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"Tangential" conical refraction in a three-dimensionally inhomogeneous weakly anisotropic medium

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The propagation of an electromagnetic wave in a three-dimensionally inhomogeneous weakly anisotropic medium with a real dielectric tensor (deformed crystal, elastically stressed isotropic material, and others) is considered. It is shown that in the vicinity of the points that would correspond in the presence of a boundary surface to internal conical reflection there should occur an intense mutual transformation of linearly polarized waves with mutually perpendicular polarization directions. For the case when the region of wave interaction is much smaller than the scale of the inhomogeneities of the medium, analytic expressions are obtained for the fields and for the transformation coefficients. It is shown that the effect takes the most complete and simplest form only in the cases when the problem has sufficient asymmetry, i.e., it is inhomogeneous, is in not fewer than two dimensions.

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INTRODUCTION

It is known that conical (internal) reflection is produced when a plane wave is incident on a homogeneous crystal in those cases when the refractive indices for the two types of normal waves are equal to each other in the given direction of the wave vector k (Ref. 1):

$$n_1(\mathbf{k}, \hat{\varepsilon}) = n_2(\mathbf{k}, \hat{\varepsilon}); \tag{1}$$

ε is the dielectric tensor of the medium. (Following Ginzburg's book,[2] we define the ordinary wave as the one corresponding to the larger refractive index.)

The purpose of the present paper is to determine the changes that occur in the classical effect if the homogeneous crystal is replaced by an anisotropic smoothly inhomogeneous medium with an inhomogeneity scale L much larger than the wavelength λ. Accordingly, the condition (1) is then satisfied not in the entire volume occupied by the wave, but only at certain points, at which the wave vector $\mathbf{k}(\mathbf{r})$ and the principal axes of the tensor $\hat{\varepsilon}(\mathbf{r})$ have a suitable relative orientation (see Sec. 1). It is clear that this gives rise to a mutual linear transformation of the ordinary and extraordinary waves (see Secs. 1 and 4), and this transformation is similar in many respects to the types of transformation which occur when high-frequency electromagnetic waves pass through a magnetoactive plasma, namely, in the region of quasitransverse propagation,[3-6] as well as in a neutral current layer[7] and in the case of propagation in a direction close to that of the magnetic field.[2]

Since there are no abrupt separation boundaries in the case considered here, the ray scattering characteristic of the classical form of the effect does not occur here and its place is taken by a specific polarization picture which is connected with the wave transformation. In some implicit form, however, the effect leaves a trace also in the ray picture (see Sec. 5).

It is useful to note that effects connected with linear transformation of vibrational and wave modes are known in many divisions of physics,[8] principally in plasma physics. Optics is an exception, and the present article should fill this gap. Linear transformation effects manifest themselves most clearly in nonstationary oscillations of lumped systems[9] and accordingly in nonstationary quantum-mechanics problems connected with adiabatic perturbations.[10] The latter includes, in particular, the umklapp process in beams of polarized particles[11,12]; the optical effect considered here is similar to it in many respects.

The inhomogeneous anisotropic medium (with real $\hat{\varepsilon}$) in which the "tangential" conical refraction takes place need not necessarily be an inhomogeneously deformed crystal. It can be also inhomogeneously deformed glass, in which the optical anisotropy is due to the elasto-optical effect. Another optically inhomogeneous anisotropic material is a moving liquid with an inhomogeneous velocity field, in which the optical anisotropy is due to the Maxwell effect. By dissolving in this liquid a substance with natural optical activity (such as sugar and others) we can obtain an optically anisotropic medium with complex tensor $\hat{\varepsilon}$.

Thus, we consider primarily the case when the tensor $\hat{\varepsilon}$ is real. The presence of gyration (nonzero Im$\hat{\varepsilon}$)

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contributes to a smoothing of the effects, and a noticable action is exerted by the presence of even small gyration (see Sec. 4).

1. GENERAL PICTURE OF THE EFFECT AND DEGENERATE CASES

A. *Localization of the effect.* It is easily seen that condition (1) is satisfied by far not on all rays. In particular, to satisfy this condition it is necessary that the two angle coordinates of the wave vector k coincide with the two angle coordinates of the optical axes. The set of points on the ray, likewise, is not two-dimensional but only one-dimensional. If on the other hand we take a one-parameter family of rays, then the appearance of a finite number of "singular" points of (1) is quite probable. Finally, if we take a two-parameter family of rays (i.e., a wave filling some region of space), then the condition (1) can already be satisfied on a certain one-parameter family of rays. The points (1) form in this case a certain "singular" curve. It is in some sufficiently close vicinity of this curve (in a "singular" zone) that the electromagnetic waves undergo a linear transformation of the type considered.

