Fluctuations in the critical region

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A qualitative theory of fluctuations in the critical region, including the critical point, is developed using the example of a classical system describable by the Ginzburg-Landau equation with a random source. The starting points are the conditions for consistency of two limits: passage to the critical point ($T \to T_c$) and the thermodynamic limit. Depending on the value of $\tau = (T - T_c)/T_c$ four subregions can be distinguished: 1) The critical point and its immediate neighborhood, in which the parameter $\epsilon \propto 1/N$ is not small. The critical indices β, γ, and ν characterize the power dependences on N. 2) The region of scale invariance. The passage to this from subregion 1 occurs when the thermodynamic limit is taken. The same critical indices determine the power dependences on $|\tau|$. In the regions 1 and 2 the renormalized Ginzburg parameter is not a small parameter. 3) The crossover region between region 2 and the region of applicability of the Landau theory. 4) The region of applicability of the Landau theory. The relationship between the critical indices in the regions 2 and 4 is determined by the "Wilson parameter" ϵ_W, the values of which lie in the interval $1 \geq \epsilon_W \geq 0$. The value 1 corresponds to the region of scale invariance and the value 0 to the region in which the Landau theory is valid. In region 4 the Ginzburg parameter is small. The respective expressions for the correlation times of fluctuations of the order parameter and the volume-averaged polarization are given. They are valid for all the regions indicated. For $T < T_c$ the dependences of these times on N and T are substantially different. The former decreases in accordance with the Curie law with increase of $|\tau|$, and the latter tends to infinity as $N \to \infty$.

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1. INTRODUCTION

The phase-transition theory developed by Landau[1] is valid only at a sufficient distance from the critical point, when fluctuations of the order paremeter η are small. In the critical region the role of fluctuations is important and the calculation of the fluctuations is one of the main problems of the modern theory of phase transitions. This theory is based on the well known work of Kadanoff, Patashinskiĭ and Pokrovskiĭ, Wilson, and Fisher (see Refs. 1–5).

According to modern ideas, in the critical region the dependence of the order paremeter η, susceptibility χ, and correlation length r_c on the quantity $\tau = (T - T_c)/T_c$ (T_c is the critical temperature) is, as in the Landau theory, a power dependence:

$$\eta \propto |\tau|^{\beta}, \quad \chi \propto |\tau|^{-\gamma}, \quad r_c \propto |\tau|^{-\nu}. \tag{1.1}$$

However, the values of the power exponents—the "critical indices"—differ substantially from the corresponding indices of the Landau theory.

Altogether, eight critical indices are introduced: $\alpha, \beta, \gamma, \delta, \nu, \varepsilon, \mu, \xi$ (see Refs. 1–5). Between them there are five general relations

$$2\beta+\gamma=2-\alpha, \quad \beta\delta=\beta+\gamma, \quad \varepsilon(\beta+\gamma)=2, \quad \mu(\beta+\gamma)=\nu, \quad \nu(2-\xi)=\gamma. \tag{1.2}$$

These, naturally, are also valid in the Landau theory.

A sixth relation can be obtained using the scale-invariance hypothesis of Kadanoff, and Patashinskiĭ and Pokrovskiĭ.[1,2] It has the form

$$\nu d=2-\alpha \quad (\text{or} \quad \nu d=\gamma+2\beta). \tag{1.3}$$

Using the relations cited we can express the eight critical indices in terms of two independent indices. Calculations of the critical indices for the model with the Landau Hamiltonian have been carried out in papers by Wilson and Fisher by the so-called ε-expansion method. The calculations carried out showed that two of the eight critical indices (the indices α and ξ) are significantly smaller than the others. In the approximation $\alpha = 0$, $\xi = 0$, from the six relations (1.2) and (1.3) we can determine the values of all the other indices. For three-dimensional space ($d = 3$) they have the following values:

$$\beta={}^1/_3, \quad \gamma={}^4/_3, \quad \nu={}^2/_3, \quad \delta=5, \quad \varepsilon=0, \quad \mu={}^2/_5. \tag{1.4}$$

For comparison we give the values of the critical indices in the Landau approximation:

$$\alpha_L=0, \quad \xi_L=0, \quad \beta_L={}^1/_2, \quad \gamma_L=1, \quad \nu_L={}^1/_2, \quad \delta_L=3, \quad \varepsilon_L=0, \quad \mu_L={}^1/_3. \tag{1.5}$$

The dependences of the critical indices *(sic)* on $|\tau|$ given by the formulas (1.1) are not valid at the critical point itself, when $|\tau| = 0$, since, e.g., the correlation length r_c cannot exceed the size of the system. In other words, it may be said that the dependences (1.1) can hold only if the thermodynamic limit has already been taken.

