforming the signal into electrical form at least preserves the number of quanta, changing only their frequency: $\omega_\mu \to \omega_e$; the same is true for the opposite process of detection. For $\Delta\omega \sim \hat{\tau}^{-1}$, the detection error, which is equal to $(\Delta E_c)_{min}$, appreciably exceeds the energy of one quantum. Thus, despite the condition $\hbar\omega_p > \varkappa T_e$ (which is usually realized in practice), the gravitational antenna admits a classical description as long as $n_0 \gg 1$, although Planck's constant already occurs in the sensitivity estimate (28) (quasiclassical treatment).

4. The operation of filtration of the signal of a gravitational antenna on the basis of the theory presented here is as follows. The high-frequency voltage of the sensor is subjected to synchronous detection (or heterodyning) with a reference voltage that has the phase of the pumping in the resonance circuit of the sensor. With increasing pumping $C_p > C_0$, the low-frequency component $\langle\hat{\eta}_{sd}\rangle$ decreases as C_0/C_p, and to preserve it the reference amplitude must be increased

at the same rate (or faster). In the case of simple quadratice detection, the low-frequency component is proportional to $C_p\langle\hat{\eta}_{sd}\rangle$, and a constant level of the useful signal is ensured automatically (as long as $\eta_{sd} \ll C_p$). The signal can then be amplified and subjected to narrow-band filtration in the neighborhood of the frequency ω_μ.

From the point of view of purely quantum limitations, these operations cannot give rise to serious difficulties: the detection is not accompanied by quantum noise[17] (here, there are no paradoxes leading to violations of the uncertainty prinicple, as in Ref. 18); low frequency amplification, even in the framework of Heffner's integral approach,[18] is accompanied by fluctuations with the "noise temperature" $T_{equ} \sim \hbar\omega_\mu/\varkappa$, which for frequencies $\omega_\mu \sim 10^4$ is negiligibly small; $T_{equ} \sim 10^{-7}$ °K. It is clear that the actual sensitivity of the antenna will be limited by the above-quantum noise of the elements of the processing, the main noise being that of the detector (mixer).

We estimate the sensitivity of a gravitational antenna on the basis of the experimental data known to us on microwave devices. For example, Ref. 19 describes an experimental model of a mixer in the region of frequencies $\omega_e \sim 10^{10} - 10^{11}$ with noise temperature $T_n \sim 1$°K. With such a device in the electronic circuit, one can detect acceleration of the detector at the level

$$(F/m) \sim \hat{\tau}^{-1}(\varkappa T_n \omega_\mu/m\omega_e)^{1/2}. \qquad (33)$$

Substituting the values $\hat{\tau} \approx 3\times10^{-4}$ sec, $\omega_\mu = 10^4$ rad/sec, $m = 10^6$ g, and $\omega_e = 10^{10}$ rad/sec, we find $(F/m) \sim 3\times10^{-11}$ cm/sec^2, which is around the lower limit of the astrophysical prediction given in the Introduction. For sapphire detectors with $m \sim 10^4$ g, the estimate (33) gives only $(F/m) \sim 3\times10^{-10}$ cm/sec^2. In this case, technical noise can be attacked by raising the pumping frequency to $\omega_e \sim 10^{12}$ (Ref. 20). The value of increasing the frequency was noted in Ref. 21, in which, to achieve sensitive detection, transition to the optical range is recommended in conjunction with the use of degenerate parametric amplifiers and photodetectors.

5. The main result of this paper is the proof that there are no quantum limitations on the accuracy with which one can measure the amplitude of a gravitational burst for a definite system of operations to separate the signal. Note that we have considered only the question of estimating the amplitude of an external force acting on the gravitational detector. The results of this analysis cannot be extended to the problem of reconstructing the profile of the external force, for which quantum limitations may be more dangerous.

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