It should be noted that for problems with sufficiently high degree of symmetry there are quite typical different degenerate cases in which the singular points are either completely absent, or fill a certain two-dimensional region ("singular" surface), or else the singular line coincides with one of the rays. By way of the simplest example, we can cite the case of propagation of a plane wave in a homogeneous anisotropic medium. If the direction of the wave vector does not coincide with the direction of any of the optical axes, then there are no singular points at all, but if it does coincide, then the singular points fill the entire volume.

We now take not a homogeneous medium but a plane-layered one (with a preferred z axis), and leave the wave plane in the x and y directions. For the sake of simplicity we assume that all the optical axes are perpendicular to the y axis. Then singular points do not exist at all at $k_y \neq 0$, but are perfectly possible (at a certain z_0) if $k_y = 0$. In this case the singular points form a singular plane $z = z_0$. By way of another example we can cite cases when an anisotropic medium has spherical symmetry while the wave has axial symmetry with an axis passing through the symmetry center. For the sake of simplicity we assume that refraction (the bending of the rays) is very small, and the wave is almost plane. It is then obvious that all the points on the ray passing through the center are singular, and there are no other singular points in this problem.

It is easy to verify that in the above examples the degeneracies are lifted when the symmetry of the problem is lowered: for example, if the plane-layered medium is replaced by a two-dimensionally inhomogeneous medium, and the spherical symmetry is replaced by axial symmetry. Analytic calculations (in Sec. 4) will be carried out only for nondegenerate cases.

B. *The polarization picture.* From qualitative considerations it is easy to visualize the picture of the transformation in the general case. It recalls in many respects the picture of linear transformation in quasi-transverse propagation in a magnetoactive plasma. It is known that the vectors of the electric induction **D** of the ordinary and extraordinary waves are directed along the principal axes of the two-dimensional part $\eta_{\alpha\beta}$ of the tensor $\hat{\eta} = \hat{\varepsilon}^{-1}$, taken in a plane perpendicular to the wave vector. One of the waves (the ordinary one) is polarized in a direction corresponding to the larger of the eigenvalues $\eta_{\alpha\beta}$, and the other in a perpendicular direction. If the properties of the medium vary sufficiently slowly, then the type of the wave is also preserved[2]: its vector **D** follows the motion of the principal axes of the tensor $\eta_{\alpha\beta}$.

We assume now that the ray passes through a singular point. When this point is approached, the anisotropic part

$$\eta_{\alpha\beta} - {}^1/_2\delta_{\alpha\beta}(\eta_{11} + \eta_{22})$$

of the tensor $\eta_{\alpha\beta}$, the part that determines the polarization direction, is proportional (in the simplest case) to the distance to the singular point, and the directions of the principal axes do not undergo a discontinuity on passing through this point, so that the direction of the wave polarization remains likewise unchanged. On the other hand, a jumplike change takes place in the correspondence between the axes and the eigenvalues: the axis corresponding to the larger eigenvalue $\eta_{\alpha\beta}$ (i.e., to the extraordinary wave) now corresponds after passing through the singular point to the smaller eigenvalue (i.e., to the ordinary wave). Thus, on passing through such a singular point the ordinary wave is completely transformed jumpwise into an extraordinary wave as a result of the actual 90° jump of the principal axes.

We assume now that the ray passes not through the singular point but close enough to it (see Fig. 1). Then on the ray segment closest to the point (having a length l of the order of the impact parameter ρ) the axes are likewise rotated through 90°. If l is much larger than the period Λ of the spatial beats of the ordinary and extraordinary waves, then the polarization of the wave

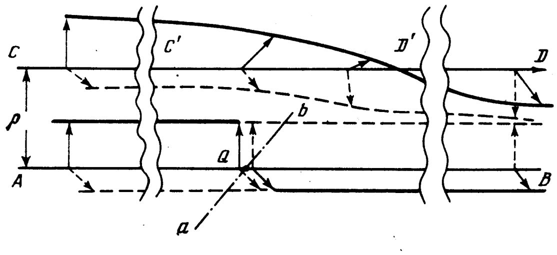

FIG. 1. Rotation of principal axes of two-dimensional real tensor $\eta_{\alpha\beta}$ in the course of motion along the beam. The solid (dashed) arrows indicate the principal-axes unit vectors corresponding to the larger (smaller) eigenvalue. The hodographs of these unit vectors are marked by solid (dashed) lines drawn through the ends of the unit vectors, AB—ray passing through the singular point Q, ab—singular line. A jumplike rotation of the unit vectors takes place at the point Q. CD—ray passing past the singular line; $C'D'$—section where the unit vectors undergo the fastest rotation. Its length is as a rule of the order of the distance ρ between CD and ab.

manages in the main to keep up with the rotation of the axes. On the other hand, if $l \ll \Lambda$, then, on the contrary, the wave hardly has time to react to the rotation of the axes, since the phase shift between the ordinary and extraordinary waves, which builds up on the indicated segment, is small. Thus, the width of the singular zone is determined by that critical value ρ_0 of the parameter ρ at which $\Lambda \sim \rho$.