In the present paper, using the example of a system of classical atomic oscillators with diffusive coupling, we give a qualitative treatment of fluctuations in the critical region, including the critical point itself. The starting point, as is customary in a number of papers,[6,7] is the Ginzburg-Landau equation for the polarization vector, with a random source included in it.

It is shown that noncontradictory results can be obtained only when we have consistency of the two limits: the limit $T \to T_c$ and the thermodynamic limit $N \to \infty$, $V \to \infty$ with $N/V = n$. The requirements of consistency of these limits at the critical point itself lead to the necessity of renormalization of the basic quantities and to the introduction of appropriate critical indices characterizing the dependences on N (or V). As we move away from the critical point in the parameter $\varepsilon_N \propto 1/N$ [cf. (2.13)] the power dependences on N change to power depen-

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dences on $|\tau|$ of the form (1.1). The critical indices here coincide with the values (1.4). To determine the small critical indices as well, the consistency conditions should be generalized (see Sec. 3).

The crossover from the region of scale invariance to the region in which the Landau theory is applicable is traced. For the approximation under consideration expressions are obtained for the correlation times (and the corresponding widths of the spectra) both for fluctuations of the order parameter and for the volume-averaged polarization vector. For $|\tau| \neq 0$ and in the thermodynamic limit the correlation time of fluctuations $\delta\eta$ tends to zero like $|\tau|^{-1}$, while the correlation time of values of the volume-averaged polarization vector (Sec. 7) tends to infinity.

2. THE INITIAL EQUATIONS. THE LANDAU-THEORY APPROXIMATION

We shall consider a system of classical atomic oscillators with characteristic frequency ω_0. They are coupled via spatial diffusion. The equation for the polarization vector $P(\mathbf{R}, t) \equiv enx(\mathbf{R}, t)$ with allowance for the Lorentz field and the anharmonicity of the individual oscillators can be written, for the description of slow processes ($\omega \ll \gamma$, where γ is the friction coefficient), as a Ginzburg-Landau equation with a random source. Here it is more convenient to use the equation for the function $x(\mathbf{R}, t)$. It has the form

$$\frac{\partial x}{\partial t}+(a+bx^2)x-g\frac{\partial^2 x}{\partial \mathbf{R}^2}=\frac{e}{m\gamma}E+y(\mathbf{R},t). \tag{2.1}$$

Below it is assumed that only one component of the polarization vector is important. The quantity a can be represented in the form

$$a=a_0\tau\equiv a_0(T-T_c)/T_c, \tag{2.2}$$

a_0, b, and g are constants, E is the external electric field, and $n = N/V$ is the average concentration of atoms. The moments of the random source y are determined by the expressions

$$\langle y\rangle=0,\ \langle y(\mathbf{R},t)y(\mathbf{R}',t')\rangle=2\frac{D}{n}\delta(\mathbf{R}-\mathbf{R}')\delta(t-t'),\ D=\frac{kT}{m\gamma}. \tag{2.3}$$

We expand the functions x and y in Fourier series, e.g.,

$$x(\mathbf{R},t)=\sum_{\mathbf{k}}x_{\mathbf{k}}(t)e^{i\mathbf{k}\mathbf{R}},\quad x_{\mathbf{k}}(t)=\int x(\mathbf{R},t)e^{-i\mathbf{k}\mathbf{R}}d\mathbf{R}/V. \tag{2.4}$$

The moments of the random source y_k are determined by the expressions

$$\langle y_{\mathbf{k}}\rangle=0,\quad \langle y_{\mathbf{k}}(t)y_{\mathbf{k}}^*(t')\rangle=2\frac{D}{N}\delta(t-t'). \tag{2.5}$$

