Stationary gravitational solitons with axial symmetry

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An application of the method of the inverse scattering problem to the integration of the gravitational equations is described. The case considered is that of stationary axially symmetric gravitational fields. The procedure for constructing soliton solutions is carried through for all metric coefficients. Axially symmetric solutions with n solitons are considered.

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1. INTRODUCTION

In a previous paper we have shown[1] that in the case in which the metric tensor depends on only two variables the gravitational equations form a system which is integrable by the method of the inverse scattering problem. The case was examined in which one of the variables is the time and the other is spacelike; this corresponds to cosmological and wave solutions of the equations of gravitation. It was pointed out that there is no difficulty in applying this method also to the case in which both the variables on which the metric tensor depends are spacelike, which corresponds to stationary gravitational fields. One possible interpretation of this case is that of a stationary gravitational field with axial symmetry. This class of solutions is important in the theory of gravitation, since it has a clear physical meaning. In this connection it is interesting to consider the case of axially symmetric stationary fields separately and to find the construction of the corresponding soliton solutions and their physical meaning. This is the purpose of the present paper. We shall also use this case as an example to carry to completion the procedure which we described earlier[1] for constructing exact soliton solutions, and deal with one important point which was left there incomplete. We shall explain the essence of the question, first introducing the metric and the corresponding Einstein equations.

Having in view the application to the case of stationary axially symmetric gravitational fields, we write the metric in the form[1)]

$$-ds^2=f(d\rho^2+dz^2)+g_{ab}dx^a dx^b, \tag{1.1}$$

where the metric coefficients f and g_{ab} are functions of only two variables, ρ and z. We use for the coordinates the notation $(x^0, x^1, x^2, x^3)=(t, \varphi, \rho, z)$. Throughout this paper the five Latin indices a, b, c, d, f run through the values 0 and 1 and correspond to the coordinates t and φ.

It is well known that in this case (by using the remaining freedom in the choice of the coordinates ρ and z) we can, without loss of generality, impose on the two-rowed matrix g (with components g_{ab}) the following supplementary condition:

$$\det g=-\rho^2. \tag{1.2}$$

It is now easy to show that the Einstein equations (in vacuum) for the metric (1.1), (1.2) separate into two groups. The first determines the matrix g and is of the form

$$(\rho g_{,\rho}g^{-1})_{,\rho}+(\rho g_{,z}g^{-1})_{,z}=0. \tag{1.3}$$

The second group of equations determines the metric coefficient f for a given solution of Eq. (1.3) and can be written in the form

$$(\ln f)_{,\rho}=-\rho^{-1}+(4\rho)^{-1}\,\mathrm{Sp}(U^2-V^2), \tag{1.4}$$

$$(\ln f)_{,z}=(2\rho)^{-1}\,\mathrm{Sp}(UV), \tag{1.5}$$

where the two-rowed matrices U and V are defined as follows:

$$U=\rho g_{,\rho}g^{-1},\quad V=\rho g_{,z}g^{-1}. \tag{1.6}$$

It is easy to see that if instead of ρ and z we introduce the pair of complex variables $\zeta=(z+i\rho)$ and $\eta=(z-i\rho)$, then in the variables ζ and η the metric (1.1) and Eqs. (1.2)–(1.6) will be formally reduced to the same form as we studied previously.[1] For this reason all of the formal side of the method for the case considered here can be obtained[2)] from the results of our earlier paper.[1] Of these results we shall present here only the basic points which are necessary for a complete exposition, and shall not go into the details of the proofs. The details can be found in Ref. 1.

Let us now turn to the point in the research which was not brought to completion in Ref. 1. As follows from what we have said, we can apply to the integration of Eqs. (1.2)–(1.6) the method given in Ref. 1, i.e., apply the method of the inverse scattering problem to the integration of the matrix equation (1.3) and thus get the major part of the metric coefficients, g_{ab}. There then remains, however, the problem of calculating the metric coefficient f, which is given in quadratures by Eqs. (1.4) and (1.5).

In Ref. 1 it was shown by direct calculations that for the simple soliton solutions given there these quadratures can be performed completely (i.e., the integrals can be calculated explicitly), and the answer for the coefficient f can be expressed explicitly in terms of the appropriate partial or background solution of the problem and elementary functions, i.e., qualitatively in the same way as the metric components g_{ab}. This suggests that the same will be true also on the general case of an n-soliton solution. It turns out that this is indeed true, and the metric coefficient f in the general n soliton

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case, like the coefficients g_{ab}, can be calculated altogether explicitly. The analysis for this point is given in Sec. 3 of the present paper.

Finally, we point out that the question of the integrability of the equations of gravitation for the case considered has also been investigated by Maison,[2] who proved the existence of an $L-A$ pair for the Einstein equations, though in a somewhat different way from that followed in Ref. 1 and here [cf. Eqs. (2.1), (2.2)]. Harrison[3] found the Bäcklund transformation for the Ernst equation corresponding to this problem.

2. THE *n*-SOLITON SOLUTION FOR THE MATRIX *g*

Using the results of Ref. 1 (as explained in the Introduction), we can easily find the $L-A$ pair for the matrix equation (1.3) in the variables ρ and z:

$$D_1\psi=\frac{\rho V-\lambda U}{\lambda^2+\rho^2}\psi,\quad D_2\psi=\frac{\rho U+\lambda V}{\lambda^2+\rho^2}\psi, \tag{2.1}$$

where the commuting differential operators D_1 and D_2 are given by

$$D_1=\partial_z-\frac{2\lambda^2}{\lambda^2+\rho^2}\partial_\lambda,\quad D_2=\partial_\rho+\frac{2\lambda\rho}{\lambda^2+\rho^2}\partial_\lambda \tag{2.2}$$

and λ is a complex spectral parameter independent of the coordinates ρ and z. It is not hard to verify that the conditions of compatibility of the equations (2.1) for the matrix function $\psi(\lambda,\rho,z)$ are identical with the original equations (1.3) and (1.6), if we rewrite them, and also the conditions for their compatibility, in terms of the matrices U and V, in the same way as this was done previously.[1] The required matrix g is the value of the matrix $\psi(\lambda,\rho,z)$ for $\lambda=0$:

$$g(\rho,z)=\psi(0,\rho,z). \tag{2.3}$$

The procedure for integrating the equations (2.1) presumes the knowledge of some particular solution of the problem. Let g_0, U_0, V_0 be some particular solution of Eqs. (1.3) and (1.6), from which, with Eq. (2.1), the corresponding solution $\psi_0(\lambda,\rho,z)$ has been found. We then seek the solution for ψ in the form

$$\psi=\chi\psi_0 \tag{2.4}$$

and for $\chi(\lambda,\rho,z)$ we get from Eq. (2.1) the following equations:

$$\begin{aligned}D_1\chi&=\frac{\rho V-\lambda U}{\lambda^2+\rho^2}\chi-\chi\frac{\rho V_0-\lambda U_0}{\lambda^2+\rho^2},\\ D_2\chi&=\frac{\rho U+\lambda V}{\lambda^2+\rho^2}\chi-\chi\frac{\rho U_0+\lambda V_0}{\lambda^2+\rho^2}.\end{aligned} \tag{2.5}$$

Now (as before[1]) it can be shown that to assure that the matrix g is real and symmetric definite supplementary conditions have to be imposed on the solutions of Eq. (2.5). For the reality of g we have the requirements

$$\bar{\chi}(\bar{\lambda})=\chi(\lambda),\quad \bar{\psi}(\bar{\lambda})=\psi(\lambda) \tag{2.6}$$

(a bar denotes the complex conjugate), and for g to be symmetric we require

$$g=\chi(-\rho^2/\lambda)g_0\tilde{\chi}(\lambda) \tag{2.7}$$

(a tilde indicates transposition). Besides this, compatibility of Eqs. (2.7) with (2.3) requires

$$\chi(\infty)=I, \tag{2.8}$$

where I is the unit matrix (here, and often from now on, we omit the arguments ρ and z of functions for simplicity).

