Short-wave asymptote of the turbulence spectrum

G. A. Kuz'min[1)] and A. Z. Patashinskiĭ

Institute of Nuclear Physics, Siberian Division of the USSR Academy of Sciences
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The turbulence spectrum decreases rapidly for scales smaller than the dissipation scale η. It is shown that the nature of this decrease is determined by the strong interaction between pulsations of different scales and by the cascade process that arises. The Green-function and diagram techniques are used in the calculations. It is shown that the response and vertex functions coincide with the bare functions in the region $\eta k \gg 1$. An equation for the spectral tensor is obtained, and the form of the solution is found up to a universal constant. The truncation of the series allows the determination for the constant of an approximate value that changes little when the next term in the series is taken into account.

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1. INTRODUCTION

A turbulent fluid is an example of a system with many degrees of freedom for which the inflow of energy without and the dissipation of that energy occur at the opposite ends of the spectrum. For the stationary case, if the dissipation scale η is small compared to the excitation scale, then the equilibrium in the small-scale region is essentially determined by the energy flux. In Kolmogorov's theory[1] the spectral density, $F(k)$, of the energy has the form

$$F(k)=\langle|\mathbf{u}(\mathbf{k})|^2\rangle\sim k^{-11/3}\psi(k\eta),$$

where $\mathbf{u}(\mathbf{k})$ is the Fourier harmonic of the velocity $\mathbf{u}(\mathbf{x})$ and the $\mathrm{Lim}\,\psi(y\to 0)=\mathrm{const}$. Attempts to obtain the behavior of the system in the $\eta k \ll 1$ region on the basis of the equations of fluid mechanics has thus far not met with complete success because of the mathematical complexity of the situation.[2] The properties of the system in the $\eta k \gg 1$ dissipation region have been investigated by Novikov,[3] using an idea first used by Townsend[4] and Batchelor[5], and based on the assumption that the deformation of the smallest vortices by the scale η plays the major role in this region. The answer for the spectrum, obtained under the assumption that the interaction between the small-scale pulsations is insignificant, has the form

$$F(k)\sim\exp[-(\eta k)^2]. \qquad (1)$$

To verify the latter assumption, let us estimate the contribution made by the interactions between pulsations whose scales are of the same order of magnitude if the pulsation spectrum has the form (1). The Navier-Stokes equation for an incompressible fluid with viscosity ν has in the Fourier representation in terms of the space coordinates the form

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)u_i(\mathbf{k},t)=-\frac{i}{2}P_{ijl}(\mathbf{k})\int d^3q\,u_j(\mathbf{q},t)u_l(\mathbf{k}-\mathbf{q},t),$$

where

$$P_{ijl}(\mathbf{k})=k_j\Delta_{il}(\mathbf{k})+k_l\Delta_{ij}(\mathbf{k}),$$
$$\Delta_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2.$$

Let us assume that $\mathbf{u}(\mathbf{k})\sim\exp[-(\eta k)^\gamma]$, where $\gamma>1$. The contribution of the region $q\sim\eta^{-1}\ll k$ is

$$P(\mathbf{k})u(\eta^{-1})\eta^{-3}u(k)\sim\exp[-(\eta k)^\gamma]$$

and coincides in order of magnitude with the left member of the equation. The contribution of the region where $q\sim|\mathbf{k}-\mathbf{q}|\sim k/2$ is

$$P(\mathbf{k})u^2(k/2)k^3\sim\exp[-(\eta k)^\gamma 2^{1-\gamma}]$$

and is large compared to the contribution of the region $q\eta\sim 1$.

Thus, the nonlinear interactions between pulsations whose scales are of the same order magnitude play an important role in the $k\eta\gg 1$ range. In order to take these interactions into account more accurately, we use in the present paper quantum-field-theory tools of the type developed by Wyld.[6] This method has been used by Kuz'min and Patashinskiĭ[7,8] to compute the exponential factor in the spectrum for $\eta k\to\infty$. They show that, although for $k\eta\gg 1$ the amplitudes of the pulsations are exponentially small, a strong-coupling regime is realized in this range, i.e., there exists an infinite subsequence of diagrams whose orders of magnitude coincide in the exponential approximation.

