the isotropic singularity and making a change of variables to the variables H, ε, σ, H_α, and σ^α, we obtain the solution (2.15). It is shown at the same time that no other solutions exist near the isotropic singularity.

1) We use a system of units in which the velocity of light and the gravitational constant are each equal to unity. The metric is written in the form $-ds^2 = g_{ik}dx^i dx^k$, where g_{ik} has the signature $(-+++)$. The Latin indices run from 0 to 3; the Greek indices, from 1 to 3.

2) In the present paper we neglect the effect of the thermal fluxes. Such fluxes do not, in fact, arise in the homogeneous models. In the more general cases it must be assumed that we are considering matter with a sufficiently small coefficient of thermal conductivity. Equations that also take account of the effects of thermal conduction can be found in Ref. 3.

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Influence of collisionless particles on the growth of gravitational perturbations in an isotropic universe

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It is shown that the kinetics of the interaction of gravitational perturbations with collisionless particles (neutrinos) in the ultrarelativistic stage of expansion of the universe leads to a behavior of long-wavelength gravitational perturbations which is qualitatively different from that obtained by Lifshitz in 1946 if the energy density of the collisionless particles is more than 5/32 of the total energy density. An important feature of the new long-wavelength asymptotic behaviors is their oscillatory nature. The asymptotic behavior is also found of high-frequency perturbations in an isotropic universe when allowance is made for the influence on the perturbations of the gas of collisionless particles.

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INTRODUCTION

In 1946, Lifshitz[1] solved the problem of the gravitational stability of the isotropic relativistic cosmological model of the universe. He assumed that the matter of a hydrodynamic model with isotropic energy–momentum tensor. The results obtained in Ref. 1 concerning the rate of growth of perturbations were subsequently widely used in studies into the theory of the formation of the large-scale structure of the universe.

In Refs. 2–5 the analogous problem was solved under the assumption that the matter of the universe can be treated in the framework of the model of a collisionless gas, i.e., a model described by a collisionless kinetic equation. This model of the matter is valid in the cases when the characteristic frequency ω of the investigated processes is much higher than the collision frequency ν of the particles of the matter ($\omega \gg \nu$). The hydrodynamic model of matter used in Ref. 1 is valid if $\omega \ll \nu$.

As is shown in Refs. 3–5, the model of a collisionless gas and the hydrodynamic model of matter lead to very different asymptotic behaviors of perturbations in an isotropic universe. This example suggests that if the universe contains not only matter described by the hydrodynamic model but also a gas of collisionless particles, then this gas could have a significant influence on the rate of growth (or damping) of perturbations.

In a hot universe, a gas of muonic and electronic neutrinos is collisionless.[6] The muonic neutrinos become collisionless $\tau = 0.01$ sec after the start of expansion of the universe, while the electronic neutrinos become collisionless at $\tau = 0.2$ sec (see Ref. 6).

Zel'dovich and Novikov[6] also give relations for the equilibrium energy density of different particles in the universe at a time close to the time of "switching off" of the muonic neutrinos:

$$\varepsilon_\gamma : \varepsilon_{e^\pm} : \varepsilon_{\nu_e, \bar\nu_e} : \varepsilon_{\nu_\mu, \bar\nu_\mu} : \varepsilon_{\mu^\pm} = 1 : {}^7/_4 : {}^7/_8 : {}^7/_8 : 10^{-4}. \tag{1}$$

These ratios remain valid until the electron–positron pairs are annihilated. It follows from the ratios (1) that the ratio α of the energy density of the collisionless particles to the energy density of the collisional particles at the times $0.01 < \tau < 0.2$ sec is

$$\alpha_1 = \frac{\varepsilon_{\nu_\mu, \bar{\nu}_\mu}}{\varepsilon_\gamma + \varepsilon_{e\pm} + \varepsilon_{\nu_e, \bar{\nu}_e}} = \frac{7}{29}. \tag{2}$$

From $\tau = 0.2$ sec and later, this ratio becomes even larger, since at $\tau = 0.2$ sec the electronic neutrinos are "switched off," i.e., they become collisionless:

$$\alpha_2 = \frac{\varepsilon_{\nu_\mu, \bar{\nu}_\mu} + \varepsilon_{\nu_e, \bar{\nu}_e}}{\varepsilon_\gamma + \varepsilon_{e\pm}} = \frac{7}{11}. \tag{3}$$

After annihilation of the electron–positron pairs, the value of α decreases to $\alpha_3 = 0.46$ (Ref. 6). Note that the influence of gravitons is not taken into account in Eqs. (2) and (3). Because of the weak interaction between the gravitons and the remaining particles, the gravitons are also collisionless. Therefore, allowance for gravitons would increase the ratios (2) and (3).

Thus, it is clearly necessary to solve the problem of the development of gravitational perturbations in an isotropic relativistic model of the universe filled with two components—a perfect liquid and a collisionless gas. The present paper is devoted to the solution of this problem.

Among the results, we mention in the introduction the following. During the ultrarelativistic stage of expansion of the universe, the asymptotic behavior of long-wavelength perturbations depends essentially on the parameter α. For $\alpha < \alpha^* = 5/27$, the perturbations behave qualitatively in the same way as in the model universe filled with perfect liquid. For $\alpha > \alpha^*$, the asymptotic behavior of the perturbations is qualitatively different from that in an isotropic universe filled with a perfect liquid. A characteristic feature of the new asymptotic behaviors is their oscillatory nature.

We see from (2) and (3) that in the real universe the condition $\alpha > \alpha^*$ is satisfied, and therefore the asymptotic behaviors of long-wavelength perturbations during the ultrarelativistic stage of expansion of the universe differ qualitatively from the behavior found in Ref. 1.

1. DERIVATION OF EQUATIONS FOR THE PERTURBATIONS

The behavior of a gravitating system consisting of a mixture of a perfect liquid and a collisionless gas is described by the system of Einstein equations in which the right-hand side is the sum of the energy–momentum tensors of the liquid and the gas. This system of equations admits a solution describing isotropic cosmological models. Because the structure of the energy–momentum tensor of a collisionless gas, whose distribution function in the unperturbed state has the form of Eq. (5) of Ref. 2, is the same as that of the energy–momentum tensor of the perfect liquid [see, for example, Eq. (6) in Ref. 2], the equation for the scale factor $a(\eta)$ of the cosmological model has the same form as for the perfect-liquid model:

$$a'^2 = \frac{8\pi k}{3c^4} a^4 \varepsilon,$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2$, in which ε_1 is the energy density of the gas, and ε_2 is the energy density of the liquid. Here and below, we shall consider only the spatially flat cosmological model with the metric

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2). \tag{4}$$

During the ultrarelavistic stage of the expansion, when $\varepsilon_1 = 3P_1 = \text{const}/a^4$ and $\varepsilon_2 = 3P_2 = \text{const}/a^4$, the solution for $a(\eta)$ has the form $a = a_1\eta$, where

$$a_1 = \left(\frac{8\pi k}{3c^4} a^4 \varepsilon\right)^{1/2} = \text{const}. \tag{5}$$

We consider small perturbations of the metric δg_{ij} and the distribution function $\delta f(x, p)$. Following Ref. 1, we impose on the perturbations of the metric the gauge conditions $\delta g_{i4} = 0$, and introduce the notation

$$h_{\alpha\beta} = -\delta g_{\alpha\beta}, \quad h_\beta^\alpha = \gamma^{\alpha\delta} h_{\delta\beta} = \frac{1}{a^2} h_{\alpha\beta} \quad (\alpha, \beta, \gamma, \delta, \ldots = 1, 2, 3).$$