The soliton solutions for the matrix g correspond, as is well known, to the presence of pole singularities of the matrix $\chi(\lambda,\rho,z)$ in the complex plane of the spectral parameter λ. Let us consider the general case, in which the matrix χ has n such poles, which we assume to be simple. The matrix $\chi(\lambda,\rho,z)$ can then be represented in the form

$$\chi=I+\sum_{k=1}^{n}\frac{R_k}{\lambda-\mu_k}, \tag{2.9}$$

where the matrices R_k and the numerical functions μ_k now depend only on the variables ρ and z.

We note that in Ref. 1 an expression analogous to Eq. (2.9) was written in a form which obviously satisfies the condition (2.6) and which emphasizes the fact that complex poles (i.e., complex μ_k) of the matrix χ can exist only as conjugate pairs. Of course these requirements still hold here, but experience shows that writing χ in the form (2.9) considerably facilitates the calculations, which it is convenient to do by neglecting the conditions (2.6) and supposing (until the final form of the solutions is reached) that we have to do with n arbitrary complex poles $\lambda=\mu_k$ $(k=1,2,\ldots,n)$. After the final form of the solution is obtained it is easy to assure that the matrix g is real by imposing definite supplementary conditions on the arbitrary constants that appear in the solution. This procedure is possible with an even number of complex poles in the sum (2.9), and is of course equivalent to introducing the complex poles at the very start as conjugate pairs. If, on the other hand, all of the μ_k in the sum (2.9) are real, then all of the matrices R_k will also be real and the matrix χ then satisfies Eq. (2.9) automatically.

Substitution of the expression (2.9) into Eq. (2.5) and the supplementary condition (2.7) completely determines the pole trajectories $\mu_k(\rho,z)$ and the matrices $R_k(\rho,z)$. The numerical functions μ_k are determined from the requirement that in the left sides of Eqs. (2.5) there are no poles of second order at the points $\lambda=\mu_k$. The result is that each function $\mu_k(\rho,z)$ (with each index $k=1,2,\ldots,n$) satisfies a pair of differential equations

$$\begin{aligned}&\mu_{k,z}+2\mu_k^2(\mu_k^2+\rho^2)^{-1}=0,\\ &\mu_{k,\rho}-2\rho\mu_k(\mu_k^2+\rho^2)^{-1}=0,\end{aligned} \tag{2.10}$$

whose solutions are the roots of a quadratic algebraic equation

$$\mu_k^2-2(w_k-z)\mu_k-\rho^2=0, \tag{2.11}$$

where w_k are arbitrary constants (in general complex).

Accordingly, for each index k (i.e., for each pole) we have its own arbitrary constant w_k, which determines two possible solutions for the trajectory of the pole $\mu_k(\rho,z)$:

$$\mu_k=w_k-z\pm[(w_k-z)^2+\rho^2]^{1/2}. \tag{2.12}$$

The matrices R_k are degenerate, and their components can be written in the form

$$(R_k)_{ab}=n_a^{(k)}m_b^{(k)} \tag{2.13}$$

The two-component vectors $m_a^{(k)}$ are found directly from Eqs. (2.5) by requiring that they be satisfied at the poles $\lambda=\mu_k$, and the vectors $n_a^{(k)}$ are then determined from the condition (2.7). The vectors $m_a^{(k)}$ can be expressed in terms of the given partial solution for the "wave" matrix $\psi_0(\lambda,\rho,z)$ taken at the value μ_k for the argument λ. These vectors are of the following form:

$$m_a^{(k)}=m_{c0}^{(k)}[\psi_0^{-1}(\mu_k,\rho,z)]_{ca}, \tag{2.14}$$

where ψ_0^{-1} denotes the matrix inverse to ψ_0. (Here and from now on summation is to be understood over repeated vector and tensor indices a, b, c, d, f, which run through the values 0 and 1. Summation over other indices occurs only when explicitly indicated.) In Eq. (2.14) the $m_{c0}^{(k)}$ are arbitrary constants.

The vectors $n_a^{(k)}$ can then be determined from the following n-th order system of algebraic equations:

$$\sum_{l=1}^{n}\Gamma_{kl}n_a^{(l)}=\mu_k^{-1}m_c^{(k)}(g_0)_{ca},\quad k,l=1,2,\ldots,n, \tag{2.15}$$

where the matrix Γ_{kl} is symmetric and its elements are

$$\Gamma_{kl}=m_c^{(k)}(g_0)_{cb}m_b^{(l)}(\rho^2+\mu_k\mu_l)^{-1} \tag{2.16}$$

[in these formulas $g_0(\rho,z)$ is a given particular solution of the original equations (1.3)]. If we introduce the symmetric matrix D_{kl} inverse to the matrix Γ_{kl}:

$$\sum_{p=1}^{n}D_{kp}\Gamma_{pl}=\delta_{kl}, \tag{2.17}$$

then we get from (2.15) for the vectors $n_a^{(k)}$

$$n_a^{(k)}=\sum_{l=1}^{n}D_{lk}\mu_l^{-1}N_a^{(l)}, \tag{2.18}$$

where

$$N_a^{(k)}=m_c^{(k)}(g_0)_{ca}. \tag{2.19}$$

According to Eqs. (2.3), (2.4), and (2.9) the required matrix g is

$$g=\psi(0)=\chi(0)\psi_0(0)=\chi(0)g_0=\Big(I-\sum_{k=1}R_k\mu_k^{-1}\Big)g_0. \tag{2.20}$$

Now, using Eqs. (2.13), (2.18), and (2.19) we get the metric components g_{ab}:

$$g_{ab}=(g_0)_{ab}-\sum_{k,l=1}^{n}D_{kl}\mu_k^{-1}\mu_l^{-1}N_a^{(k)}N_b^{(l)}. \tag{2.21}$$

With the expression (2.21) the matrix g is obviously symmetric. Let us now consider the question of its being real. If all of the functions $\mu_k(\rho,z)$ are real, the components g_{ab} are automatically real, if we take all of the arbitrary constants appearing in the solution to be real. In fact, the particular solution $\psi_0(\lambda,\rho,z)$ is always taken to satisfy the second of the conditions (2.6), and consequently $\psi_0(\lambda)$ is real on the real axis of the λ plane, i.e., at the points $\lambda=\mu_k$. It can now be seen from Eq. (2.14) that the arbitrary constants $m_{c0}^{(k)}$ that occur in the vectors $m_a^{(k)}$ must be taken real, and then the vectors $m_a^{(k)}$ will also be real. It then follows that all the other quantities from which the matrix g is constructed are real. We now suppose that there are also complex values among the functions $\mu_1, \mu_2, \ldots, \mu_n$. The conditions (2.6) then require that all the complex poles appear only as conjugate pairs; for each complex pole $\lambda=\mu$ its conjugate $\lambda=\bar{\mu}$ must also appear. Suppose there is such a pair of poles $\lambda=\mu_p$ and $\lambda=\mu_q$, with $\mu_q=\bar{\mu}_p$. To these poles there correspon vectors $m_a^{(p)}$ and $m_a^{(q)}$, which according to Eq. (2.14) are given by

$$m_a^{(p)}=m_{c0}^{(p)}[\psi_0^{-1}(\mu_p,\rho,z)]_{ca},$$
$$m_a^{(q)}=m_{c0}^{(q)}[\psi_0^{-1}(\mu_q,\rho,z)]_{ca}.$$