Following Bogolyubov,[8] besides the usual averages we shall use quasi-averages. We denote them by $\langle\cdots\rangle_{E=0}$. The notation indicates that we must first perform the calculation with $E\neq 0$ and then pass to the limit $E = 0$. We also introduce the notation

$$\eta_{\mathbf{k}}(t)=\langle x_{\mathbf{k}}(t)\rangle_{E=0}=\eta\delta_{\mathbf{k},0}. \tag{2.6}$$

Here it is assumed that the quasi-averages are nonzero only for the fundamental mode $\mathbf{k}=0$. It follows from the formulas (2.4) that the order parameter

$$\eta=\langle x_{\mathbf{k}=0}(t)\rangle=\int\langle x(\mathbf{R},t)\rangle_{E=0}d\mathbf{R}/V\equiv\langle x(t)\rangle_{E=0},$$

i.e., it coincides with the quasi-average of the quantity $\int x(\mathbf{R}, t)d\mathbf{R}/V$. Neglecting fluctuations, from Eq. (2.1) we obtain

$$\eta=(|a|/b)^{1/2}. \tag{2.7}$$

From the equation for the order-parameter fluctuation

$$\delta\eta_{\mathbf{k}}=x_{\mathbf{k}}-\eta\delta_{\mathbf{k},0} \tag{2.8}$$

in the linear approximation there follows the well known expression for the susceptibility,

$$\chi(\omega,\mathbf{k})=[-i\omega+a+3b\eta^2+gk^2]^{-1}, \tag{2.9}$$

and also expressions for the variance of the order parameter

$$\langle(\delta\eta)^2\rangle=\frac{D}{N}\frac{1}{a+3b\eta^2}=\frac{D}{N}\chi(0,0) \tag{2.10}$$

and the spatial correlation of the fluctuations

$$\langle(\delta\eta)^2\rangle_r=\frac{V\langle(\delta\eta)^2\rangle}{4\pi r_c^2 r}e^{-r/r_c}. \tag{2.11}$$

Here we have introduced for the correlation length the notation

$$r_c=(g\chi(0,0))^{1/2}. \tag{2.12}$$

We note that the notation used differs from that adopted in the book by Landau and Lifshitz.[1] To go over to the corresponding formulas of Ref. 1 it is necessary to use the correspondence formulas

$$en\eta\leftrightarrow\eta_{LL},\quad \frac{e^2n}{m\gamma}\chi\leftrightarrow\chi_{LL}.$$

Then, e.g.,

$$\langle(\delta\eta)_{LL}{}^2\rangle=e^2n^2\frac{D}{N}\chi=\frac{kT}{V}\chi_{LL}.$$

Using the results cited we can introduce two dimensionless parameters. The first of these characterizes the relative variance:

$$\frac{\langle(\delta\eta)^2\rangle}{\eta^2}\sim\frac{Db}{Na^2}=\varepsilon_N,\quad a=a_0|\tau|. \tag{2.13}$$

At a finite distance from the critical point ($|a| \neq 0$) the parameter ε_N tends to zero in the thermodynamic limit.

Following Patashinskiĭ and Pokrovskiĭ,[2] we shall call the second parameter Gi the Ginzburg parameter (more consistently, the Levanyuk–Ginzburg parameter). It defines the ratio of the correlation function at the point $r = r_c$ [the point at which the function $\langle(\delta\eta)^2\rangle_r r^2$ is a maximum] to η^2. Thus,

$$\mathrm{Gi}\sim\frac{\langle(\delta\eta)^2\rangle_{r=r_c}}{\eta^2}\sim\frac{V}{r_c^3}\frac{\langle(\delta\eta)^2\rangle}{\eta^2}\sim\frac{V}{r_c^3}\varepsilon_N. \tag{2.14}$$

It follows from the definitions (2.13) and (2.14) that for $|a| \neq 0$ the Ginzburg parameter has a finite value in the thermodynamic limit.

From the formulas given it can be seen that the static susceptibility and correlation length tend to infinity as $T\to T_c$. Because of this, as $T\to T_c$ (but with finite N, i.e., without taking the thermodynamic limit), the variance of the order parameter also tends to infinity.

However, at the critical point the actual values of χ and r_c can tend to infinity only when the thermodynamic limit is taken. In order to remove this contradiction we first carry out a very simple generalization of the results of the Landau theory.