2. THE DIAGRAM EQUATIONS

The diagram technique for the theory of turbulence has been expounded in a number of papers.[6–11] In our case the form of the equations for the theory's quantities that is similar to that of the unitarity conditions for the S matrix of quantum theory is convenient.[8] For the derivation of these equations, we use the method of partial summation of the diagrams, it being the simplest and most graphic. The expansion of the spectral tensor $F_{ij}(k,t-t')$ has the following form[6,2];

(2)

Here and below the spectral tensor is represented by a heavy wavy line, while the Green tensor $G_{ij}(k,t-t')$ is represented by a heavy arrow. The corresponding bare quantities $F_{ij}^{(0)}$, $G_{ij}^{(0)}$ are represented by thin lines. The bare vertex $-\frac{1}{2}iP_{ijl}(k)$ is associated with a point. The spectral tensor of the external exciting force is denoted by Φ_{ij}.

Let us introduce complete vertices, defined as the sums of all possible diagrams each with one exit and a certain number of entrances. We shall represent them by hatched polygons. They have a simple physical meaning. Let us make a small nonrandom correction, $h_i(\mathbf{k},t)$, to the external exciting force $f_i(\mathbf{k},t)$, expand the velocity response in a series in powers of $\mathbf{h}$, and aver-

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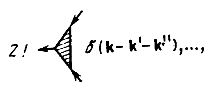
age the result:

$$\langle \delta u_i(\mathbf{k},t)\rangle = \int \left\langle \frac{\delta u_i(\mathbf{k},t)}{\delta h_j(\mathbf{k}',t')}\bigg|_{\mathbf{h}=0} \right\rangle h_j(\mathbf{k}',t')\,d\mathbf{k}'\,dt'$$
$$+\int \left\langle \frac{\delta^2 u_i(\mathbf{k},t)}{\delta h_j(\mathbf{k}',t')\,\delta h_m(\mathbf{k}'',t'')}\bigg|_{\mathbf{h}=0} \right\rangle h_j(\mathbf{k}',t')\,h_m(\mathbf{k}'',t'')\,d\mathbf{k}'\,d\mathbf{k}''\,dt'\,dt''+\ldots$$

Let us stubstitute into the variational derivatives $u_i(\mathbf{k},t)$ in the form of a functional expansion in terms of $\mathbf{f}$.[6] Term-by-term averaging leads to the relations

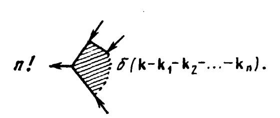
$$\left\langle \frac{\delta u_i(\mathbf{k},t)}{\delta h_j(\mathbf{k}',t')}\bigg|_{\mathbf{h}=0} \right\rangle = \leftarrow \delta(\mathbf{k}-\mathbf{k}'),$$

$$\left\langle \frac{\delta^2 u_i(\mathbf{k},t)}{\delta h_j(\mathbf{k}',t')\,\delta h_m(\mathbf{k}'',t'')}\bigg|_{\mathbf{h}=0} \right\rangle$$

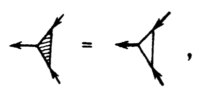


(3)

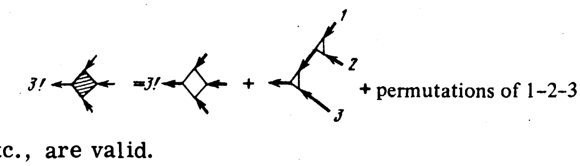
$$\left\langle \frac{\delta^n u_i(\mathbf{k},t)}{\delta h_{i_1}(\mathbf{k}_1,t_1)\ldots\delta h_{i_n}(\mathbf{k}_n,t_n)}\bigg|_{\mathbf{h}=0} \right\rangle$$



The Green tensor describes the averaged velocity response in the linear approximation in $\mathbf{h}(\mathbf{k},t)$; the tiangle, in the second-order approximation, etc. Let us introduce nodal vertices—sums of diagrams each of which cannot be cut by a single line. We shall represent them by unhatched polygons. It is easy to see that the relations

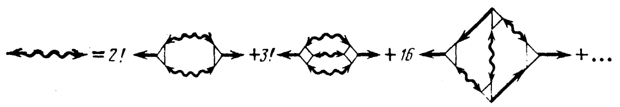


(4)



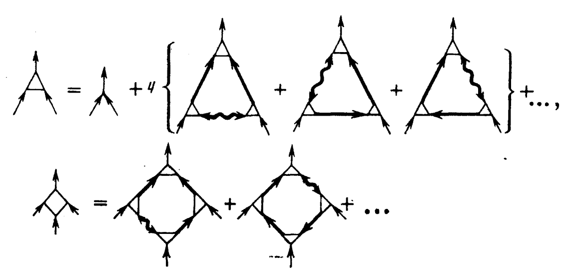
etc., are valid.