A simple analysis shows that the matrix g will be real if for each such pair of complex-conjugate poles the arbitrary constant $m_{c0}^{(p)}$ and $m_{c0}^{(q)}$ are taken conjugate to each other. This means that the vectors $m_a^{(p)}$ and $m_a^{(q)}$ corresponding to each pair of conjugate poles are also conjugate to each other $[m_a^{(q)}=\bar{m}_a^{(p)}]$, since the function $\psi_0(\lambda,\rho,z)$ satisfies the condition $\psi_0(\bar{\lambda})=\bar{\psi}_0(\lambda)$. Accordingly, we can formulate the following rule that determines the choice of the arbitrary constants $m_{c0}^{(k)}$ in Eq. (2.14): To assure that the matrix g is real, it is necessary to choose the arbitrary $m_{c0}^{(k)}$ in Eq. (2.14) so that the vectors $m_a^{(k)}$ corresponding to real poles $\lambda=\mu_k$ are real and the vectors $m_a^{(p)}$ and $m_a^{(q)}$ corresponding to each pair of complex-conjugate poles $\lambda=\mu_p$ and $\lambda=\mu_q=\bar{\mu}_p$ are complex conjugate to each other.

Satisfying the requirements that g be real and symmetric is still not enough. It must not be forgotten that g must also satisfy the supplementary condition (1.2). We now calculate the determinant of the matrix g. The form (2.21) is not convenient for this calculation, and we use a different representation of our solution. We note that the process of perturbing the background solution g_0 and obtaining from it the n-soliton solution g, as described above, is formally equivalent to the introduction of the n solitons one at a time successively. The first step is to go from the background metric g_0 to the metric g_1 containing one soliton, corresponding to the presence in the matrix χ (which we at this stage call χ_1) only one pole $\lambda=\mu_1$.

This one-soliton solution is easily obtained from the results given above. The matrix $\chi_1(\lambda)$ and its inverse $\chi_1^{-1}(\lambda)$ can be written in the following form

$$\chi_1=I+(\mu_1^2+\rho^2)\mu_1^{-1}(\lambda-\mu_1)^{-1}P_1,$$
$$\chi_1^{-1}=I-(\mu_1^2+\rho^2)(\rho^2+\lambda\mu_1)^{-1}P_1, \tag{2.22}$$

where the matrix P_1 has the elements

$$(P_1)_{ab}=m_c^{(1)}(g_0)_{ca}m_b^{(1)}/m_d^{(1)}(g_0)_{df}m_f^{(1)} \tag{2.23}$$

and accordingly has the following properties:

$$P_1^2=P_1,\quad \mathrm{Sp}\,P_1=1,\quad \det P_1=0. \tag{2.24}$$

The quantities μ_1 and $m_a^{(1)}$ are given by Eqs. (2.12) and (2.14) with $k=1$. We now get for the matrix g_1:

$$g_1=\chi_1(0)g_0=[I-(\mu_1^2+\rho^2)\mu_1^{-2}P_1]g_0. \tag{2.25}$$

It is not hard to calculate the determinant of g_1. Owing to the general relation

$$\det(I+F)=1+\mathrm{Sp}\,F+\det F$$

(which holds for an arbitrary two-rowed matrix F) and the properties (2.24) we get

$$\det[I-(\mu_1^2+\rho^2)\mu_1^{-2}P_1]=-\rho^2\mu_1^{-2} \tag{2.26}$$

and consequently

$$\det g_1=-\rho^2\mu_1^{-2}\det g_0. \tag{2.27}$$

We can now take the solution g_1 as a new particular or background solution and repeat the operation of adding a soliton to it, that corresponding to the pole $\lambda=\mu_2$. To do this we form the new background matrix function $\psi_1=\chi_1\psi_0$, take its inverse ψ_1^{-1} and calculate it at the point $\lambda=\mu_2$, and then find the corresponding vector $M_a^{(2)}$:

$$M_a^{(2)}=M_{c0}^{(2)}[\psi_1^{-1}(\mu_2,\rho,z)]_{ca}$$

after which we construct the matrix P_2, in analogy with Eq. (2.23):

$$(P_2)_{ab}=M_c^{(2)}(g_1)_{ca}M_b^{(2)}/M_d^{(2)}(g_1)_{df}M_f^{(2)},$$

which matrix has the same properties (2.24) as the matrix P_1.

When we now construct the matrix $\chi_2(\lambda)$ [this matrix is calculated from the same formulas (2.22), with the index 1 replaced with 2], we get the two-soliton solution g_2:

$$g_2=[I-(\mu_2^2+\rho^2)\mu_2^{-2}P_2][I-(\mu_1^2+\rho^2)\mu_1^{-2}P_1]g_0.$$

Continuing this process, we get the n-soliton solution (2.21) in the form

$$g=\left\{\prod_{k=1}^{n}[I-(\mu_k^2+\rho^2)\mu_k^{-2}P_k]\right\}g_0, \tag{2.28}$$

where all of the matrices P_k satisfy the same conditions as the matrix P_1 does:

$$P_k^2=P_k,\quad \operatorname{Sp}P_k=1,\quad \det P_k=0. \tag{2.29}$$

Naturally the explicit form of the matrices P_k rapidly becomes cumbersome as k increases, and therefore this way of calculating solutions is less convenient than the one previously described. But the representation of the solution in the form (2.28) is useful for the study of some particular questions, and especially for calculating the determinant of the matrix g. The important thing for this is only that the matrices P_k have the properties (2.29), not their specific form. The contribution from each factor in Eq. (2.28) to the determinant of g can be calculated trivially, and the result is

$$\det g=(-1)^n\rho^{2n}\left(\prod_{k=1}^{n}\mu_k^{-2}\right)\det g_0. \tag{2.30}$$

If we take the particular solution g_0 as satisfying by definition the condition $\det g_0=-\rho^2$, then it follows from Eq. (2.30) that the number of solitons n must always be even, since an odd number would change the sign of $\det g$ and violate the physical signature of the metric. Accordingly (in contrast with the case investigated earlier[1]) on a physical background all stationary axially symmetric solitons (even those which correspond to real poles $\lambda=\mu_k$) can appear only in pairs forming bound two-soliton states.[3)]

We still have to construct an n-soliton solution g which not only satisfies Eq. (1.3) but also the supplementary condition (1.2). We shall call such a solution a physical one and denote it by $g^{(ph)}$. Constructing it is simple if we note that $\det g$ for any solution g of Eq. (1.3) satisfies the equation

$$\rho^{-1}[\rho(\ln\det g)_{,\rho}]_{,\rho}+(\ln\det g)_{,zz}=0.$$

Then it is easy to verify that the matrix

$$g^{(ph)}=-\rho(-\det g)^{-1/2}g \tag{2.31}$$

also satisfies Eq. (1.3), and also the condition $\det g^{(ph)}=-\rho^2$. Now supposing the number n of solitons is even and $\det g_0=-\rho^2$, we get from Eqs. (2.30) and (2.31) the final expression for the metric tensor:

$$g^{(ph)}=-\rho^{-n}\left(\prod_{k=1}^{n}\mu_k\right)g,\quad \det g^{(ph)}=-\rho^2, \tag{2.32}$$

where the matrix g is given by Eq. (2.21).