For this we note that, in the critical region, $\eta \sim \delta\eta$; therefore, to generalize the formulas (2.9), (2.10), and (2.12) to the critical region we make in them the replacement

$$\eta^2 \to \langle x^2_{k=0}(t)\rangle = \langle x^2(t)\rangle, \quad x(t) = \int x(\mathbf{R}, t)\, d\mathbf{R}/V. \tag{2.15}$$

We denote the new characteristics by $\tilde{\chi}$, $\tilde{r}_c$, and $(\tilde{\delta}\eta)^2$. Then, e.g.,

$$\tilde{\chi} = [a + 3b\langle x^2(t)\rangle]^{-1}. \tag{2.16}$$

To find an equation for the quantity $\langle x^2\rangle$ we use the Langevin equation for the function $x(t)$. From it we find an equation for the function x^2:

$$\frac{1}{2}\frac{dx^2}{dt} + (a + bx^2)x^2 = y_x(t) \equiv x(t)y(t). \tag{2.17}$$

Here, with the aim of obtaining a qualitative result, only the contribution determined by the fundamental mode has been kept.

Using the equality $\langle y_x\rangle = D/N$, with the aid of Eq. (2.17) we find the equation

$$a\langle x^2\rangle + b\langle x^4\rangle = D/N. \tag{2.18}$$

Naturally, this equation is not closed, since, besides the function $\langle x^2\rangle$, the fourth moment appears in it. The simplest approximation which at the same time takes into account features of the behavior at the critical point is

$$\langle x^4\rangle = \langle x^2\rangle\langle x^2\rangle \tag{2.19}$$

and leads to the closed equation

$$(a + b\langle x^2\rangle)\langle x^2\rangle = D/N. \tag{2.20}$$

From this, with neglect of fluctuations, the result (2.7) follows, and at the critical point,

$$\langle x^2\rangle = (D/bN)^{1/2} \propto N^{-1/2}. \tag{2.21}$$

Of course, this result is true only in order of magnitude.

Using the latter result, with the aid of formulas (2.16) and (2.12) we find the values of $\tilde{\chi}$ and $\tilde{r}_c$ at the critical point:

$$\tilde{\chi}(0,0) \sim \left(\frac{N}{Db}\right)^{1/2}, \quad \tilde{r}_c = (g\tilde{\chi})^{1/2} \sim \left(\frac{Ng^2}{Db}\right)^{1/4} \propto V^{1/4}. \tag{2.22}$$

It follows from this that at the critical point the relative volume

$$\tilde{r}_c^3/V \propto V^{-1/4} \propto N^{-1/4} \tag{2.23}$$

and is, therefore, equal to zero in the thermodynamic limit.

On the other hand, the corresponding value for the Ginzburg parameter at the critical point

$$\widetilde{\mathrm{Gi}} \sim \frac{V}{\tilde{r}_c^3}\frac{\langle(\delta\eta)^2\rangle}{\langle x^2\rangle} \sim \frac{V}{r_c^3} \propto V^{1/4} \propto N^{1/4} \tag{2.24}$$

and, consequently, tends to infinity in the thermodynamic limit.

We see that what would appear to be the natural generalization of the Landau theory leads to the unsatisfactory results (2.23, 24).

It is interesting that the contradiction (2.23) is removed when $d = 4$, i.e., when we consider four-dimensional space. This illustrates the well known fact that four-dimensional space plays a special role in mean-field theory.

We show now that the position can be improved (for $d = 3$, of course) if we satisfy the requirements that the above two limits be consistent.

3. CONSISTENCY OF THE LIMITS

We shall start from two conditions.

1) At the critical point in the thermodynamic limit the correlation volume is proportional to the volume of the system, i.e.,

$$r_c^{*3}/V \propto N^0 \propto V^0. \tag{3.1}$$

We shall indicate renormalized quantities by an asterisk.

2) At the critical point the renormalized Ginzburg parameter remains finite in the thermodynamic limit, i.e.,

$$\mathrm{Gi}^* \propto N^0. \tag{3.2}$$

Here we shall assume first that the relation between the renormalized parameters r_c^* and χ^* retains the previous form (2.12), i.e.,

$$r_c^* \sim (g\chi^*)^{1/2}. \tag{3.3}$$

In the following we shall consider the more general case when the equality $\xi = 0$ entailed by the relation (3.3) does not hold.