Calculating the perturbations of the components of the energy-momentum tensor of the gas and the liquid, and also the perturbations of the left-hand side of the Einstein equations, we obtain, as in Refs. 1, 2, and 7, the system of linearized Einstein equations and the linearized kinetic equation for $\delta f(x, p)$:

$$h_{\alpha,\beta}^{\beta,\alpha} - h_{,\alpha}^{,\alpha} + 2\frac{a'}{a} h' = \frac{16\pi k}{c^3 a^2} \int d^3q\, (m^2c^2a^2 + q^2)^{1/2} f + \frac{16\pi k}{c^4} a^2 \delta\varepsilon_2,$$

$$h^{\cdot\alpha} - h_\beta^{\cdot\alpha,\beta} = \frac{16\pi k}{c^3 a^2} \int d^3q\, q^\alpha f + \frac{16\pi k}{c^4} a^3 (\varepsilon_2 + P_2)\, \delta u^\alpha,$$

$$h_{\alpha,\gamma}^{\gamma,\beta} + h_{\gamma,\alpha}^{\beta,\gamma} - h_{\alpha,\gamma}^{\beta,\gamma} - h_{,\alpha}^{,\beta} + h_\alpha^{\beta\,\prime\prime} + 2\frac{a'}{a} h_\alpha^{\beta\,\prime} - (h_{\gamma,\delta}^{\delta,\gamma} - h_{,\gamma}^{,\gamma})\delta_\alpha^\beta - h''\delta_\alpha^\beta - 2\frac{a'}{a} h'\delta_\alpha^\beta \tag{6}$$

$$= \frac{16\pi k}{c^3 a^2} \int \frac{d^3q\, q_\alpha q^\beta}{(m^2c^2a^2 + q^2)^{1/2}} f + \frac{16\pi k}{c^4} a^2 \delta_\alpha^\beta \frac{dP_2}{d\varepsilon_2} \delta\varepsilon_2,$$

$$(m^2c^2a^2 + q^2)^{1/2} \frac{\partial f}{\partial \eta} + q^\alpha \frac{\partial f}{\partial x^\alpha} = \frac{1}{2q} \frac{df_0}{dq} (m^2c^2a^2 + q^2)^{1/2} h_\beta^{\alpha\,\prime} q_\alpha q^\beta.$$

Here, we have introduced the notation $h = h_\alpha^\alpha$, $q^\alpha = a^2 p^\alpha$, $q_\alpha = \delta_{\alpha\beta} q^\beta$, $q^2 = q_\alpha q^\alpha$, p^α are the spatial momentum components of the particles, k is the gravitational constant, and δu^α are the perturbations of the velocity components of the liquid. Both superscripts and subscripts following the comma denote simple derivatives with respect to the coordinates x^α. The prime denotes differentiation with respect to η. In addition, in (6) we have introduced the function f:

$$f = \delta f(x, p) - \frac{1}{2q} h_\beta^\alpha q_\alpha q^\beta \frac{df_0}{dq},$$

where f_0 is the distribution function of the collisionless gas in the unperturbed state. As is shown in Ref. 8, the function f_0 depends only on q. In all expressions, summation over repeated indices is understood.

Below, we consider perturbations during the ultrarelativistic stage of the expansion of the universe, when $\varepsilon_1 = 3P_1$, $\varepsilon_2 = 3P_2$, $a = a_1\eta$. In this case, we can set $q \gg mca$ in Eqs. (6).

Using a dependence of the perturbations on the spatial coordinates in the form $\exp(in_\alpha x^\alpha)$, where $n_\alpha = \text{const}$, and bearing in mind the explicit form of scalar, vector, and tensor perturbations,[1] we obtain from (6) in the ultrarelativistic limit a system of equations for all types of perturbation.

For scalar perturbations, the system (6) acquires in the ultrarelativistic limit the form

$$\frac{\partial \Phi_s}{\partial t} + ix\Phi_s = \frac{\alpha}{1+\alpha}(\dot{\lambda} + \dot{\mu} - 3x^2\dot{\lambda}), \tag{7a}$$

$$t\dot{\mu}+\frac{t^2}{3}(\mu+\lambda)=-\int_{-1}^{+1}dx\Phi_s+3\frac{\delta\varepsilon_2}{\varepsilon}, \tag{7b}$$

$$i(\dot{\mu}+\dot{\lambda})t^2=-3\int_{-1}^{+1}dx\,x\Phi_s+\frac{12}{1+\alpha}\left(a\frac{n_\alpha}{n}\delta u^\alpha\right), \tag{7c}$$

$$t^2\ddot{\lambda}+2t\dot{\lambda}-\frac{t^2}{3}(\mu+\lambda)=-\int_{-1}^{+1}dx(1-3x^2)\Phi_s, \tag{7d}$$

$$t^2\ddot{\mu}+2t\dot{\mu}+\frac{t^2}{3}(\mu+\lambda)=\int_{-1}^{+1}dx\Phi_s-3\frac{\delta\varepsilon_2}{\varepsilon}. \tag{7e}$$

In deriving these equations, we have introduced the function

$$\Phi_s(x,t)=-\frac{8\pi k}{c^3a_1{}^2}\int_0^{2\pi}d\varphi\int_0^\infty dq\,q^3f(\eta,q,\theta,\varphi)$$

and used the expression for a_1 [see (5)], from which it follows that

$$\frac{8\pi k}{3c^4a_1{}^2}\varepsilon_1a^4=\frac{\alpha}{1+\alpha}$$

where $\alpha=\varepsilon_1/\varepsilon_2$ is the ratio of the energy density of the collisionless gas to the energy density of the perfect liquid. In addition in (7) we have used the notation $x=\cos\theta$, $t=n\eta$, $n^2=n_\alpha n^\alpha=\delta^{\alpha\beta}n_\beta$. The dot in (7) denotes differentiation with respect to t; θ and φ are spherical coordinates in the momentum space. In particular, $n_\alpha q^\alpha=nq\cos\theta$. The functions $\mu(t)$ and $\lambda(t)$ were introduced earlier in Ref. 1.

We note the differential consequences of the system (7), which are important for the following calculations:

$$4\dot{\vartheta}_{\parallel}+i(1+\alpha)g=0, \tag{7f}$$

$$2\dot{\mu}+4i\vartheta_{\parallel}+3(1+\alpha)\dot{g}=0. \tag{7g}$$

Here and below, we use the notation $g=\delta\varepsilon_2/\varepsilon$ and also $\vartheta_{\parallel}$ for the projection of the velocity of the perfect liquid onto the direction of propagation of the wave: $\vartheta_{\parallel}=a(n_\alpha/n)\delta u^\alpha$.