The partial summation of the diagrams in (2) leads to the equation



(5)

The term with Φ_{ij} has been dropped in this equation, since it is assumed below that the spectrum of the external force is bounded from above, and the system of equations is studied in the region of wave numbers much higher than the reciprocal of the principal turbulence scale, which, in the present paper, will be assumed to coincide with the Kolmogorov scale η. Similar equations can be written down for the nodal vertices:



(6)

etc.

3. ANALYSIS OF THE EQUATIONS

Let us consider Eqs. (5) and (6) in the region where the wave numbers of the external lines are high in comparison with the quantity η^{-1}. These equations contain the spectral tensor and the nodal vertices, which describe the damping of the external perturbation introduced into the turbulent flux. We shall, to begin with, study the behavior of the nodal vertices in the region in question. The viscosity-induced-damping time of the introduced perturbation is equal to $(\nu k^2)^{-1}$, while the time needed for its transport by the motions of the large scales over the distance k^{-1} is of the order of $(\mathbf{k}\cdot\mathbf{u}_\eta)^{-1}$. If the wave number is sufficiently high, then over a period of time equal to the viscous-damping time of the perturbations a region of dimension k^{-1} can be considered to be at rest relative to the principal turbulence scale. Therefore, the translational interactions, which significantly complicates the analysis of the problem in the inertial range,[12-14] are unimportant in the energy-dissipation range. At the same time, for $\eta k \gg 1$, the amplitudes of the turbulent pulsations are exponentially small, and cannot have any effect on the rate of damping of the introduced perturbation. In other words, in the energy-dissipation range the nonlinearity should be taken into account only so far as it is the only energy source. When $\eta k \gg 1$, the external perturbation attenuates regardless of the motion of the fluid. In the linear approximation the resonance is described by the bare Green tensor.

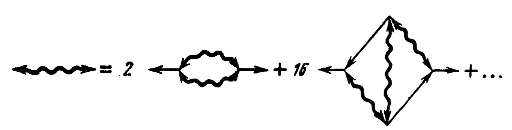
Let us show that, in the limit as $k\eta \to \infty$, the bare vertex does not become renormalized. We shall seek the solution to the equation for the spectral tensor for $k\eta \gg 1$ in the form

$$F_{ij}(\mathbf{k}) = \Delta_{ij}(\mathbf{k})\frac{\psi(k)}{4\pi k^2}\exp[-(\eta k)^\gamma], \qquad (7)$$

where $\gamma > 1$ and $\psi(k)$ is a function that varies, when $k\eta \gg 1$, not more rapidly than a power function. On account of the rapid decrease of the spectrum when $\eta k > 1$, the dominant contribution to the integrals of Eqs. (6) is made by the region where the wave numbers of the F lines are of the order of η^{-1}. The parameter of the series expansion for the nodal vertices is thus the quantity

$$u_\eta^2 k^2/\nu^2 k^4 \sim (\eta k)^{-2}.$$

In the $\eta k \to \infty$ limit the expansion parameter is small, and the bare vertex is not renormalized. The remaining nodal vertices are small in the same parameter. Equation (5) for the spectral tensor assumes the form



(8)

The arguments that led to the truncation of the series for the nodal vertices are inapplicable here, since the wave numbers of the F lines in each diagram are connected by the relation

$$\sum_{i=1}^{n}\mathbf{q}_i = \mathbf{k}$$

and cannot all be small at the same time.