The first condition cited permits us to determine one of the critical indices. Indeed, according to the requirement (3.1) we can write the following relation for dimensionless quantities:

$$nr_c^{*3} \sim (n\tilde{r}_c^3)^{2\nu} \propto V. \tag{3.4}$$

The convenience of this way of introducing the index ν will become clear from the following discussion.

Using for $\tilde{r}_c$ the relation (2.22) and equating the powers of N in the left- and right-hand sides, we find that

$$\nu = 2/3 \tag{3.5}$$

and, consequently,

$$r_c^* \sim n^{1/9}\tilde{r}^{4/3}. \tag{3.6}$$

We now use the second of the above conditions. For this we first represent the parameter Gi [cf. (2.14)] in the form of a product of three dimensionless parameters:

$$\mathrm{Gi} \sim \frac{1}{nr_c^3}\frac{\chi(Db)^{1/2}}{\eta^2(D/b)^{-1/2}}. \tag{3.7}$$

We then go over to the quantities $\tilde{r}_c$, $\tilde{\chi}$, and $\langle x^2\rangle$ [cf. (2.16)] and define the renormalized Ginzburg parameter by the expression

$$\mathrm{Gi}^* \sim \left(\frac{1}{n\tilde{r}_c^3}\right)^{2\nu}\frac{(\chi(Db)^{1/2})^{\gamma}}{(\langle x^2\rangle/(D/b)^{1/2})^{2\beta}}. \tag{3.8}$$

Here we have introduced two new critical indices γ and β.

In accordance with the condition that the parameter

Gi* be finite at the critical point in the thermodynamic limit, we find one more relation between the power exponents:

$$3\nu=\gamma+2\beta. \tag{3.9}$$

For $d=3$ this coincides with the second equality of (1.3) and, consequently, corresponds to the scale-invariance condition.

Finally, from the equality (3.3) follows one more equality

$$2\nu=\gamma. \tag{3.10}$$

From the three relations (3.5, 9, 10) we obtain for the three indices β, γ, and ν the values (1.4); therefore, (under the condition that the dimensionalities of all the characteristics are conserved in the renormalization) the renormalized quantities r_c^*, χ^*, and $\langle x^2\rangle^*$ are connected with $\tilde{r}_c$, $\tilde{\chi}$, and $\langle x^2\rangle$ by the relations

$$r_c^*\sim n^{1/6}\tilde{r}_c^{4/3},\quad \chi^*\sim(Db)^{1/6}(\tilde{\chi})^{4/3},\quad \langle x^2\rangle^*\sim(D/b)^{1/6}\langle x^2\rangle^{2/3}. \tag{3.11}$$

Thus, we have obtained the index values corresponding to zero values of the small indices. In the more general case the relation (3.9) remains unchanged, but in place of (3.5) and (3.10) we have the more general relations

$$3\nu=2-\alpha,\quad (2-\xi)\nu=\gamma. \tag{3.12}$$

To obtain, e.g., the second of these equalities, it is necessary to replace the condition 2) by the more general condition that the spatial correlations at the critical point be finite for all values of r in the region $r<r_c$. For this, in (2.11) we replace

$$\frac{V\langle(\delta\eta)^2\rangle}{r_c^2 r}=\frac{nD\chi}{r_c^2 r}\to\frac{nD\chi^*}{(r_c^*)^{2-\xi}r^{1+\xi}}. \tag{3.13}$$

Here we have introduced the new index ξ, characterizing the deviation from a $1/r$ dependence. We then put $\chi^*\sim\tilde{\chi}^\gamma$ and $r_c^*\sim(\tilde{r}_c)^{2\nu}$ and take into account the dependence of (2.22) on N. After this, from the condition that the correlations at the critical point be finite for all r, we arrive at the second relation (3.12). It replaces the previous relation (3.10). Naturally, for $\xi=0$ the two equalities coincide.

Since for nonzero values of α and ξ the critical indices depend on the details of the model—in particular on the dimensionality of the "spin," the qualitative analysis performed above proves to be inadequate in this case.

The relations obtained will be considered now for different regions of values of the parameters.