The system of equations for the vector perturbations during the ultrarelativistic stage of expansion of the universe has the form

$$\partial\Phi_v/\partial t+ix\Phi_v=\frac{3\alpha}{1+\alpha}\dot{\sigma}x(1-x^2)^{1/2}, \tag{8a}$$

$$it^2\dot{\sigma}=2\int_{-1}^{+1}dx(1-x^2)^{1/2}\Phi_v-\frac{8}{1+\alpha}a\frac{S_\alpha}{S}\delta u^\alpha, \tag{8b}$$

$$t^2\ddot{\sigma}+2t\dot{\sigma}=-2\int_{-1}^{+1}dx\,x(1-x^2)^{1/2}\Phi_v, \tag{8c}$$

where we have used the notation

$$\Phi_v=-\frac{8\pi k}{c^3a_1{}^2}\int_0^{2\pi}d\varphi\cos\varphi\int_0^\infty dq\,q^3f.$$

A differential consequence of the system of equations (8) is

$$\vartheta_\perp=\text{const}, \tag{8d}$$

where $\vartheta_\perp=a(S_\alpha/S)\delta u^\alpha$ is the projection of the velocity of the liquid onto the polarization vector of the vector perturbation. The result (8d) agrees completely with Lifshitz's result in Ref. 1, in which the physical component of the velocity of the liquid also remains constant for the vector perturbations.

Finally we write down the system of equations for tensor perturbations during the ultrarelativistic stage of expansion of the universe:

$$\frac{\partial\Phi_t}{\partial t}+ix\Phi_t=\frac{3}{2}\frac{\alpha}{1+\alpha}\dot{\nu}(1-x^2), \tag{9a}$$

$$t^2\ddot{\nu}+2t\dot{\nu}+t^2\nu=-\int_{-1}^{+1}dx(1-x^2)\Phi_t. \tag{9b}$$

Here, we have introduced the notation

$$\Phi_t=-\frac{8\pi k}{c^3a_1{}^2}\int_0^{2\pi}d\varphi\sin2\varphi\int_0^\infty dq\,q^3f.$$

To investigate the obtained systems of equations, we use the operation method. We introduce the Laplace transforms

$$z_p=\int_{t_0}^\infty dt\,e^{-pt}z(t),\qquad z(t)=\frac{1}{2\pi i}\int_{-i\infty+b}^{+i\infty+b}dp\,e^{pt}z_p, \tag{10}$$

where $z(t)$ is any of the functions $\mu,\lambda,\nu,Q,=\sigma,g,\vartheta_{\parallel},\vartheta_\perp$.

Determining Φ from the first equations in the systems (7)–(9) and substituting the result in the right-hand side of the remaining equations, we obtain integrodifferential equations for the functions $\mu,\lambda,Q,\nu,g,\vartheta_{\parallel}$, and $\vartheta_\perp$. After the transition in these equations to the Laplace transforms, omitting a number of lengthy but simple calculations (in the case $\alpha\to\infty$, these calculations are given in Ref. 3), we obtain equations for μ_p,λ_p,Q_p, and ν_p. In deriving the equations for μ_p and λ_p, we have used the equations (7f) and (7g).

As a result, the equations for the Laplace transforms in the case of scalar perturbations could be reduced to a second-order differential equation for the new unknown $R(p)$:

$$R''+\left\{\frac{\alpha}{1+\alpha}\left[\frac{3}{(1+p^2)^2}+\frac{4}{1+p^2}+6-18(1+p^2)+9ip(1+p^2)\ln\left(\frac{p-i}{p+i}\right)\right]+\frac{1}{1+\alpha}\left[\frac{3}{(1+p^2)^2}+\frac{4}{1+p^2}-\frac{12}{1+3p^2}\right]\right\}R=p(1+p^2)^{1/2}e^{-pt_0}\psi(p), \tag{11}$$

where

$$\psi(p)=\frac{18}{p}\int_{-1}^{+1}\frac{dx\Phi_c(x,t_0)}{p+ix}-i\frac{\alpha}{1+\alpha}\ln\left(\frac{p-i}{p+i}\right)\left[9\mu_0t_0p^2+\frac{18(\mu_0+\lambda_0)}{p}+3Ap\right]$$
$$+\frac{3\alpha}{1+\alpha}\left[2-ip\ln\left(\frac{p-i}{p+i}\right)\right](9\mu_0t_0p+18\lambda_0+(2+3p^2)A)-\frac{30\mu_0t_0}{p}+18\mu_0t_0^2$$
$$-3\mu_0t_0^3p-At_0^2+\frac{4}{p}t_0A+\frac{54}{1+\alpha}\left[\frac{2p}{1+3p^2}\left(\frac{1}{3}\mu_0t_0+\frac{A}{9p}\right)\right.$$
$$\left.-\frac{1}{(1+3p^2)}(2\mu_0+12i\vartheta_{\parallel,0}p+3(1+\alpha)g_0)+\frac{4i\vartheta_{\parallel,0}}{p}\right], \tag{12}$$

$$A=6\mu_0+9g_0-t_0^2(\mu_0+\lambda_0)-3\int_{-1}^{+1}dx\,\Phi_s(x,t_0),$$

$\mu_0,\lambda_0,g_0,\vartheta_{\parallel,0}$ and $\Phi_s(x,t_0)$ are the values of the perturbations at the time $t_0=n\eta_0$.

The functions μ_p and λ_p can be expressed in terms of R as follows:

$$\mu_p=\frac{1}{9p}y_p'+e^{-pt_0}\left(\frac{1}{3}\mu_0t_0+\frac{A}{9p}\right),\quad \mu_p+\lambda_p=\frac{1}{2}y_p-\frac{3}{2}p^2\mu_p,$$
$$y_p=p(1+p^2)\int dp\frac{R}{p^2(1+p^2)^{1/2}}. \tag{13}$$

From the system of equations for the vector perturbations we obtain, after a Laplace transformation, the equation

$$Q_p''+\frac{\alpha}{1+\alpha}\left[8+12p^2-6ip(1+p^2)\ln\left(\frac{p-i}{p+i}\right)\right]Q_p$$
$$=-2ie^{-pt_0}\int_{-1}^{+1}dx\frac{(1-x^2)^{1/2}\Phi_v(x,t_0)}{p+ix}+\frac{8i\vartheta_\perp}{1+\alpha}\frac{1}{p}e^{-pt_0}. \tag{14}$$

From the system of equations for the tensor perturbations, we obtain the equation

$$U_p''+\left\{\frac{\alpha}{1+\alpha}\left[-\frac{1}{(1+p^2)^2}+1-3(1+p^2)+\frac{3}{2}ip(1+p^2)\ln\left(\frac{p-i}{p+i}\right)\right]\right.$$
$$\left.-\frac{1}{1+\alpha}\frac{1}{(1+p^2)^2}\right\}U_p=e^{-pt_0}\frac{\gamma(p)}{(1+p^2)^{1/2}}, \tag{15}$$

where

$$v_p=U_p/(1+p^2)^{1/2},\quad \gamma(p)=-\int_{-1}^{+1}dx\frac{(1-x^2)\Phi_t(x,t_0)}{p+ix}+t_0^2(\dot v_0+pv_0)$$
$$+\frac{\alpha}{1+\alpha}\left[-5p-3p^3+\frac{3}{2}i(1+p^2)^2\ln\left(\frac{p-i}{p+i}\right)\right]v_0,\quad \dot v_0=\left(\frac{dv}{dt}\right)_{t=t_0}.$$

2. FINDING OF THE ASYMPTOTIC BEHAVIOR OF PERTURBATIONS FOR $t=n\eta\ll 1$

Thus, the finding of small perturbations in an isotropic universe has been reduced to the solution of the linear second-order differential equations (11), (14), and (15) and integration in the complex plane in accordance with the second equation of (10).

To find the perturbations in the limit $t=n\eta\ll 1$, we replace the path of integration $\mathrm{Re}\,p=b$ in (10) by $\mathrm{Re}\,p=A$, where A is a sufficiently large positive number. For $\mathrm{Re}\,p>b$, the functions z_p are analytic, and therefore the value of the integral (10) is not changed by such a displacement of the contour. To calculate the integral (10) along $\mathrm{Re}\,p=A$, it is sufficient to know the expansion of z_p in the limit $|p|\to\infty$.