3. CALCULATION OF THE METRIC COEFFICIENT f

It is also convenient to do the calculation of the coefficient f in two stages. First we calculate the value of f that follows from Eqs. (1.4) and (1.5) when we substitute in them the nonphysical solution g given by Eq. (2.21), which does not satisfy the condition $\det g=-\rho^2$, and then use a simple procedure to find the physical value of the coefficient, $f^{(ph)}$, which is obtained from these same Eqs. (1.4) and (1.5) when $g^{(ph)}$ is substituted in them instead of g.

To calculate f we must determine from Eqs. (2.5) the matrices U and V; this can be done by equating the left and right sides of these equations at the poles $\lambda=i\rho$ and $\lambda=-i\rho$ (cf. the analogous procedure in the previous paper[1]). Then calculating the traces $\operatorname{Sp}(U^2-V^2)$ and $\operatorname{Sp}(UV)$ and substituting them in Eqs. (1.4) and (1.5), we find f by direct integration. It is a remarkable fact that this integration can actually be carried out. The key point in calculating the coefficient f corresponding to an n-soliton solution is to determine it for a one-soliton solution (which coefficient we denote as f_1), described by Eqs. (2.22)–(2.27). Having done the necessary calculations with the scheme indicated above (in analogy with the way this was done in Ref. 1), we get the following result for the one-soliton solution:

$$f_1=C_1f_0\rho\mu_1^2(\mu_1^2+\rho^2)^{-1}\Gamma_{11}, \tag{3.1}$$

where C_1 is an arbitrary constant, f_0 is the particular (background) solution for the coefficient t, which corresponds to the solution g_0, and Γ_{11} is the single component of the matrix (2.16), which is all that exists in this case ($k=1$ and $l=1$):

$$\Gamma_{11}=(\mu_1^2+\rho^2)^{-1}m_c^{(1)}(g_0)_{cb}m_b^{(1)} \tag{3.2}$$

(the vector $m_a^{(1)}$ follows from Eq. (2.14) for $k=1$).

The next step in the calculations is that, taking the solution g_1, f_1 as a new particular solution and repeating the operation just performed (as was explained in the foregoing section in connection with finding the matrix g_2), we get the coefficient f_2 that corresponds to the two-soliton solution with the poles $\lambda=\mu_1$ and $\lambda=\mu_2$. At this second step we already have to deal with only calculations of an algebraic nature, since the need for integration appears in the whole procedure only once, in the transition from the background solution g_0, f_0 to the solution g_1, f_1, which contains one soliton.

Omitting the details of the calculation, we give only the result:

$$f_2=C_2 f_0 \rho^2 \mu_1^2 \mu_2^2 (\mu_1^2+\rho^2)^{-1} (\mu_2^2+\rho^2)^{-1} (\Gamma_{11}\Gamma_{22}-\Gamma_{12}^2). \tag{3.3}$$

Here C_2 is an arbitrary constant, f_0 is the same background solution as in Eq. (3.1), and Γ_{11}, Γ_{22}, and Γ_{12} are the components of the matrix (2.16). We now have three independent components of Γ_{kl}, since the indices k and l can take two values, 1 and 2.

Equations (3.1) and (3.3) suggest that in the general n-soliton case the coefficient f is given by the expression

$$f_n = C_n f_0 \rho^n \left(\prod_{k=1}^{n} \mu_k^2\right) \left[\prod_{k=1}^{n} (\mu_k^2+\rho^2)\right]^{-1} \det \Gamma_{kl} \tag{3.4}$$

(where $k,l=1,2,\ldots,n$). Since we see from Eqs. (3.1) and (3.3) that this formulat indeed holds for $n=1$ and $n=2$, we can prove that it holds in the general case by using the method of mathematical induction. This proof is given in the Appendix to the present paper, and shows that Eq. (3.4) is indeed correct in general.

Now we must determine the physical value $f_n^{(\mathrm{ph})}$ of the coefficient, i.e., the value that would be obtained from Eqs. (1.4) and (1.5) if we substituted in them the physical matrix $g^{(\mathrm{ph})}$ of Eq. (2.32) instead of g. From Eq. (2.31) we get the obvious relations

$$U^{(\mathrm{ph})}=\rho g^{(\mathrm{ph})}_{,\rho} g^{(\mathrm{ph})-1} = U + [1 - {}^1/_2\rho(\ln\det g)_{,\rho}]I,$$

$$V^{(\mathrm{ph})}=\rho g^{(\mathrm{ph})}_{,z} g^{(\mathrm{ph})-1} = V - {}^1/_2\rho(\ln\det g)_{,z}I.$$

When we now substitute in Eqs. (1.4) and (1.5) the matrices $U^{(\mathrm{ph})}$ and $V^{(\mathrm{ph})}$ instead of U and V, we find that the physical coefficient $f_n^{(\mathrm{ph})}$ is given by the formula

$$f_n^{(\mathrm{ph})}=f_n \rho^{1/2} Q^{-1}, \tag{3.5}$$

where f_n is the value of this coefficient which is given by Eq. (3.4) and the function Q is defined by the equations

$$(\ln Q)_{,z}={}^1/_4\rho(\ln\det g)_{,\rho}(\ln\det g)_{,z},$$

$$(\ln Q)_{,\rho}={}^1/_8\rho[(\ln\det g)_{,\rho}^2-(\ln\det g)_{,z}^2].$$

On substituting here the expression (2.30) for $\det g$ (with the condition $\det g_0=-\rho^2$), we find that these equations can be integrated easily, and the answer can be written in the form

$$Q^{-1}=\mathrm{const}\cdot\rho^{-(n^2+2n+1)/2}\left(\prod_{k=1}^{n}\mu_k\right)^{n-1} \times\left[\prod_{k=1}^{n}(\mu_k^2+\rho^2)\right]\prod_{\substack{k,l=1\\k>l}}^{n}(\mu_k-\mu_l)^2. \tag{3.6}$$

From this and Eqs. (3.4) and (3.5) we get the final expression for the physical value of the coefficient f:

$$f_n^{(\mathrm{ph})}=C_n^{(\mathrm{ph})} f_0 \rho^{-n^2/2}\left(\prod_{k=1}^{n}\mu_k\right)^{n+1}\left[\prod_{\substack{k,l=1\\k>l}}^{n}(\mu_k-\mu_l)^2\right]^{-1}\det\Gamma_{kl} \tag{3.7}$$

[$C_n^{(\mathrm{ph})}$ is an arbitrary constant].

For clarity we point out that the product

$$\prod_{\substack{k,l=1\\k>l}}^{n}(\mu_k-\mu_l)^2$$

is equal to 1 for $n=1$, to $(\mu_2-\mu_1)^2$ for $n=2$, to $(\mu_3-\mu_2)^2 (\mu_3-\mu_1)^2 (\mu_2-\mu_1)^2$ for $n=3$, and so on. In deriving Eq. (3.7) we have assumed that no two of the quantities μ_1, μ_2, $\ldots$, μ_n are equal.

Accordingly, the final form of the n-soliton solution can be written in the form

$$-ds^2=f_n^{(\mathrm{ph})}(d\rho^2+dz^2)+g_{ab}^{(\mathrm{ph})}dx^a dx^b, \tag{3.8}$$

where $f_n^{(\mathrm{ph})}$ is given by Eq. (3.7) and the matrix elements $g_{ab}^{(\mathrm{ph})}$ are determined by Eqs. (2.32) and (2.21).