4. THE CRITICAL POINT

The region of temperature values in which the formulas (3.11) are valid can be divided into two subregions. The first of these includes the critical point itself and its immediate neighborhood, in which the parameter ε_N is not small. At the critical point, using the formulas (2.21) and (2.22) we obtain from (3.11) the following expressions:

$$r_c^*\sim n^{1/6}\left(\frac{g^2N}{Db}\right)^{1/3},\quad \chi^*\sim\frac{N^{2/3}}{(Db)^{1/2}},\quad \langle x^2\rangle\sim\left(\frac{D}{b}\right)^{1/2}N^{-1/3}. \tag{4.1}$$

The corresponding formula for the Ginzburg parameter Gi* has the form

$$\mathrm{Gi}^*\sim Db/g^2n^{4/3}. \tag{4.2}$$

Here, as we see, there is no dependence on N. Thus, the expression (4.2) defines a new dimensionless parameter characterizing the values of the correlations at the critical point.

5. THE REGION OF THE SCALE INVARIANCE

We shall consider the region at a distance from the critical point such that the parameter

$$\varepsilon_N=Db/Na^2\ll 1\quad(\tau^2\gg Db/Na_0^2). \tag{5.1}$$

However, the parameter Gi is not small.

According to (2.20), in the zeroth approximation in the parameter ε_N the quantity $\langle x^2\rangle=\eta^2$ and, consequently, the formulas for $\tilde{\chi}$ and $\tilde{r}_c$ coincide with the corresponding expressions of the Landau theory, i.e., with the formulas (2.9) and (2.12). Thus, for $a<0$,

$$\tilde{\chi}=\chi=\frac{1}{2a_0|\tau|},\quad \tilde{r}_c=r_c=\left(\frac{g}{2a_0|\tau|}\right)^{1/2},\quad \langle E\rangle=\eta^2=\frac{|a|}{b} \tag{5.2}$$

and, consequently, the expressions (3.11) take the form

$$\chi^*\sim\frac{(Db)^{1/6}}{(2a_0)^{4/3}}|\tau|^{-4/3},\quad \eta^*\sim\left(\frac{a_0}{b}\right)^{1/3}\left(\frac{D}{b}\right)^{1/12}|\tau|^{1/3},$$
$$r_c^*\sim n^{1/6}\left(\frac{g}{2a_0}\right)^{2/3}|\tau|^{-2/3}. \tag{5.3}$$

Thus, the power dependences (1.1) hold, with the exponents (1.4).

For the region $a>0$, instead of (5.2) we must use the expressions

$$\tilde{\chi}=1/a,\quad \tilde{r}_c=(g/a)^{1/2},\quad \langle x^2\rangle=D/Na. \tag{5.4}$$

These follow from the formulas (2.16) and (2.20) for $a>0, b=0$. Then the formulas (3.11) take the form

$$\chi^*\propto\tau^{-4/3},\quad r_c^*\propto\tau^{-2/3}. \tag{5.5}$$

We see that the values of the critical indices above and below the critical point are the same. This result corresponds to general conclusions of the theory of phase transitions.

The Ginzburg parameter Gi* for the subregion under consideration, as follows from the formulas (3.8) and (5.2), is again determined by the expression (4.2).

6. CROSSOVER TO THE RESULTS OF THE LANDAU THEORY

With increase of the temperature difference $|T-T_c|$ the next region is the crossover region between the region of scale invariance and the region in which the Landau theory is applicable.

Since the Landau theory corresponds to the self-consistent field approximation, in which the correlations are assumed to be negligibly small, for the parameter characterizing the crossover region it is natural to use a parameter that is related in some way to the Ginzburg parameter. We shall denote it by ε_W (the Wilson parameter) and define it as follows:

$$\varepsilon_W=\mathrm{Gi}/(1+\mathrm{Gi}). \tag{6.1}$$

Since the parameter Gi depends on $|\tau|$ and increases like $|\tau|^{-1/2}$ as the critical point is approached, the parameter ε_W varies within the limits

$$1 \geqslant \varepsilon_W \geqslant 0. \tag{6.2}$$

Large values of Gi (the region of scale invariance) correspond to values of ε_W close to unity, and small values of Gi correspond to small values of ε_W. In the latter case, as is well known, the Landau theory[1,2] is valid.

In the calculations of critical indices by the ε-expansion method the index values calculated in a space of four dimensions ($d=4$) are taken as the starting point. They coincide with the critical indices of the Landau theory. The required indices are calculated in a space with $d=4-\varepsilon_W$ dimensions, under the condition $\varepsilon_W \ll 1$, and are represented in the form of series in ε_W. Thus, the zeroth approximation corresponds to the Landau theory.