We seek the solution of Eqs. (11), (14), and (15) as $|p|\to\infty$ as follows. We expand the coefficients of Eqs. (11), (14), and (15) in powers of $1/p$, after which these equations take the form

$$V''+\frac{1}{p^2}\left(\sum_{k=0}^{\infty}\frac{a_k}{p^{2k}}\right)V=e^{-pt_0}\left(\sum_{n=0}^{\infty}\frac{b_n}{p^{n+l}}\right), \tag{16}$$

where $V=R$, $l=-3$ for scalar perturbations, $V=Q_p$, $l=1$ for vector perturbations, and $V=U_p$, $l=0$ for tensor perturbations. In all three cases, the coefficient a_0 is $8\alpha/5(1+\alpha)$.

We write down some of the first coefficients b_n. For scalar perturbations,

$$b_0=-3\mu_0t_0^3,\quad b_1=(18\mu_0-A)t_0^2,$$
$$b_2=-\frac{114\alpha+90}{5(1+\alpha)}\mu_0t_0+4t_0A+\frac{1}{2}b_0,$$
$$b_3=-6t_0^2(\mu_0+\lambda_0)-\frac{18\alpha+10}{5(1+\alpha)}A+\frac{1}{2}b_1,$$
$$b_4=-18i\int_{-1}^{+1}dx\,x\Phi_s(x,t_0)-\frac{108\alpha+140}{35(1+\alpha)}\mu_0t_0+\frac{72i\vartheta_{\parallel,0}}{1+\alpha}+\frac{1}{2}b_2-\frac{3}{8}b_0.$$

For vector perturbations

$$b_0=-2i\int_{-1}^{+1}dx(1-x^2)^{1/2}\Phi_v(x,t_0)+\frac{8i\vartheta_\perp}{1+\alpha},$$
$$b_1=-2\int_{-1}^{+1}dx\,x(1-x^2)^{1/2}\Phi_v(x,t_0).$$

For tensor perturbations,

$$b_0=t_0^2v_0,\quad b_1=t_0^2\dot v_0.$$

The expressions for the coefficients a_k are

$$a_k=\frac{(-1)^{k+1}}{1+\alpha}\left\{\frac{36\alpha}{(2k+3)(2k+5)}+3k(\alpha-1)-4\alpha-1+\frac{4}{3^k}\right\},\quad k\geq 1,$$
$$a_0=8/5,$$

in the case of Eq. (11);

$$a_k=\frac{24\alpha}{1+\alpha}\frac{(-1)^k}{(2k+3)(2k+5)}$$

in the case of Eq. (14); and

$$a_k=\frac{\alpha}{1+\alpha}(-1)^k\left[k+2-\frac{6}{(2k+3)(2k+5)}\right]+\frac{(-1)^kk}{1+\alpha}$$

in the case of Eq. (15).

We find first two independent solutions of the homogeneous equation corresponding to Eq. (16). For $\alpha\neq 5/27$, these solutions have the form

$$V_\pm=\sum_{m=0}^{\infty}\frac{A_m^\pm}{p^{2m+q_\pm}}, \tag{17}$$

where

$$q_\pm=-{}^1/_2\pm i\beta\quad\text{for}\quad\alpha>{}^5/_{27},$$
$$q_\pm=-{}^1/_2\pm\beta\quad\text{for}\quad\alpha<{}^5/_{27},$$
$$\beta=\left|\frac{27\alpha-5}{20(1+\alpha)}\right|^{1/2}.$$

For $\alpha=5/27$, the solutions of the homogeneous equation corresponding to (16) have a form that differs from (17). For the time being, we shall not consider this case.

The coefficients $A_m^\pm$ in (17) are determined by the recursion relations

$$A_m^\pm=-\sum_{l=1}^{m}a_lA_{m-l}^\pm\Big/\left[(2m+q_\pm)(2m+q_\pm+1)+\frac{8}{5}\frac{\alpha}{1+\alpha}\right]. \tag{18}$$

We find the general solution of Eq. (16), which in the limit $|p|\to\infty$ has the form $\alpha(p)\exp(-pt_0)$, where $\alpha(p)\to 0$ as $\mathrm{Re}\,p\to+\infty$ [this requirement follows directly from the definition for the Laplace transforms (10)], in accordance with the formula

$$V=V_-\int_\infty^p\frac{V_+\Omega}{W}dp-V_+\int_\infty^p\frac{V_-\Omega}{W}dp,$$

where

$$\Omega=e^{-pt_0}\sum_{n=0}^{\infty}\frac{b_n}{p^{n+l}}\qquad W=V_+V_-'-V_+'V_-.$$

Finally, the expressions for μ_p and λ_p can be reduced to the form ($a_{-1}^\pm=0$)

$$\mu_p=(1-\hat{S})\Big\{\sum_{\substack{l,R=0\\(l+k\geqslant 1)}}^{\infty}\frac{(2l+q_-+1)(a_{l-1}^-+a_l^-)}{9p^{2l+q_-+3}}\int\limits_{\infty}^{p}dp\,p^{1+q_-}\Big[\frac{F_k^+}{p^{2k}}+\frac{E_k^+}{p^{2k+1}}\Big]e^{-pt_0}$$

$$+\frac{(1+q_-)(3+q_-)a_0^-}{9t_0p^{3+q_-}}\Big[E_0^++\frac{4+q_-}{t_0}F_0^+\Big]\int\limits_{\infty}^{p}dp\,p^{2+q_-}e^{-pt_0}\Big\}$$

$$+\frac{(1+3p^2)}{9p}\sum_{\substack{l,k=0\\(l+k\geqslant 1)}}^{\infty}\int\limits_{\infty}^{p}dp\Big[\frac{\Delta_{lk}^F}{p^{2l+2k}}+\frac{\Delta_{lk}^E}{p^{2l+2k+1}}\Big]e^{-pt_0}$$

$$+\frac{1}{9p}\int\limits_{\infty}^{p}dp\Big[\Delta_{00}^F+\frac{\Delta_{00}^E}{p}\Big]e^{-pt_0}-\frac{\Delta_{00}^E}{3t_0}p\int\limits_{\infty}^{p}\frac{dp}{p^2}e^{-pt_0}+\frac{A}{9p}e^{-pt_0},\quad(19)$$

$$\mu_p+\lambda_p=(1-\hat{S})\Big\{\sum_{\substack{l,k=0\\(l+k\geqslant 2)}}^{\infty}\frac{(2l+q_++4)(a_{l-1}^++a_l^+)}{6p^{2l+q_++1}}\int\limits_{\infty}^{p}dp\,p^{1+q_+}\Big[\frac{F_k^-}{p^{2k}}+\frac{E_k^-}{p^{2k+1}}\Big]e^{-pt_0}$$

$$+\frac{D^+}{p^{1+q_+}}\int\limits_{\infty}^{p}dp\,p^{q_+}e^{-pt_0}+\frac{E^+}{p^{2+q_+}}\int\limits_{\infty}^{p}dp\,p^{2+q_+}e^{-pt_0}\Big\}$$

$$+\frac{1}{3}p\sum_{\substack{l,k=0\\(l+k\geqslant 1)}}^{\infty}\int\limits_{\infty}^{p}dp\Big[\frac{\Delta_{lk}^F}{p^{2l+2k}}+\frac{\Delta_{lk}^E}{p^{2l+2k+1}}\Big]e^{-pt_0}-\frac{\Delta_{00}^E}{3t_0}p\int\limits_{\infty}^{p}\frac{dp}{p^2}e^{-pt_0}.\quad(20)$$