4. TWO-SOLITON SOLUTION ON A FLAT BACKGROUND

In this and the following sections we consider the application of the results presented above to the case in which the background metric g_0, f_0 is flat and given by the interval

$$-ds^2=-dt^2+\rho^2 d\varphi^2+d\rho^2+dz^2. \tag{4.1}$$

That is, $f_0=1$ and $g_0=\mathrm{diag}(-1,\rho^2)$ with the obvious property $\det(g_0)=-\rho^2$. The matrix V_0 is equal to zero, and for the matrix U_0 we have $U_0=\mathrm{diag}(0,2)$. From Eq. (2.1) we get the corresponding solution for $\psi_0(\lambda,\rho,z)$:

$$\psi_0=\begin{pmatrix}-1 & 0\\ 0 & \rho^2-2z\lambda-\lambda^2\end{pmatrix}, \tag{4.2}$$

which satisfies the requirement $\psi_0(0)=g_0$. From this and Eq. (2.14), using Eq. (2.11), we easily find the components of the vectors $m_a^{(k)}$:

$$m_0^{(k)}=C_0^{(k)},\quad m_1^{(k)}=C_1^{(k)}\mu_k^{-1}, \tag{4.3}$$

where $C_0^{(k)}$ and $C_1^{(k)}$ are arbitrary constants.

Now from Eq. (2.16) we get the elements of the matrix Γ_{kl}:

$$\Gamma_{kl}=(-C_0^{(k)}C_0^{(l)}+C_1^{(k)}C_1^{(l)}\mu_k^{-1}\mu_l^{-1}\rho^2)(\rho^2+\mu_k\mu_l)^{-1}. \tag{4.4}$$

From Eq. (2.19) we get the components of the vectors $N_a^{(k)}$:

$$N_0^{k}=-C_0^{(k)},\quad N_1^{(k)}=C_1^{(k)}\mu_k^{-1}\rho^2. \tag{4.5}$$

Together with the expressions (2.12) for the functions μ_k we now have everything necessary for constructing n-soliton solutions on a flat-space background.

Let us now consider the simplest case of all. As was already stated at the end of Sec. 2, solitons on a physical background (with either complex or real poles) can appear only in pairs. Consequently, the simplest case will be a two-soliton solution, corresponding to two poles, $\lambda=\mu_1$ and $\lambda=\mu_2$. It is not hard to show by direct calculation that what we have here is just the Kerr-NUT solution. In our previous paper[1] it was already pointed out that a double stationary soliton on a flat background, corresponding to a pair of complex-conjugate poles, gives a Kerr-NUT solution with an "anomalously large" rotational moment (i.e., a solution without horizons and with a bare singularity). In fact, here we get precisely this situation for $\mu_2=\bar{\mu}_1$. On the other hand, if both functions, μ_1 and μ_2, are real, the solution corresponds to the "normal" situation, with the singularity hidden from an outside observer by horizons.

These assertions can be verified by direct calculation of the metric. Let us represent the constant w_1 and w_2

that appear in the relations (2.11) and (2.12) in the form

$$w_1=z_1+\sigma,\quad w_2=z_1-\sigma, \tag{4.6}$$

where z_1 and σ are new arbitrary constants. We now introduce instead of ρ and z new coordinates r and θ:

$$\rho=[(r-m)^2-\sigma^2]^{1/2}\sin\theta,\quad z-z_1=(r-m)\cos\theta, \tag{4.7}$$

where m is an arbitrary constant whose meaning will be clear later. Then from Eq. (2.12) it is easy to express the quantities μ_1 and μ_2 in terms of the new variables r and θ. In this calculation we can choose the signs in the formula (2.12) either the same for μ_1 and μ_2, or else opposite. It is not hard to show that both cases lead to the same metric (to within linear transformations of the two coordinates t, φ in terms of each other, and a trivial conformal transformation, multiplication of the interval with a constant).

Let us consider first the case of like signs. If we choose the plus[4)] sign in Eq. (2.12) for both values μ_1 and μ_2, then substituting the expressions (4.6) and (4.7), we get

$$\mu_1=2(r-m+\sigma)\sin^2\frac{\theta}{2},\quad \mu_2=2(r-m-\sigma)\sin^2\frac{\theta}{2}. \tag{4.8}$$

From this [using the expression (4.7) for ρ] and from Eq. (4.5) we find the components of the vectors $N_a^{(1)}$ and $N_a^{(2)}$, and from Eq. (4.4) we find the matrix Γ_{kl} and its inverse D_{kl} (in this case $k,l=1,2$). After this we get from Eqs. (2.32) and (2.21) the components of the metric tensor $g_{ab}^{(\mathrm{ph})}$ and from Eq. (3.7) the metric coefficient $f_2^{(\mathrm{ph})}$. Substitution of these quantities in the interval (3.8) gives the final form of the solution, which can be reduced by simple linear transformations of the coordinates to the standard form of the Kerr-NUT solution in Boyer–Lindquist coordinates.

Omitting details, we point out that without loss of generality we can subject the arbitrary constants $C_0^{(k)}$ and $C_1^{(k)}$ that appear in the expressions (4.3) for the vectors $m_a^{(k)}$ to two conditions:

$$C_1^{(1)}C_0^{(2)}-C_0^{(1)}C_1^{(2)}=\sigma,\quad C_1^{(1)}C_1^{(2)}+C_0^{(1)}C_0^{(2)}=-m, \tag{4.9}$$

which are equivalent to the requirement that the variable r indeed be the Boyer–Lindquist radial coordinate. We then introduce two arbitrary constants a and b defined by

$$C_1^{(1)}C_1^{(2)}-C_0^{(1)}C_0^{(2)}=-b,\quad C_1^{(1)}C_1^{(2)}+C_0^{(1)}C_0^{(2)}=a. \tag{4.10}$$

From Eqs. (4.9) and (4.10) it follows that

$$\sigma^2=m^2-a^2+b^2. \tag{4.11}$$

Now the metric (3.8) contains only the constants m, a, b and takes the form

$$\begin{aligned}-ds^2=&\,\omega\Delta^{-1}dr^2+\omega d\theta^2-\omega^{-1}\{(\Delta-a^2\sin^2\theta)d\tau^2\\&-[4\Delta b\cos\theta-4a\sin^2\theta(mr+b^2)]d\tau d\varphi+[\Delta(a\sin^2\theta+2b\cos\theta)^2\\&-\sin^2\theta(r^2+b^2+a^2)^2]d\varphi^2\},\end{aligned} \tag{4.12}$$

where the variable τ is connected with t (the original coordinate $x^0=t$) by the relation

$$\tau=t+2a\varphi, \tag{4.13}$$

and the quantities ω and Δ are

$$\omega=r^2+(b-a\cos\theta)^2,\quad \Delta=r^2-2mr+a^2-b^2. \tag{4.14}$$

It can be seen from this that the Kerr-NUT solution with horizons corresponds to real poles $\lambda=\mu_1$ and $\lambda=\mu_2$, since in this case the constant σ is real ($m^2+b^2>a^2$), and the constants w_1 and w_2 and the functions μ_1 and μ_2 are real along with σ. If the quantity σ is imaginary ($m^2+b^2<a^2$), then the constants w_1 and w_2 and the functions μ_1 and μ_2 are complex and conjugate to each other. This case corresponds to a solution without horizons. Furthermore the metric (4.12) and the constants m, a, b are of course still real, but the original constants $C_a^{(k)}$, as Eqs. (4.9) and (4.10) show, must be taken complex and related by $C_a^{(2)}=\bar{C}_a^{(1)}$, which, we see from Eq. (4.3), means that also $m_a^{(2)}=\bar{m}_a^{(1)}$. This agrees with the rule for choosing real solutions with a complex-conjugate pair of poles that were formulated earlier in Sec. 2.