In the final results the parameter ε_W is put equal to unity. A formal transition to ordinary three-dimensional space is achieved in this way, since $d=4-\varepsilon_W=3$ when $\varepsilon_W=1$.

On the basis of the above account we shall try to give a qualitative physical idea of the crossover from the region of scale invariance to the region of applicability of the Landau theory.

We note first of all that in four-dimensional space the correlation volume must be defined as r_c^4. Then, taking (2.22) into account, we obtain

$$\frac{r_c^4}{V_{d=4}} \sim \frac{N}{V_{d=4}} \frac{g^2}{Db} \tag{6.3}$$

and, consequently, this ratio is finite in the thermodynamic limit. For this reason, in four-dimensional space there is no reason to distinguish a region of scale invariance. This circumstance has been known for a long time. It cannot yet serve as an adequate foundation for the Wilson ε-expansion, but it suggests a way of finding the relationship between the critical indices ν, β, and γ in three-dimensional space and the corresponding critical indices ν_L, β_L, and γ_L of the Landau theory [cf. (1.5)]. This consists in replacing the quantities d by $4-\varepsilon_W$ in the formulas (1.3) (with $\alpha=0$). Then the critical indices ν, γ, and β can be represented in the form

$$\nu = \frac{2}{4-\varepsilon_W}, \quad \gamma = \frac{4}{4-\varepsilon_W}, \quad \beta = \frac{2-\varepsilon_W}{4-\varepsilon_W}. \tag{6.4}$$

With $\varepsilon_W=1$ they coincide with the values of the critical indices for the region of scale invariance [cf. (1.4)], and with $\varepsilon_W=0$ they coincide with the values of the critical indices of the Landau theory. Intermediate values of the indices correspond to the crossover region.

Naturally, the formulas (6.4) give only a very simple qualitative idea of the relationship between the critical indices. More-exact results can be obtained only by solving the initial equation (2.1).

7. CORRELATION TIMES AND SPECTRAL WIDTHS

We shall show that the results described lead to expressions for the correlation times of the order-parameter fluctuations $\delta\eta_{\mathbf{k}=0}$ [cf. (2.8)] and the quantity $x(t) = \int x(\mathbf{R}, t)d\mathbf{R}/V$. We shall denote these by τ_η and τ_x, respectively. The meaning of the quantity τ_x requires clarification.

The definition used above for the order parameter was based on the introduction of quasi-averages and, consequently, assumes that the choice of the positive or negative direction along the given axis (we recall that we are considering only one component of the polarization vector) is determined by an external field.

Naturally, a phase transition in the system is also possible in the absence of an external field. Then, in an ensemble of identical systems, one direction or the other will be chosen with equal probability, and therefore the average value $\langle x \rangle = 0$. For this reason the presence of a phase transition can be judged from e.g., the character of the behavior of the correlation $\langle x(t)x(t-\tau)\rangle$.

In fact, at temperatures $T>T_c$ the correlation time τ_x is finite for all values of N, including $N\to\infty$. On the other hand, at temperatures $T<T_c$ the correlation time should depend on N in such a way that $\tau_x\to\infty$ when the thermodynamic limit is taken. This behavior of the correlation time as we pass through the critical point serves as an indication of the presence of a phase transition.

We shall denote the corresponding spectral widths by $\Delta\omega_\eta$ and $\Delta\omega_x$. It follows from what has been said that for $T<T_c$ the spectral width $\Delta\omega_x$ tends to zero in the thermodynamic limit. A state with an infinitesimally narrow spectrum implies the existence of an order parameter in the system.

Naturally, in the absence of an external field there is degeneracy with respect to the possible values of the direction of the order parameter. There is in this a certain analogy with the process of passing across the generation threshold, e.g., in lasers, in which, in the absence of an external field, there is degeneracy with respect to the phase.