Here, $\hat{S}$ is an operator which interchanges the indices + and −. For example,

$$(1-\hat{S})A^-F_k^+q_-=A^-F_k^+q_--A^+F_k^-q_+,$$

etc. The coefficients $F_k^\pm$ and $E_k^\pm$ are determined from the relations

$$\sum_{m=0}^{l}A_m^\pm b_{2(l-m)}=\sum_{m=0}^{l}\sum_{n=0}^{l-m}(2m-2n+q_+-q_-)A_m^+A_n^-F_{l-m-n}^\pm,\quad(21a)$$

$$\sum_{m=0}^{l}A_m^\pm b_{2(l-m)+1}=\sum_{m=0}^{l}\sum_{n=0}^{l-m}(2m-2n+q_+-q_-)A_m^+A_n^-E_{l-m-n}^\pm,\quad(21b)$$

and the coefficients $a_l^\pm$ and Δ_{lk} from

$$a_l^\pm=\sum_{n=0}^{l}\frac{(-1)^n(2n+1)!!}{2^nn!}\frac{A_{l-n}^\pm}{(2l+q_\pm+4)},\quad(21c)$$

$$\Delta_{lk}^F=a_l^-F_k^++a_l^+F_k^-,\ \Delta_{lk}^E=a_l^-E_k^+-a_l^+E_k^-.\quad(21d)$$

The coefficients D^+ and E^+ have the form

$$D^+=\frac{(4+q_+)(1+q_+)a_0^-}{6t_0^4}[(4+q_+)(3+q_+)(2+q_+)F_0^-$$
$$+(3+q_+)(2+q_+)E_0^-t_0+(2+q_+)F_1^-t_0^2+E_1^-t_0^3],$$
$$E^+=\frac{(6+q_+)(3+q_+)(a_0^++a_1^+)}{6t_0^2}[(4+q_+)F_0^-+t_0E_0^-].$$

We also write down an expansion in the limit $|p|\to\infty$ for the functions Q_p and ν_p $[(-1)!!\equiv 1]$:

$$Q_p=(1-\hat{S})\sum_{m,n=0}^{\infty}\frac{A_m^-}{p^{2m+q_-}}\int\limits_{\infty}^{p}dp\,p^{q_-}\Big[\frac{F_k^+}{p^{2k}}+\frac{E_k^+}{p^{2k+1}}\Big]e^{-pt_0},\quad(22)$$

$$\nu_p=(1-\hat{S})\sum_{n,m,k=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^nn!}\frac{A_m^-}{p^{2m+2n+1+q_-}}\cdot$$
$$\int\limits_{\infty}^{p}dp\,p^{1+q_-}\Big[\frac{F_k^+}{p^{2k}}+\frac{E_k^+}{p^{2k+1}}\Big]e^{-pt_0}.\quad(23)$$

Here, the coefficients $F_k^\pm$ and $E_k^\pm$ can also be determined from the expressions (21a) and (21b), in which it is necessary to substitute the coefficients b_n and a_k corresponding to the considered case.

Integrating term by term the terms of the series (19), (20), (22), and (23) by means of the relation

$$\frac{1}{2\pi i}\int\limits_{-i\infty+b}^{+i\infty+b}dp\,e^{pt}\frac{1}{p^{n+q}}\int\limits_{\infty}\frac{dp}{p^{m-q}}e^{-pt_0}=-t_0^{m-q-1}\int\limits_{t_0}^{t}dt_1\int\limits_{t_0}^{t_1}dt_2\ldots\int\limits_{t_0}^{t_{m+n-2}}\frac{dt_{m+n-1}}{t_{m+n-1}^{m-q}}$$

and restricting ourselves to the first terms of the resulting expansions (in powers of $t=n\eta$), we arrive at the following results in the limit $t=n\eta\ll 1$ (here, we omit a number of lengthy calculations which repeat almost literally the calculations given for the case $\alpha\to\infty$ in Ref. 5); in particular, for the given case ($\alpha\neq\infty$) Eqs. (5.5), (5.6), and (5.8) of Ref. 5 remain valid, and in (5.7) for $Q(t)$ there is added the term $8i\vartheta_\perp/t_0^2(1+\alpha)$; using (18), (21), and the expressions for b_n and a_k to calculate the coefficients $\mathscr{L}^\pm$, $\varphi^\pm$, and $\psi^\pm$ in Eqs. (5.5)–(5.8) of Ref. 5, we arrive at Eqs. (24)–(31).

1) In the case $\alpha<5/27$

$$\mu=\mu_0+\frac{it_0(I_1-4\vartheta_{\parallel,0}/(1+\alpha))}{(9-4\beta^2)(49-4\beta^2)}\Big\{160\Big[\Big(\frac{t}{t_0}\Big)^{1/2}\mathrm{ch}\Big(\beta\ln\frac{t}{t_0}\Big)-1\Big]$$
$$-\frac{168+32\beta^2}{\beta}\Big(\frac{t}{t_0}\Big)^{1/2}\mathrm{sh}\Big(\beta\ln\frac{t}{t_0}\Big)\Big\},\quad(24)$$

$$\mu+\lambda=\mu_0+\lambda_0+\frac{3i}{\beta t_0}\Big(I_1-\frac{4\vartheta_{\parallel,0}}{1+\alpha}\Big)\Big(\frac{t_0}{t}\Big)^{1/2}\mathrm{sh}\Big(\beta\ln\frac{t}{t_0}\Big),\quad(25)$$

$$\sigma=\sigma_0+\Big[\frac{4I_3}{\beta(1-4\beta^2)}-\frac{2iI_2}{\beta t_0}+\frac{8i\vartheta_\perp}{\beta t_0(1+\alpha)}\Big]\Big(\frac{t_0}{t}\Big)^{1/2}\mathrm{sh}\Big(\beta\ln\frac{t}{t_0}\Big)$$
$$+\frac{8I_3}{1-4\beta^2}\Big[\Big(\frac{t_0}{t}\Big)^{1/2}\mathrm{ch}\Big(\beta\ln\frac{t}{t_0}\Big)-1\Big],\quad(26)$$

$$\nu=\nu_0\Big(\frac{t_0}{t}\Big)^{1/2}\mathrm{ch}\Big(\beta\ln\frac{t}{t_0}\Big)+\frac{1}{2\beta}(2t_0\dot{\nu}_0+\nu_0)\Big(\frac{t_0}{t}\Big)^{1/2}\mathrm{sh}\Big(\beta\ln\frac{t}{t_0}\Big),\quad(27)$$

where

$$I_1=\int\limits_{-1}^{+1}dx\,x\Phi_s(x,t_0),\quad I_2=\int\limits_{-1}^{+1}dx(1-x^2)^{1/2}\Phi_\nu(x,t_0),$$
$$I_3=\int\limits_{-1}^{+1}dx\,x(1-x^2)^{1/2}\Phi_\nu(x,t_0).$$

In the limit $\alpha\to 0$ we obtain from (24)–(27), up to "fictitious" transformations (see Ref. 1), the solutions obtained in Ref. 1.