Let us, to begin with, set $t=t'$ in the spectral tensor F on the left-hand side of the equation. The exponential factor

$$\exp\left[-\sum_{i=1}^{n}(\eta q_i)^{\gamma}\right];\quad \sum_{i=1}^{n}\mathbf{q}_i=\mathbf{k}$$

enters into the integrand of the diagram containing n wavy lines. Furthermore, there is a power factor that arises from the vertices of the bare Green functions and the integrations over the time differences. The dominant contribution to the integral is made by the region where the index of the exponential function has its maximum value. For $\gamma>1$, the maximum of the index of the exponential function lies in the region where $\mathbf{q}_1=\mathbf{q}_2=\cdots=\mathbf{q}_n=\mathbf{k}/n$, and the diagram is proportional to $\exp[-(\eta k)^{\gamma}n^{1-\gamma}]$. Thus, all the diagrams on the right-hand side of the equation are exponentially large as compared to the spectral tensor (7), and Eq. (8) for $\gamma>1$ cannot be satisfied. For $\gamma<1$, the index of the exponential function is a maximum when the wave numbers of all the F lines, except one, are small. This corresponds to the case in which the dominant role is played by the interactions of the short-wave pulsations directly with the pulsations of the principal scale. It has, however, been shown[3-5] with the aid of other methods that such interactions lead to a solution with $\gamma=2$, and not with $\gamma<1$. Therefore, the only γ value that is not at variance with the equations is $\gamma=1$.

Let us now study the rate of damping of the time correlations in the energy-dissipation range. Let us assume that the attenuation of the correlations in time is also exponential:

$$F(k,\tau)=F(k)\exp[-\alpha k^{\mu}\tau]\sim\psi(k)\exp[-\eta k-\alpha k^{\mu}\tau]. \tag{9}$$

Substituting (9) into Eq. (8), we again find that both sides of the equation have the same asymptotic form for $k\to\infty$ only when $\mu=1$.

Thus, although the interactions in the energy-dissipation range occur in cascade fashion, a significant randomization does not occur here, and the lifetime of the correlations on the k^{-1} scale is long compared to the lifetime of the introduced nonrandom perturbation.

Let us compute approximately the preexponential factor $\psi(k)$ with the aid of Eq. (8), on the right-hand side of which we retain only the first diagram. Substituting $F_{ij}(k,\tau)=\Delta_{ij}(k)E(k,\tau)/(4\pi k^2)$ and $G_{ij}(\mathbf{k},\tau)=\delta_{ij}\exp(-\nu k^2\tau)$ into Eq. (8), and computing the trace, we obtain the equation

$$E(k,t-t')=\frac{1}{4\pi}\int d^3q\,a(\mathbf{k},\mathbf{p},\mathbf{q})\int_{-\infty}^{t}dt_1\int_{-\infty}^{t'}dt_2\exp[-\nu k^2(t+t'-t_1-t_2)]\times E(q,t_1-t_2)E(p,t_1-t_2)[pq/k^2]^{-2}, \tag{10}$$

where

$$\mathbf{p}=\mathbf{k}-\mathbf{q};\quad a(\mathbf{k},\mathbf{p},\mathbf{q})=\frac{1}{4k^2}P_{ijl}(\mathbf{k})P_{i\alpha\beta}(\mathbf{k})\Delta_{j\alpha}(\mathbf{p})\Delta_{l\beta}(\mathbf{q}).$$

Since the damping time of the correlations is long compared to $(\nu k^2)^{-1}$, we can set the time differences in the E functions in (10) equal to zero and perform the integration over t_1 and t_2. Equation (10) assumes the form

$$E(k)=\frac{1}{4\pi\nu^2k}\int\frac{d^3q}{k^3}\frac{E(p)E(q)}{(pq/k^2)^2}a(\mathbf{k},\mathbf{p},\mathbf{q}). \tag{11}$$

Substituting into this equation $E(k)=\psi(k)\exp(-\eta k)$, we have

$$\psi(k)=\frac{1}{4\pi\nu^2k}\int\frac{d^3q}{k^3}a(\mathbf{k},\mathbf{p},\mathbf{q})\exp\left[-\eta k\left(\frac{p+q}{k}-1\right)\right]\frac{\psi(p)\psi(q)}{(pq/k^2)^2}. \tag{12}$$

The exponent of the exponential function contains the large factor ηk, and the dominant contribution to the integral is made by the region where $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{k}$ are almost collinear. The effective width of the integration domain at right angles to the vector $\mathbf{k}$ is of the order of $(k/\eta)^{1/2}$. In this region $a(\mathbf{k},\mathbf{p},\mathbf{q})\sim(\eta k)^{-1}$. Assuming that the dominant contribution is made by the region where $q\sim k/2$, we obtain that

$$\psi(k)\sim\psi^2(k/2)/[\nu^2k(\eta k)^2].$$

This indicates that (12) has the power solution

$$\psi(k)=C\nu^2k(\eta k)^2, \tag{13}$$

where C is some constant. The possibility of such a solution was first pointed out by Kraichnan.[15] Let us show that such a solution does indeed exist, and let us compute the quantity C.