Let us now look at the second possibility for choosing the solutions of Eq. (2.11), the one that corresponds to using different signs in Eq. (2.12). Choosing the plus sign for μ_1 and the minus sign for μ_2, we get

$$\mu_1=2(r-m+\sigma)\sin^2\frac{\theta}{2},\quad \mu_2=-2(r-m+\sigma)\cos^2\frac{\theta}{2}. \tag{4.15}$$

Calculations like the foregoing ones show that in this case we again arrive at a Kerr-NUT metric, the only difference being that instead of the variables τ, φ we will now have certain new coordinates τ' and φ', connected with the original variables $x^0=t$ and $x^1=\varphi$ by a linear transformation different from that in Eq. (4.13). The new relations are $\tau'=c_1t+c_2\varphi$, $\varphi'=c_3\varphi$, where the coefficients are real only if the constant σ is real (i.e., if μ_1 and μ_2 are real), and become complex when σ is imaginary. This means that for imaginary σ the matrix is complex in the original coordinates t, φ; this is quite natural, since in this case, as can be seen from Eq. (4.15), the poles $\lambda=\mu_1$ and $\lambda=\mu_2$ do not compose a complex-conjugate pair.

Besides this, the connection between the arbitrary constants $C_a^{(k)}$ and the parameters m, a, b are now different:

$$\begin{aligned}&C_1^{(1)}C_1^{(2)}+C_0^{(1)}C_0^{(2)}=\sigma,\quad C_1^{(1)}C_1^{(2)}-C_0^{(1)}C_0^{(2)}=-m,\\&C_1^{(1)}C_0^{(2)}-C_0^{(1)}C_1^{(2)}=a,\quad C_1^{(1)}C_0^{(2)}+C_0^{(1)}C_1^{(2)}=-b,\end{aligned} \tag{4.16}$$

but the relation (4.11) between σ and the constants m, a, b is still valid.

In conclusion we point out that the only actual physical solution is that of Kerr, since the presence of the NUT parameter b makes the metric no longer asymptotically Euclidean and produces a number of nonphysical properties of the solution (the relevant analysis has been given by Misner[4]).

5. THE n-SOLITON SOLUTION ON A FLAT BACKGROUND

In this section we consider some general properties of the n-soliton solution, confining ourselves to one of its possible types. We shall assume that on the background of a flat space with the metric (4.1) an even number n of solitons are introduced, corresponding to the poles $\lambda=\mu_1$, $\lambda=\mu_2$, ..., $\lambda=\mu_n$. We divide all of the functions μ_k ($k=1,2,\ldots,n$) into pairs and introduce the Greek index γ, which will number these pairs and takes only the odd values from 1 to $n-1$: $\gamma=1,3,\ldots n-1$. We thus have $n/2$ pairs of pole trajectories $(\mu_\gamma, \mu_{\gamma+1})$.

To understand the physical meaning of the solution it is helpful to examine first a special case which corresponds to a diagonal matrix g, i.e., to a static n-soliton field remaining after the rotation has been turned off. To obtain such a special case we set all of the arbitrary constants $C_0^{(k)}$ in Eq. (4.3) equal to zero, and then all the $m_0^{(k)}$ also equal to zero. It now follows from Eq. (2.15) that all the $n_0^{(k)}=0$ and the matrices R_k as we can see from Eq. (2.15) take the form

$$R_k=\begin{pmatrix}0 & 0\\ 0 & n_1^{(k)}m_1^{(k)}\end{pmatrix}.$$

This means that all the matrices P_k in the representation (2.28) of the solution take the form

$$P_k=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

in accordance with the conditions (2.29). Then from Eqs. (2.28) and (2.32) we get the following solution for the diagonal case under consideration:

$$g_{00}^{(\text{ph})}=\rho^{-n}\prod_{k=1}^{n}\mu_k,\quad g_{01}^{(\text{ph})}=0,\quad g_{11}^{(\text{ph})}=-\rho^2/g_{00}^{(\text{ph})}. \tag{5.1}$$

The metric coefficient $f_n^{(\text{ph})}$ can be found from Eq. (3.7); to do so we must calculate the determinant of the matrix Γ_{kl} (with $C_0^{(k)}=0$). It is simpler, however, to determine $f_n^{(\text{ph})}$ directly from Eqs. (1.4) and (1.5), since the solution (5.1) is simple and easy to integrate. The result is

$$f_n^{(\text{ph})}=\text{const}\cdot\rho^{(n^2+2n)/2}\Big[\prod_{\substack{k,l=1\\k>l}}^{n}(\mu_k-\mu_l)^2\Big]\Big(\prod_{k=1}^{n}\mu_k\Big)^{1-n}\Big[\prod_{k=1}^{n}(\mu_k^2+\rho^2)\Big]^{-1}. \tag{5.2}$$

We now determine from Eqs. (2.11) and (2.12) the functions μ_k, which we have arranged in the pairs $(\mu_\gamma, \mu_{\gamma+1})$. Confining our treatment to the case when the signs in Eq. P2.12) are chosen differently for the functions of each pair, we have

$$\begin{aligned}\mu_\gamma&=w_\gamma-z+[(w_\gamma-z)^2+\rho^2]^{1/2},\\ \mu_{\gamma+1}&=w_{\gamma+1}-z-[(w_{\gamma+1}-z)^2+\rho^2]^{1/2}.\end{aligned} \tag{5.3}$$

Instead of each pair of arbitrary constants w_γ and $w_{\gamma+1}$, we introduce new constants z_γ and m_γ, setting

$$w_\gamma=z_\gamma-m_\gamma,\quad w_{\gamma+1}=z_\gamma+m_\gamma. \tag{5.4}$$

If we now introduce $n/2$ pairs of functions $r_\gamma(\rho, z)$ and $\theta_\gamma(\rho, z)$ (giving to each pair of poles its own "radial and angular coordinates") through the relations

$$\rho=[r_\gamma(r_\gamma-2m_\gamma)]^{1/2}\sin\theta_\gamma,\quad z-z_\gamma=(r_\gamma-m_\gamma)\cos\theta_\gamma, \tag{5.5}$$

we get from Eq. (5.3)

$$\begin{aligned}\mu_\gamma&=2(r_\gamma-2m_\gamma)\sin^2\frac{\theta_\gamma}{2},\\ \mu_{\gamma+1}&=-2(r_\gamma-2m_\gamma)\cos^2\frac{\theta_\gamma}{2}.\end{aligned} \tag{5.6}$$

Using these expressions for ρ and μ_k, we get from Eq. (5.1) the component $g_{00}^{(\text{ph})}$ as the following product of $n/2$ factors:

$$g_{00}^{(\text{ph})}=-(1-2m_1r_1^{-1})(1-2m_3r_3^{-1})\ldots(1-2m_{n-1}r_{n-1}^{-1}). \tag{5.7}$$

For the case of the two-soliton solution Eq. (5.7) will have only one factor, the Schwarzschild expression for the coefficient $g_{00}^{(\text{ph})}$. Calculating from Eq. (5.2) the coefficient $f_2^{(\text{ph})}$ for this case and writing out the interval, we indeed get the standard expression for the Schwarzschild metric with radial coordinate r_1 and polar angle θ_1. This result also follows, of course, from the general form of the two-soliton Kerr-NUT solution, given in the preceding section [case (4.15), (4.16)] with $C_0^{(1)}=C_0^{(2)}=0$.