2) In the case $\alpha>5/27$

$$\mu=\mu_0+\frac{it_0(I_1-4\vartheta_{\parallel,0}/(1+\alpha))}{(9+4\beta^2)(49+4\beta^2)}\Big\{160\Big[\Big(\frac{t}{t_0}\Big)^{1/2}\cos\Big(\beta\ln\frac{t}{t_0}\Big)-1\Big]$$
$$-\frac{168-32\beta^2}{\beta}\Big(\frac{t}{t_0}\Big)^{1/2}\sin\Big(\beta\ln\frac{t}{t_0}\Big)\Big\},\quad(28)$$

$$\mu+\lambda=\mu_0+\lambda_0+\frac{3i}{\beta t_0}\Big(I_1-\frac{4\vartheta_{\parallel,0}}{1+\alpha}\Big)\Big(\frac{t_0}{t}\Big)^{1/2}\sin\Big(\beta\ln\frac{t}{t_0}\Big),\quad(29)$$

$$\sigma=\sigma_0+\Big[\frac{4I_3}{\beta(1+4\beta^2)}-\frac{2iI_2}{\beta t_0}+\frac{8i\vartheta_\perp}{\beta t_0(1+\alpha)}\Big]\Big(\frac{t_0}{t}\Big)^{1/2}\sin\Big(\beta\ln\frac{t}{t_0}\Big)$$
$$+\frac{8I_3}{1+4\beta^2}\Big[\Big(\frac{t_0}{t}\Big)^{1/2}\cos\Big(\beta\ln\frac{t}{t_0}\Big)-1\Big],\quad(30)$$

$$\nu=\nu_0\Big(\frac{t_0}{t}\Big)^{1/2}\cos\Big(\beta\ln\frac{t}{t_0}\Big)+\frac{1}{2\beta}(2t_0\dot{\nu}_0+\nu_0)\Big(\frac{t_0}{t}\Big)^{1/2}\sin\Big(\beta\ln\frac{t}{t_0}\Big).\quad(31)$$

In the limit $\alpha \to \infty$, we obtain from (28)–(31) the asymptotic behaviors obtained previously in Refs. 4 and 5 for long-wavelength perturbations in an isotropic universe consisting of a collisionless gas. (In the expression for μ obtained in Refs. 4 and 5 there are some minor misprints.)

In the foregoing calculations, we have not considered yet the case $\alpha = 5/27$. This case can be treated separately by calculations similar to those made above. However, the results for $\alpha = 5/27$ can be obtained directly from (24)–(27) or (28)–(31) by going to the limit $\beta \to 0$. As a result, for $\alpha = 5/27$ we have

$$\mu = \mu_0 + \frac{160}{441} it_0 \left(I_1 - \frac{27}{8}\vartheta_{\parallel,0}\right)\left[\left(\frac{t}{t_0}\right)^{3/2} - 1\right] - \frac{56}{147} it_0 \left(I_1 - \frac{27}{8}\vartheta_{\parallel,0}\right)\left(\frac{t}{t_0}\right)^{3/2} \ln\frac{t}{t_0}, \tag{32}$$

$$\mu + \lambda = \mu_0 + \lambda_0 + \frac{3i}{t_0}\left(I_1 - \frac{27}{8}\vartheta_{\parallel,0}\right)\left(\frac{t_0}{t}\right)^{1/2} \ln\frac{t}{t_0}, \tag{33}$$

$$\sigma = \sigma_0 + 8I_3\left(\left(\frac{t_0}{t}\right)^{1/2} - 1\right) + \left[4I_3 - \frac{2iI_2}{t_0} + \frac{27}{4}\frac{i\vartheta_\perp}{t_0}\right]\left(\frac{t_0}{t}\right)^{1/2} \ln\frac{t}{t_0}, \tag{34}$$

$$\nu = \nu_0\left(\frac{t_0}{t}\right)^{1/2} + \frac{1}{2}(2t_0\dot{\nu}_0 + \nu_0)\left(\frac{t_0}{t}\right)^{1/2} \ln\frac{t}{t_0}. \tag{35}$$

We see from Eqs. (24), (28), and (32) that the scalar perturbations contain terms that grow in accordance with the law η^γ, where γ varies in the range from 3/2 to 2 depending on the parameter α. However, the conditions of applicability of the linearized Einstein equations are

$$\mu \ll 1, \quad \lambda \ll 1, \quad \sigma \ll 1, \quad \nu \ll 1, \tag{36a}$$

$$\dot{\mu} \ll \dot{a}/a, \quad \dot{\lambda} \ll \dot{a}/a, \quad \dot{\sigma} \ll \dot{a}/a, \quad \dot{\nu} \ll \dot{a}/a. \tag{36b}$$

Using these conditions for $\eta = \eta_0$, we obtain the inequalities

$$I_1 - 4\vartheta_{\parallel,0}/(1+\alpha) \ll t_0, \quad I_2 - 4\vartheta_\perp/(1+\alpha) \ll t_0, \tag{36c}$$

from which it follows that long-wavelength perturbations do not attain large values even at the upper limit ($t \approx 1$) of applicability of the expressions (24)–(35). For example, from (28) we obtain $(\mu)_{t=1} \ll t_0^{1/2} \ll 1$.

In Refs. 4 and 5, I did not take into account the conditions (36b) and this led to the incorrect conclusion that there is appreciable growth of scalar perturbations.

Using the asymptotic behaviors for the perturbations of the metric, one can calculate the perturbations of the macroscopic characteristics of the liquid and the gas. In particular, from (7f) and (7g) we obtain for the projection of the vector of the macroscopic velocity of the liquid onto the vector $\boldsymbol{n}_\alpha$ (the longitudinal component of the velocity) and for the perturbation of the energy density of the liquid

$$\vartheta_\parallel = \frac{3^{1/2}}{4}\left\{e^{it/3^{1/2}} \int_{t_0}^{t} dt\, e^{-it/3^{1/2}} \dot{\mu} - e^{-it/3^{1/2}} \int_{t_0}^{t} dt\, e^{it/3^{1/2}} \dot{\mu}\right\} + \vartheta_{\parallel,0} \cos\frac{t - t_0}{3^{1/2}} - \frac{3^{1/2}}{4}(1+\alpha) i g_0 \sin\frac{t - t_0}{3^{1/2}}, \tag{37}$$

$$\frac{\delta\varepsilon_2}{\varepsilon} = g = -\frac{1}{1+\alpha}\left\{e^{it/3^{1/2}} \int_{t_0}^{t} dt\, e^{-it/3^{1/2}} \dot{\mu} + e^{-it/3^{1/2}} \int_{t_0}^{t} dt\, e^{it/3^{1/2}} \dot{\mu}\right\} - \frac{4i\vartheta_{\parallel,0}}{3^{1/2}(1+\alpha)} \sin\frac{t - t_0}{3^{1/2}} + g_0 \cos\frac{t - t_0}{3^{1/2}}. \tag{38}$$

For the transverse velocity $\vartheta_\perp = a(S_\alpha/S)\delta u^\alpha$, we have the result (8d).

From the relations (37) and (38), which hold for all values of $t = n\eta$, we see that in the limit $t \ll 1$ the perturbations of the energy density of the liquid and the longitudinal velocity component of the liquid, like the perturbations of the metric, undergo for $\alpha > 5/27$ slow oscillations as the sine or cosine of the argument $\beta \ln(t/t_0)$. The amplitude of the oscillations for g increases in accordance with the law $\eta^{3/2}$, but this growth does not lead to very large perturbations of the energy density of the liquid because of the conditions (36c).