It is convenient to perform the integration in a coordinate system in which one of the axes is parallel to the vector $\mathbf{k}$:

$$d^3q=dq_{\parallel}d^2q_{\perp}=2\pi q_{\perp}dq_{\perp}dq_{\parallel}.$$

Let us introduce the dimensionless integration variables $q_{\parallel}=sk$ and $q_{\perp}=wk$. Then $d^3q=2\pi k^3 w\,dw\,ds$. Let us perform the integration with the aid of the Laplace method.[16] For this purpose, let us expand the factor $a(\mathbf{k},\mathbf{p},\mathbf{q})$ and the index of the exponential function in powers of the ratios of the transverse components of the vectors $\mathbf{p}$ and $\mathbf{q}$ to the longitudinal components, and limit ourselves to the lowest-order terms:

$$a(\mathbf{k},\mathbf{p},\mathbf{q})\approx\frac{w^2}{2}\left[\frac{1}{s^2}+\frac{1}{(1-s)^2}-\frac{1}{s(1-s)}\right], \tag{14}$$

$$\eta(p+q)\approx\eta k\left[1+\frac{w^2}{2s(1-s)}\right]. \tag{15}$$

Substituting (14) and (15) into (12), and integrating over w, we obtain an equation for C:

$$C=C^2\int_0^1 ds\,s^3(1-s)^3\left[\frac{1}{s^2}+\frac{1}{(1-s)^2}-\frac{1}{s(1-s)}\right]=\frac{C^2}{30}, \tag{16}$$

whence

$$E(k)=30\nu^2k(\eta k)^2\exp(-\eta k). \tag{17}$$

The contribution of the ends of the integration range to the integral (16) is small; therefore, the expansion of $a(\mathbf{k},\mathbf{p},\mathbf{q})$ and the index of the exponential function in powers of w/s and $w/(1-s)$ is admissible.

Thus, the equation of the lowest approximation for the spectrum has the analytical asymptotic solution (17). Let us discuss the situation that arises in the higher-order diagrams. We have already noted above that each diagram contains the same exponential factor which cancels out on both sides of the equation. As in the case of the diagrams of the lowest approximation, the dimension of the domain of integration over the wave numbers along **k** is of the order of k. The width of the integration range in the transverse direction is of the order of $(k/\eta)^{1/2}$. The angular factor $a(\mathbf{k}, \mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) \sim 1/(k\eta)^{n-1}$. The quantity $\psi(k)/[\nu^2 k(\eta k)^2] \sim \text{const}$ thus serves as the expansion parameter. Consequently, the smallness in the higher-order diagrams can, if it exists at all, only be numerical.

In order to estimate how far the solution (17) is accurate, it is necessary to compute the higher-order diagrams in the expansion (8). Let us compute the spectral function with allowance for the diagram of the next order in Eq. (8). There appears on the right-hand side of Eq. (12) the additional term

$$D=\frac{1}{16\pi^2\nu^4k^2}\int\frac{d^3pd^3q\,A(\mathbf{k},\mathbf{p},\mathbf{q},\mathbf{z})\,\psi(q)\,\psi(p)\,\psi(z)}{k^6((\mathbf{p}+\mathbf{q})/k)^2((\mathbf{q}+\mathbf{z})/k)^2(pqz/k^3)^2}\times\exp\left[-\eta k\left(\frac{p+q+z}{k}-1\right)\right], \qquad (18)$$

$$A(\mathbf{k},\mathbf{p},\mathbf{q},\mathbf{z})=\frac{1}{2k^4}P_{i\alpha\beta}(\mathbf{k})P_{i\mu\nu}(\mathbf{k})P_{\alpha\gamma\delta}(\mathbf{p}+\mathbf{q})P_{\nu\sigma\rho}(\mathbf{q}+\mathbf{z})\Delta_{\beta\rho}(\mathbf{z})\Delta_{\delta\mu}(\mathbf{p})\Delta_{\gamma\sigma}(\mathbf{q}),$$