To interpret the n-soliton static solution with the "potential" (5.7) we must choose a suitable radial variable. Any one of the functions $r_\gamma(\rho, z)$ could now serve as a radial coordinate, but it is most natural to define the radial variable in such a way that the dipole moment relative to it vanishes in the expansion at infinity of the Newtonian potential of the system in question. As is well known, the Newtonian potential here is $\Phi=1+g_{00}^{(\text{ph})}$, and from Eq. (5.7) we have

$$\Phi=1-(1-2m_1r_1^{-1})(1-2m_3r_3^{-1})\ldots(1-2m_{n-1}r_{n-1}^{-1}). \tag{5.8}$$

Let us try to define the "true" radial coordinate r and polar angle θ by relations of the same form as Eq. (5.5):

$$\rho=[r(r-2m)]^{1/2}\sin\theta,\quad z-z_0=(r-m)\cos\theta, \tag{5.9}$$

but with new constants m and z_0, which are subject to definition. From Eqs. (5.9) and (5.5) we can find functions $r_\gamma(r,\theta)$ and $\theta_\gamma(r,\theta)$ and obtain their asymptotic expansions for $r\to\infty$ (in the first approximation we have for $r\to\infty$ simply $r_\gamma=r$ and $\theta_\gamma=\theta$). Substituting these expansions into Eq. (5.8), we find the expansion of the potential Φ, and from the condition that it must contain no dipole term we can determine the constants m and z_0. In this way we get

$$m=\sum_{\gamma=1}^{n-1}m_\gamma,\quad z_0=\Big(\sum_{\gamma=1}^{n-1}m_\gamma z_\gamma\Big)\Big/\sum_{\gamma=1}^{n-1}m_\gamma, \tag{5.10}$$

and then the expansion for Φ takes the form

$$\Phi=2mr^{-1}+q(3\cos^2\theta-1)r^{-3}+\ldots, \tag{5.11}$$

where q is the quadrupole moment of the system. For the case of a four-soliton solution, for example, (where the index γ takes only the two values 1 and 3) we have

$$q=m_1m_3[(z_1-z_3)^2-m^2](m_1+m_3)^{-1}.$$

These results show that the n-soliton static solution is a localized perturbation in an asymptotically flat space. For a sufficiently remote observer such a field can be regarded as an external field produced by $n/2$ localized axially symmetric structures, each of which has its own mass m and its center of mass lying on the axis of symmetry at the point with coordinate z. The equations (5.10) show that the total mass of all these $n/2$ objects (or pairs of solitons) is equal to the sum of their masses, and the coordinate z_0 of their common center of gravity is given by the usual expression of the mechanics of particles. All of the multipole moments of the field can also be expressed in definite ways in terms of the constants m_γ and z_γ.

If we now suppose that in this system there appear "rotational motions either of the whole or of the individual elements" around the axis of symmetry, the resulting case will correspond to a nondiagonal metric with $g_{01}^{(\mathrm{ph})}\neq 0$. In the special case of a two-soliton system considered in the preceding section, this change corresponds to the change from the Schwarzschild solution to that of Kerr. Just as in that special case, we must also here assure that the solution with n solitons is asymptotically Euclidean. In the two-soliton case this made it necessary to set the NUT parameter equal to zero. This means that the off-diagonal component $g_{01}^{(\mathrm{ph})}$ of the metric must decrease like r^{-1} as $r\to\infty$ [in the Kerr-NUT solution, $g_{01}^{(\mathrm{ph})}\sim b\cos\theta+O(r^{-1})$ for $r\to\infty$]. Then the coefficient of r^{-1} in $g_{01}^{(\mathrm{ph})}$ gives the total rotational moment of the system.

It is not hard to find the behavior of the components $g_{ab}^{(\mathrm{ph})}$ for $r\to\infty$ in the general case of an n-soliton metric. As in the two-soliton case, we must introduce the notations (4.6) for each pair of constants w_γ, $w_{\gamma+1}$ and for each pair of functions μ_γ, $\mu_{\gamma+1}$ we must introduce a pair of "coordinates" r_γ, θ_γ by the formulas (4.7). After this we get from Eq. (5.3) expressions for μ_γ and $\mu_{\gamma+1}$ of the form (4.15). At infinity all of the variables r_γ, θ_γ coincide, so that it is immaterial which pair we take as spherical coordinates r, θ, if we are concerned only with the first terms of the expansion for $r\to\infty$.

Now from Eq. (4.3) we get the asymptotic form of the vectors $m_a^{(k)}$, and from Eqs. (4.4) and (2.15), that of the vectors $n_a^{(k)}$. From these it is now easy to find the behavior of the components $g_{ab}^{(\mathrm{ph})}$. The result shows that the asymptotic behavior of the metric coefficients $g_{ab}^{(\mathrm{ph})}$ for $r\to\infty$ is precisely the same as in the two-soliton case:

$$g_{00}^{(\mathrm{ph})}\to-1,\quad g_{11}^{(\mathrm{ph})}\to r^2\sin^2\theta,\quad g_{01}^{(\mathrm{ph})}\to b_1\cos\theta+b_2+O(r^{-1}),\qquad(5.12)$$

where b_1 and b_2 are constants which can be expressed in terms of $C_0^{(k)}$ and $C_1^{(k)}$. For the metric to be asymptotically Euclidean for $r\to\infty$ the parameter b_1 must be zero, which gives a supplementary condition connecting the constants $C_a^{(k)}$:

$$b_1(C_0^{(k)},C_1^{(k)})=0.\qquad(5.13)$$

After this the constant b_2 can be eliminated from the asymptotic form of the metric coefficient $g_{01}^{(\mathrm{ph})}$ with a linear transformation of the form $t=t'=b_2\varphi$.

APPENDIX

We shall now prove the validity of Eq. (3.4). As was already stated in Sec. 3, we have only to show that it holds for the case $m+1$, on the assumption that it holds for the case m. We suppose that we have some solution g_n, f_n, ψ_n of our problem, and on it as background we construct a solution g_{n+m}, f_{n+m}, ψ_{n+m} by introducing m solitons corresponding to poles $\lambda=\mu_{n+1}$, $\lambda=\mu_{n+2}$, ..., $\lambda=\mu_{n+m}$. We assume that for such a "case m" Eq. (3.4) is true, and consequently the coefficient f is of the form

$$f_{n+m}=C_{n+m}f_n\rho^m\left(\prod_{k=1}^{m}\mu_{n+k}^2\right)\left[\prod_{k=1}^{m}(\mu_{n+k}^2+\rho^2)\right]^{-1}D_{n+m},\qquad(\mathrm{A}.1)$$

where C_{n+m} is an arbitrary constant and D_{n+m} is the determinant of the matrix $\Gamma_{n+k,n+l}$ (relative to the indices $k,l=1,2,\ldots,m$).

$$\Gamma_{n+k,n+l}=m_a^{(n+k)}(g_n)_{ab}m_b^{(n+l)}(\rho^2+\mu_{n+k}\mu_{n+l})^{-1}.\qquad(\mathrm{A}.2)$$

Here and for what follows we have adopted the following conventions about indices: n and m are given constants; the letters k and l are used to denote running indices which go through the values $1,2,\ldots,m$; and the Greek letters α, β denote indices (appearing later) that go through the $m+1$ values $0,1,2,\ldots,m$.