The perturbations of the components of the energy-momentum tensor of the gas can be calculated by means of Eq. (16) in Ref. 2, and also by means of the linearized Einstein equations obtained at the start of this paper. From these last, the integrals of f can be expressed in terms of the perturbations of the metric and the perturbations of the velocity and energy density of the liquid.

As we see from Eqs. (24)–(31), there exists a critical ratio of the energy density of the collisionless gas to the energy density of the liquid, this being equal to $\alpha^* = 5/27$. If the ratio α is smaller than the critical value, the long-wavelength perturbations behave in qualitatively the same way as in the case $\alpha = 0$ considered in Ref. 1: The perturbations can be represented as a sum of power-law functions of the form $\eta^{(3/2)+\beta}$, $\eta^{(3/2)-\beta}$, $\eta^{(-1/2)+\beta}$, $\eta^{(-1/2)-\beta}$, where the parameter β varies in the range $0 < \beta \leq \frac{1}{2}$. A difference from the solutions of Lifshitz[1] is a slight change in the exponent of η.

For $\alpha > 5/27$, when the energy density of the collisionless gas is more than 15.6% of the total energy density, the perturbations acquire an oscillatory nature, i.e., they differ qualitatively from the solutions obtained in Ref. 1. As follows from Eqs. (2) and (3), the energy density of collisionless neutrinos in the universe is between 16.6% (for $0.01 < \tau < 0.2$ sec) and 38.8% (for $\tau < 0.2$ sec) of the total energy density. Therefore, during the ultrarelativistic stage in the expansion of the universe the asymptotic behaviors of the long-wavelength perturbations have the form (28)–(31), these commencing after the "switching off" of the muonic neutrinos ($\tau = 0.01$ sec). The results for long-wave perturbations during the ultrarelativistic stage obtained in Ref. 1 from the hydrodynamic description of the matter can hold only for $\tau < 0.01$ sec.

Note that during the complete period of the ultrarelativistic stage of expansion of the universe the long-wavelength perturbations can pass through only a few oscillations, since the argument of the sines and cosines in (28)–(31), which is equal to $\beta \ln(t/t_0) = \frac{1}{2}\beta \ln(\tau/\tau_0)$ for $\tau_0 = 0.01$ sec, does not reach values appreciably exceeding 2π during the time of the ultrarelativistic stage.

3. ASYMPTOTIC BEHAVIOR OF THE PERTURBATIONS FOR $t = n\eta \gg 1$

To find the asymptotic behaviors for $t = n\eta \gg 1$, we use the method of integration in the second equation in (10) proposed by Landau to solve the problem of oscillations of an electron plasma (see Ref. 9). We integrate in (10) along a contour displaced far to the left of the points $p = p_k$, where p_k are the singular points of the functions μ_p, λ_p, Q_p and ν_p, and surrounding these points. For sufficiently large $t = n\eta$, the most important contribution to the integral (10) is made by the integration over the part of the contour in the immediate proximity of the points $p = p_k$. Therefore, to calculate the integrals (10) by Landau's method it is sufficient to know the expansions of the functions μ_p, λ_p, Q_p and ν_p in the neighborhood of their singularities. The singular points of the solutions of Eqs. (11), (14), and (15) coincide with the singular points of the coefficients of these equations (see Ref. 10). Therefore, the functions μ_p and λ_p have singularities at the points $p = \pm i$, $p = \pm i/3^{1/2}$, and, as is readily seen from Eqs. (13), at the point $p = 0$. The functions Q_p have singularities at the points $p = \pm i$ and $p = 0$, and the function ν_p at the points $p = \pm i$.

To find the solutions of Eqs. (11), (14), and (15) in the neighborhood of the points $p = \pm i$, we introduce the variables $s = 1 \pm ip$ and expand the coefficients in powers of s. We find the general solutions of the homogeneous equations corresponding to (11), (14), and (15). These solutions have the form

$$\sum_{n,m=0}^{\infty} A_n^m s^{n-1/2}(s^3 \ln s)^m$$

for Eq. (11) and

$$\sum_{n,m=0}^{\infty} A_n^m s^n (s^3 \ln s)^m$$

for Eq. (14). For Eq. (15), the one independent solution has the form

$$U_1 = \sum_{n,m=0}^{\infty} A_n^m s^{n+1/2}(s^3 \ln s)^m,$$

and the second

$$U_2 = U_1 \ln s + \sum_{n,m=0}^{\infty} B_n^m s^{n+1/2}(s^3 \ln s)^m.$$

After this, we find particular solutions of Eqs. (11), (14), and (15) in the form of the series

$$\sum_{n,m=0}^{\infty} A_n^m s^{n+r}(s^3 \ln s)^m.$$

The coefficients A_n^m can be found by substitution of the series in (11), (14), and (15) and by equating the coefficients of equal powers of s and $s^3 \ln s$.

Restricting ourselves to the first terms of the series for μ_p, λ_p, Q_p, and ν_p, we obtain the following expansions in the neighborhood of $s = 0$:

$$\mu_p^\pm = \frac{1}{2} C_1^\pm (1-s) s^2 \ln s + \frac{2}{3} C_2^\pm s^3 \ln s + O^\cdot(s^4 \ln s) + M^\pm, \tag{39}$$

$$\mu_p^\pm + \lambda_p^\pm = \frac{3}{4} C_1^\pm (1-2s) s^2 \ln s + C_2^\pm s^3 \ln s + O^\cdot(s^4 \ln s) + N^\pm, \tag{40}$$

$$Q_p^\pm = \frac{B_1^\pm}{6i} s^3 \ln s + \frac{B_2^\pm}{4!} s^4 \ln s + O^\cdot(s^5 \ln s) + K^\pm, \tag{41}$$

$$\nu_p^\pm = -D_1^\pm \left(1 + \frac{\alpha}{1+\alpha} s\right) \ln s + O^\cdot(s^2 \ln s) + F^\pm, \tag{42}$$

where $M^\pm, N^\pm, K^\pm, F^\pm$ are power series in s, and $C_1^\pm, C_2^\pm, B_1^\pm, B_2^\pm, D_1^\pm$ are constants. The superscript + is appended to solutions in the neighborhood of the points $p = i$; in this case, $s = 1 + ip$. The superscript − is appended to solutions in the neighborhood of the points $p = -i$, and then $s = 1 - ip$.

It remains to determine the expansions of the functions μ_p and λ_p in the neighborhood of the points $p = 0$ and $p = \pm i/3^{1/2}$ and of the function Q_p in the neighborhood of the point $p = 0$. In the neighborhood of the point $p = 0$, the solution of Eq. (11) has the form of a power series in s. Substituting in (13)

$$R = \sum_{n=0}^{\infty} f_n p^n,$$

we obtain

$$\mu_p = \frac{f_1(1+3p^2)}{9p} \ln p + \frac{A}{9p} + L_1, \tag{43a}$$

$$\mu_p + \lambda_p = \frac{f_1}{3} p \ln p + L_2, \tag{43b}$$

where L_1 and L_2 are power series in s.

To solve Eq. (11) in the neighborhood of the points $p = \pm i/3^{1/2}$, we introduce the variables $s = 1 \pm i \cdot 3^{1/2} p$, and expand the coefficients of Eq. (11) in powers of s. The general solution of Eq. (11) can be represented in the form of a linear combination of the solutions R_1 and R_2 of the homogeneous equation plus the particular solution

$$R = C_1 R_1 + C_2 R_2 + R_3,$$

where

$$R_1 = \sum_{n=0}^{\infty} A_n s^{n+1}, \quad R_2 = R_1 \ln s + \sum_{n=0}^{\infty} B_n s^n, \quad R_3 = \sum_{n=0}^{\infty} D_n s^{n+1}.$$

The coefficients A_n, B_n, and D_n are found by substituting these series in (11) and equating the coefficients of equal powers of s.