We shall start by considering the quantities τ_η and $\Delta\omega_\eta$. For this, using the linear equation for the fluctuation $\delta\eta$, we find expressions for the spatial-temporal and spatial-spectral densities:

$$(\delta\eta\delta\eta)_{\omega,\mathbf{k}} = 2\frac{D}{N}\frac{1}{\omega^2+(a+3b\eta^2+g\mathbf{k}^2)^2}, \qquad (\delta\eta\delta\eta)_{\mathbf{k}} = \frac{D}{N}\frac{1}{a+3b\eta^2+g\mathbf{k}^2}. \tag{7.1}$$

It can be seen from these formulas that the quantities $\Delta\omega_\eta$ and τ_η can be expressed in terms of the static susceptibility $\chi(0, \mathbf{k})$ [cf. (2.9)]:

$$\tau_\eta(\mathbf{k}) = 1/\Delta\omega_\eta = \chi(0, \mathbf{k}). \tag{7.2}$$

Hence, using the second formula (3.11), we find the correlation time of the order-parameter fluctuations $\delta\eta_{\mathbf{k}=0}$ with allowance for the renormalization:

$$\tau_\eta \sim \chi^* \sim (Db)^{1/6}(\tilde{\chi})^{4/3}. \tag{7.3}$$

The function $\tilde{\chi}$ is determined by the expression (2.16).

From this, using the formulas (4.1) and (5.3), we find

$$\tau_\eta \sim \frac{N^{2/3}}{(Db)^{1/3}} \text{ for } a=0, \quad \tau_\eta \sim \frac{(Db)^{1/3}}{(2a_0)^{4/3}} |\tau|^{-4/3}. \quad (7.4)$$

We see that at the critical point the correlation time $\tau_\eta \propto N^{2/3}$. In the region of scale invariance the time τ_η decreases like $|\tau|^{-4/3}$ with increase of $|\tau|$, and in the region of applicability of the Landau theory decreases like $|\tau|^{-1}$ (Ref. 9).

To find the quantities τ_x and $\Delta\omega_x$ we use the equation for the function $\langle x(t)x(t-\tau)\rangle$. In the same approximation in which Eq. (2.20) was obtained, from the initial equation (2.1) we find

$$\frac{d}{d\tau}\langle x(t)x(t-\tau)\rangle + (a+b\langle x^2\rangle)\langle x(t)x(t-\tau)\rangle = 0. \quad (7.5)$$

As the initial solution (with $\tau=0$) we must use the solution of Eq. (2.20).

Thus, the required correlation time (as yet without renormalization, i.e., without the replacement $\langle x^2\rangle \to \langle x^2\rangle^*$) is determined by the expression

$$\tau_x = \frac{1}{\Delta\omega_x} = \frac{1}{a+b\langle x^2\rangle} = \frac{\langle x^2\rangle}{D} N. \quad (7.6)$$

Here we have used Eq. (2.20). By virtue of Eq. (2.20) the two definitions in (7.6) are equivalent. However, the second definition is the more convenient, since it contains the explicit dependence on the strength D/N of the random source.

In the region of applicability of the Landau theory, when $\langle x^2\rangle = |a|/b$, it follows from (7.6) that $\tau_x \propto N$. Thus, when we take the thermodynamic limit the correlation time tends to infinity. Correspondingly, the spectral width tends to zero. This indicates the presence of the phase transition.

To describe the behavior of τ_x in the critical region we replace $\langle x^2\rangle \to \langle x^2\rangle^*$. As a result,

$$\tau_x \sim \langle x^2\rangle^* N/D. \quad (7.7)$$

The quantity $\langle x^2\rangle^*$ is determined by the last of the formulas (3.11). At the critical point, according to (4.1),

$$\tau_x \sim N^{2/3}(Db)^{-1/3}, \quad (7.8)$$

i.e., it is of the same order as τ_η [cf. (7.4)].

Thus, the use of the function $\langle x(t)x(t-\tau)\rangle$ permits us to describe the phase transition without defining the order parameter as a quasi-average [cf. (2.6)]. It can be shown that the susceptibility characterizing the fluctuation $\delta x^2 = x^2 - \langle x^2\rangle$ behaves qualitatively similarly to the susceptibility χ. There is an analogous correspondence for the dependences of the correlation times τ_{x^2} and τ_η on the values of N and T.

This account gives a qualitative idea of the behavior of various characteristics of the system in the critical region in the whole range of temperatures and for a finite number of particles. A more detailed description requires a more detailed study of the solutions of the initial equation (2.1). As yet it is still unclear whether the methods used at present to calculate the critical indices in the region of scale invariance can be applied to solve the more general problem of the calculation of the fluctuations in the entire critical region. At present, apparently, such calculations have been carried out only for the two-dimensional Ising model.

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