As we have already said,

$$D_{n+m}=\det\Gamma_{n+k,n+l}.\qquad(\mathrm{A}.3)$$

The vectors $m_a^{(n+k)}$ in Eq. (A.2) are constructed according to the rule (2.14):

$$m_a^{(n+k)}=m_{c0}^{(n+k)}[\psi_n^{-1}(\mu_{n+k},\rho,z)]_{ca}.\qquad(\mathrm{A}.4)$$

Let us now consider that the solution g_n, f_n, ψ_n was obtained from another solution g_{n-1}, f_{n-1}, ψ_{n-1} by adding to it one soliton, corresponding to the pole $\lambda=\mu_n$. In this case, according to Eqs. (2.22) and (2.25) we have

$$\psi_n=[I+(\mu_n^2+\rho^2)\mu_n^{-1}(\lambda-\mu_n)^{-1}P_n]\psi_{n-1},$$

$$\psi_n^{-1}=\psi_{n-1}^{-1}[I-(\mu_n^2+\rho^2)(\rho^2+\lambda\mu_n)^{-1}P_n].\qquad(\mathrm{A}.5)$$

$$g_n=\psi_n(0)=[I-(\mu_n^2+\rho^2)\mu_n^{-2}P_n]g_{n-1}.\qquad(\mathrm{A}.6)$$

The matrix P_n is constructed from ψ_{n-1} and g_{n-1} according to the law (2.23):

$$P_n=l_c^{(n)}(g_{n-1})_{ca}l_b^{(n)}/l_f^{(n)}(g_{n-1})_{fd}l_d^{(n)},\qquad(\mathrm{A}.7)$$

where the vector l_a is given by the expression

$$l_a^{(n)}=l_{c0}^{(n)}[\psi_{n-1}^{-1}(\mu_n,\rho,z)]_{ca}.\qquad(\mathrm{A}.8)$$

Besides the vector $l_a^{(n)}$ we need the vectors $l_a^{(n+k)}$ ($k=1,2,\ldots,n$), which are given by

$$l_a^{(n+k)}=m_{c0}^{(n+k)}[\psi_{n-1}^{-1}(\mu_{n+k},\rho,z)]_{ca},\qquad(\mathrm{A}.9)$$

where $m_{c0}^{(n+k)}$ are the same arbitrary constants as appear in Eq. (A.4).

Now from Eq. (A.4), (A.5), and (A.7) we can obtain an expression for the vectors $m_a^{(n+k)}$ in terms of the vectors $l_a^{(n)}$ and $l_a^{(n+k)}$:

$$m_a^{(n+k)}=l_a^{(n+k)}-(E_{n,n})^{-1}E_{n,n+k}l_a^{(n)},\qquad(\mathrm{A}.10)$$

where we have introduced the matrix $E_{n+\alpha,n+\beta}$ ($\alpha,\beta=0,1,2,\ldots,m$):

$$E_{n+\alpha,n+\beta}=l_c^{(n+\alpha)}(g_{n-1})_{cb}l_b^{(n+\beta)}(\rho^2+\mu_{n+\alpha}\mu_{n+\beta})^{-1}.\qquad(\mathrm{A}.11)$$

Then, substituting Eqs. (A.10) and (A.6) in (A.2), we find an expression for the matrix $\Gamma_{n+k,n+l}$ in terms of the matrix $E_{n+\alpha,n+\beta}$:

$$\Gamma_{n+k,n+l}=E_{n+k,n+l}-(E_{n,n})^{-1}E_{n,n+k}E_{n,n+l}.\qquad(\mathrm{A}.12)$$

From Eq. (A.12) it follows that the determinants of the matrices $E_{n+\alpha,n+\beta}$ and $\Gamma_{n+k,n+l}$ are connected by the relation

$$\det E_{n+\alpha,n+\beta}=E_{n,n}\det\Gamma_{n+k,n+l}.\qquad(\mathrm{A}.13)$$

Now from Eqs. (3.1) and (3.2) we bet a connection between f_n and f_{n-1}:

$$f_n=C_nf_{n-1}E_{n,n}\rho\mu_n^2(\mu_n^2+\rho^2)^{-1}\qquad(\mathrm{A}.14)$$

(C_n is an arbitrary constant). Substituting this expression in Eq. (A.1) and using Eqs. (A.3) and (A.13), we get

$$f_{n+m}=\mathrm{const}\cdot f_{n-1}\rho^{m+1}\left(\prod_{\alpha=0}^{m}\mu_{n+\alpha}^2\right)\left[\prod_{\beta=0}^{m}(\mu_{n+\beta}^2+\rho^2)\right]^{-1}\det E_{n+\alpha,n+\beta}.\qquad(\mathrm{A}.15)$$

This result, together with the expressions (A.8), (A.9), and (A.11) for the matrix $E_{n+\alpha,n+\beta}$ is nothing other than the formula (3.4) itself, except that it is for the "case $m+1$," with the solution g_{n+m}, f_{n+m}, ψ_{n+m} being obtained from g_{n-1}, f_{n-1}, ψ_{n-1} by adding $m+1$ solitons to the latter. This analysis completes the proof that Eq. (3.4) is valid.

[1])A system of units is used in which the speed of light is equal to unity. The interval is written in the form $-ds^2 = g_{ik}dx_i dx_k$, where g_{ik} has the signature $-+++$.

[2])We may indicate that the formal transformations from the variables ζ, η, α, β and matrices A, B which we used previously to the variables ρ, z and matrices U, V of the present paper are of the form $\zeta = (z+i\rho)/2$, $\eta = (z-i\rho)/2$, $\alpha = i\rho$, $\beta = z$, $A = -U - iV$, $B = -U + iV$.

[3])Nevertheless we can obtain physical solutions with an odd number of solitons, but for this it is necessary to take a background solution with a nonphysical signature, for which $\det g_0 = \rho^2$.

[4])We point out that the indicated choice of "signs" here has a precise meaning only for sufficiently large positive values of the variable r and real values of the constants w_1 and w_2. In the general case there is only a choice of one branch or the other of the solution of the quadratic equation (2.11).

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Search for unusual decays of superdense nuclei using two-meter hydrogen and propane bubble chambers

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Preliminary results are reported on the determination of the upper limits of the cross sections for production of superdense nuclei, by detection of their unusual decays occurring with times in the millisecond range. A special mode of bubble-chamber operation is proposed. It is shown that by use of this technique it is possible to determine comparatively simple cross sections at the level 10^{-33}–10^{-35} cm^2 per nucleus.

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In recent years, especially with the appearance of A. B. Migdal's theory of the pion condensate, great interest has arisen in the search for superdense nuclei. It is expected that they can have a very large binding energy, and therefore it is possible in principle to observe the decays of such nuclei, which occur with a large energy release. On the other hand it is known that for decays of ordinary nuclei in the case when the decay electrons are relativistic we have the relation[1] $\tau_c \sim 1/E_{\max}^5$, where τ_c is the lifetime of the nucleus and $E_{\max}$ is the maximum energy of the decay electrons. If we assume that this relation will be satisfied also for decays of superdense nuclei, then for a maximum decay-particle energy $E_{\max} = 18$ MeV, τ_c will be 1.6 times less than the lifetime of N_7^{12} and will amount to ≈ 6.7 nsec, and for $E_{\max} = 36$ MeV $\tau_c = 0.2$ nsec, etc. Measurement of these lifetimes and decay energies can be carried out very satisfactorily by means of bubble chambers.

The advantages of the bubble-chamber technique are a 4π geometry, the possibility of detecting decay particles of various types ($e^\pm$, γ, heavier particles), the accurate measurement of their energies, and also the possibility of observing "explosions" of superdense nuclei which result in stars recorded in the chamber. A major advantage of this experimental arrangement is the absence of any ordinary physical process imitating the effect.

Up to the present time there has been no experimental proof of the existence of superdense nuclei. Kulikov and Pontecorvo[2] presented some data obtained by an electronic technique on determination of the upper limits of the cross sections for production of superdense nuclei, as a function of their lifetime. It is evident from these data that the region of lifetimes $\tau_c \leq 5$ msec has not yet been investigated.

The experiments described in the present paper were intended to search for unusual decays, which can arise from superdense nuclei, with energy more than 16.4 MeV (the maximum energy of the decays known up to this time) and occurring with lifetimes 0.5–1000 msec.

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