The first terms of the series in the neighborhood of $p = \pm i/3^{1/2}$ for μ_p and λ_p have the form ($s = 1 \pm i \cdot 3^{1/2} p$)

$$\mu_p^\pm = \mp \frac{i}{3^{1/2}} \left(A_1^\pm + \frac{1+3\alpha}{2(1+\alpha)} A_1^\pm s\right) s \ln s + O^\cdot(s^3 \ln s) + F_1^\pm, \tag{44}$$

$$\mu_p^\pm + \lambda_p^\pm = \mp \frac{i}{2 \cdot 3^{1/2}} \left(A_1^\pm + \frac{\alpha}{1+\alpha} A_1^\pm s\right) s \ln s + O^\cdot(s^3 \ln s) + F_2^\pm, \tag{45}$$

where $A_1^\pm = \text{const}$, and $F_1^\pm$ and $F_2^\pm$ are power series in s.

Equation (14) for Q_p in the neighborhood of the point $p = 0$ has general solution of the form

$$Q_p = \sum_{k=0}^{\infty} c_k p^k + \sum_{k=0}^{\infty} b_k p^k \ln p. \tag{46}$$

The first terms of the expansion (46) have the form

$$Q_p = \frac{8i\vartheta_\perp}{1+\alpha} \left(1 - \frac{4\alpha}{3(1+\alpha)} p^2\right) p \ln p + F_3, \tag{47}$$

where F_3 is a power series in p.

To find the asymptotic behavior of the perturbations for $t = n\eta \gg 1$, we substitute the expansions (39)–(45) and (47) in (10) and calculate the integrals around the contour indicated above. Using the expressions

$$\frac{1}{2\pi i} \int_{C_\pm} dp\,(1 \pm i\gamma p)^n \ln(1 \pm i\gamma p) e^{pt} = \frac{(\mp i)^{n+2} \gamma^n n!}{t^{n+1}} e^{\pm it/\gamma}, \tag{48}$$

where $\gamma=1$ and $3^{1/2}$, and $C_\pm$ are the parts of the contours surrounding the points $p=\pm i/\gamma$, and the expression

$$\frac{1}{2\pi i}\int\limits_{C_0} dp\, e^{pt}p^n \ln p = \frac{(-1)^{n+1}n!}{t^{n+1}}, \tag{49}$$

where C_0 is the part of the contour surrounding the point $p=0$, we arrive at the following results when $t=n\eta\gg 1$:

$$\mu=\frac{1}{t^2}\left[\left(1-i\frac{3^{1/2}(1+3\alpha)}{1+\alpha}\frac{1}{t}\right)A_1^+e^{it/3^{1/2}}\right.$$
$$\left.+\left(1+i\frac{3^{1/2}(1+3\alpha)}{1+\alpha}\frac{1}{t}\right)A_1^-e^{-it/3^{1/2}}\right]+\frac{1}{t^3}(C_1^+e^{it}+C_1^-e^{-it}), \tag{50}$$

$$\mu+\lambda=\frac{1}{2t^2}\left[\left(1-i\frac{2\cdot 3^{1/2}\alpha}{1+\alpha}\frac{1}{t}\right)A_1^+e^{it/3^{1/2}}\right.$$

$$\left.+\left(1+i\frac{2\cdot 3^{1/2}\alpha}{1+\alpha}\frac{1}{t}\right)A_1^-e^{-it/3^{1/2}}\right]+\frac{3}{2t^3}(C_1^+e^{it}+C_1^-e^{-it}), \tag{51}$$

$$Q=\dot\sigma=\frac{8i\theta_\perp}{1+\alpha}\frac{1}{t^2}\left[1-\frac{8\alpha}{1+\alpha}\frac{1}{t^2}\right]+\frac{1}{t^4}(B_1^+e^{it}+B_1^-e^{-it}), \tag{52}$$

$$\nu=\frac{1}{t}(D_1^+e^{it}+D_1^-e^{-it})-\frac{i\alpha}{1+\alpha}\frac{1}{t^2}(D_1^+e^{it}-D_1^-e^{-it}). \tag{53}$$

In obtaining the asymptotic behaviors for μ_p and λ_p, we have not taken into account the contribution from the integrals over the part C_0 of the contour surrounding the point $p=0$, since allowance for the integrals around C_0 of (43) leads to additional terms in (50) and (51) corresponding to fictitious changes of the metric,[1] which, following Ref. 1, we omit.

The first terms in (50) and (51) describe the propagation, with the velocity of sound $c/3^{1/2}$, of scalar high-frequency oscillations in the liquid. The second terms in (50) and (51) describe the propagation with the velocity of light of high-frequency scalar oscillations in the collisionless gas. The amplitude of the oscillations in the gas is damped in accordance with a faster law, $1/\eta^3$, than in the liquid $(1/\eta^2)$. If the liquid is absent, then for scalar perturbations in the collisionless gas we have the results that follow from (50) and (51) for $A_1^\pm=0$ (and not the results (24) and (25) of Ref. 3 and (5.1) and (5.2) of Ref. 5, which were obtained on the basis of the incorrect assumption that the coefficient G_1^0 in (1.4) and (1.5) in Ref. 3 is nonzero). In the absence of the collisionless gas, the results are obtained from (50) and (51) for $C_1^\pm=\alpha=0$ and agree with Lifshitz's well-known results.[1]

Note that for large t the first terms become dominant in (50) and (51). Therefore, a certain time after the occurrence of the initial perturbation the asymptotic behaviors for μ and λ will virtually coincide with the asymptotic behaviors for $t\gg 1$ obtained in Ref. 1 without allowance for the influence of collisionless particles.

As we see in (52), the asymptotic behaviors for vector perturbations for $t\gg 1$ also consist of sums of two terms. The second, oscillating, term in (52) is due to the interaction of the perturbations with the collisionless gas; the first term is due to the interaction with the liquid. For sufficiently large t, after the initial perturbation, the first term in (52) becomes predominant.

Thus, allowance for the influence of collisionless particles in the universe on the development of gravitational perturbations has shown that the asymptotic behaviors of the scalar and vector perturbations in the isotropic universe obtained in Ref. 1 in the limit $n\eta\gg 1$ are the principal terms of the expansions for $n\eta\gg 1$ of perturbations in a universe consisting of a liquid and a collisionless gas. Allowance for the influence of the collisionless gas leads to corrections that are damped much more rapidly than the principal terms of the expansions.

A collisionless gas present in the universe has its most pronounced influence on long-wavelength $(n\eta\ll 1)$ perturbations in an isotropic universe. The asymptotic behaviors of long-wavelength perturbations in an isotropic universe are qualitatively different from those of Ref. 1 if the energy density of the collisionless gas is more than 5/32, i.e., more than 15.6%, of the total energy density.

A characteristic feature of the obtained long-wavelength asymptotic behaviors is the presence in them of oscillating factors of the form $\cos[\beta\ln(\eta/\eta)]$ or $\sin[\beta\ln(\eta/\eta_0)]$, as a result of which the long-wavelength perturbations can pass through one or two oscillations during the ultrarelativistic stage.

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