$$\mathbf{p}+\mathbf{q}+\mathbf{z}=\mathbf{k}.$$

The dominant contribution to the integral in (18) is made by the region where **k**, **p**, **q**, and **z** are almost collinear. Let us substitute (13) into (18) and evaluate the integral over the transverse components of the vectors **p** and **q** with the aid of the Laplace method (see the Appendix). As a result, we obtain in place of (16) the equation

$$1=C/30+2IC^2, \qquad (19)$$

where

$$I=\int_0^1 ds\int_0^{1-s} dw\,\frac{s^2w^2(1-s-w)^2}{(1-s)^2(1-w)^2(s+w)^2}\left[5s^2-8s-1+3(s+w)(1-w)+\frac{2s(1-s-s^2)}{(s+w)(1-w)}\right]=\frac{1641}{96}\pi^2-168-\frac{51}{72}\approx 2.07\cdot 10^{-4}. \qquad (20)$$

The region where s, w, or $1-s-w$ is small makes a small contribution to the integral I; therefore, the expansion in powers of the ratios of the transverse components of the vectors **p**, **q**, and **z** to the longitudinal components, which is performed in the Appendix, is justified. The value of the constant C obtained with the aid of Eq. (19) is equal to $C \approx 23$, which is not too different from the value $C = 30$ obtained above.

Thus, the relative error that results from the neglect of the second term in Eq. (8) is small, it being ~0.2. One may hope that the series of the theory are asymptotic series and that the found value of C is close to the true value.

APPENDIX

The substitution of (13) into (18) yields

$$D=\frac{C^3\nu^2k(\eta k)^6}{16\pi^2}\int\frac{d^3pd^3q}{k^6}A(\mathbf{k},\mathbf{p},\mathbf{q},\mathbf{z})\,\frac{pqz}{k^3}\,\frac{\exp[-\eta k((p+q+z)/k-1)]}{((\mathbf{p}+\mathbf{q})/k)^2((\mathbf{q}+\mathbf{z})/k)^2}.$$

The index of the exponential function contains the large factor $\eta k \gg 1$, and, outside the region where $p+q+z \approx k$ (i.e., where **k**, **p**, **q**, and **z** are almost collinear), the integrand is exponentially small. Therefore, the integration over the transverse components of the vectors **p** and **q** can be performed, using the Laplace method.

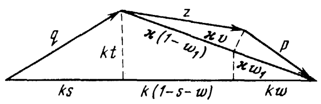


FIG. 1.

Let us first perform the integration over **p** in the plane perpendicular to the vector $\mathscr{H}=\mathbf{p}+\mathbf{z}$ (see Fig. 1) and then integrate the expression over **q** in the plane perpendicular to **k**. Let us introduce the dimensionless integration variables

$$v=p_\perp/\varkappa, \quad w_1=p_\parallel/\varkappa, \quad t=q_\perp/k, \quad s=q_\parallel/k.$$

Then

$$d^3p=2\pi p_\perp dp_\perp dp_\parallel=2\pi\varkappa^3 v\,dv\,dw_1, \quad d^3q=2\pi q_\perp dq_\perp dq_\parallel=2\pi k^3 t\,dt\,ds.$$

We can, with satisfactory accuracy, also set $\mathscr{H} \approx (1-s)k$ and $w_1 \approx w/(1-s)$. The index of the exponential function and the angular factor assume the forms

$$-\eta k\left(\frac{p+q+z}{k}-1\right)\approx-\frac{\eta k}{2}\left[\frac{(1-s)^3v^2}{w(1-s-w)}+\frac{t^2}{s(1-s)}\right],$$

$$A(\mathbf{k},\mathbf{p},\mathbf{q},\mathbf{z})\approx\frac{t^4}{(1-s)^4s^2}\left[5s^2-8s-1+3(s+w)(1-w)+\frac{2s(1+s-s^2)}{(s+w)(1-w)}\right].$$

Substituting all these expressions into the expression for D, and performing the dv and dt integrations within the limits $(0, \infty)$, we obtain

$$D=2\nu^2k(\eta k)^2C^3I,$$

where I is given by the expression (20).

1)Member of staff of the Institute of Thermal Physics, Siberian Division of the USSR Academy of Sciences.

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