The quantity ρ_0 can be easily estimated by first representing the tensor $\hat{\varepsilon}$ in the form

$$\varepsilon_{ij}=\varepsilon_0\delta_{ij}+\Delta\varepsilon_{ij},$$

where ε_0 is a certain scalar function that coincides with the quantity $(1/3)\mathrm{Tr}\hat{\varepsilon}$ or is close to it. If we now recognize that

$$\Lambda\sim\lambda n_1|n_1-n_2|^{-1}\sim k^{-1}(\rho\varepsilon_0^{-1/2}\max_{i,j,p}|\partial\Delta\varepsilon_{ij}/\partial x_p|)^{-1}\quad(k=\omega/c),$$

then we obtain for ρ_0 the estimate

$$\rho_0\sim\varepsilon_0^{1/4}k^{-1/2}(\max_{i,j,p}|\partial\Delta\varepsilon_{ij}/\partial x_p|)^{-1/2} \tag{2}$$

C. *Possibility of calculating the effect.* A sufficiently compact (geometrical-optics) method of calculating the described effect can be developed if the medium is weakly anisotropic, i.e., when

$$\Delta\varepsilon=\max_{i,j}|\Delta\varepsilon_{ij}|\ll\varepsilon_0.$$

In this case $\Delta\varepsilon/\varepsilon_0$ is an additional small parameter of the problem, accurate to which the calculation is carried out. For geometrical-optics equations, in turn, it is possible to construct also analytic solutions—in the known particular case when the effect is localized, i.e., when the width ρ_0 of the singular zone is much less than the inhomogeneity scale L:

$$\rho_0^2\max_{i,j,k,l}|\partial^2\Delta\varepsilon_{ij}/\partial x_k\partial x_l|\ll\max_{i,j}|\Delta\varepsilon_{ij}|. \tag{3}$$

For this case, in particular, the estimate (2) will also be confirmed.

2. GEOMETRICAL OPTICS OF WEAKLY ANISOTROPIC MEDIA

If the parameter $\Delta\varepsilon/\varepsilon_0$ is small, then, generally speaking, we can use the quasi-isotropic approximation.[13,5] The rays in this case are assumed to be the same as in an isotropic medium with permittivity $\varepsilon_0(\mathbf{r})$, and the field is calculated from the formulas

$$\mathbf{E}=\Phi_0(\Gamma_1\mathbf{q}_1+\Gamma_2\mathbf{q}_2)e^{i\varphi_0},\quad \varphi_0=k\int\left(\varepsilon_0^{1/2}+\frac{\varepsilon_{11}+\varepsilon_{22}-2\varepsilon_0}{4\varepsilon_0^{1/2}}\right)d\sigma, \tag{4}$$

where $\mathbf{q}_1$ and $\mathbf{q}_2$ are arbitrary real unit vectors that form, together with the tangential unit vector $\mathbf{t}$ an orthogonal right-handed triad; Φ_0 satisfies the energy-flux conservation law

$$\mathrm{div}(\varepsilon_0^{1/2}\Phi_0^2\mathbf{t})=0;$$

$\varepsilon_{11},\ldots,\varepsilon_{22}$ are the components of the tensor ε_{ij} along the unit vectors $\mathbf{q}_1$ and $\mathbf{q}_2$; the quantities Γ_1 and Γ_2, which determine the polarization of the field, are obtained from the equations

$$\frac{d\Gamma_1}{d\sigma}=\frac{ik}{2\varepsilon_0^{1/2}}(\nu_{11}\Gamma_1+\nu_{12}\Gamma_2)+\frac{\Gamma_2}{T_{\mathrm{eff}}},$$
$$\frac{d\Gamma_2}{d\sigma}=\frac{ik}{2\varepsilon_0^{1/2}}(\nu_{21}\Gamma_1+\nu_{22}\Gamma_2)-\frac{\Gamma_1}{T_{\mathrm{eff}}}. \tag{5}$$

Here

$$\nu_{11}=-\nu_{22}={}^1/_2(\varepsilon_{11}-\varepsilon_{22}),\ \nu_{12}=\varepsilon_{12},\ \nu_{21}=\varepsilon_{21},$$
$$T_{\mathrm{eff}}^{-1}=T^{-1}+d\psi/d\sigma,\quad \psi=\arg[\mathbf{q}_1\times(\mathbf{n}+i\mathbf{b})], \tag{6}$$

where n and b are the normal and binormal to the ray, and T is the ray torsion radius ($T^{-1}=\mathbf{b}\cdot d\mathbf{n}/d\sigma$).

Although Eqs. (4)–(6) do make it possible, in principle, to describe the linear wave transformation, in contrast to the traditional geometrical-optics methods,[14,15] their application to our application to our problem entails certain difficulties. The point is that these equations determine the difference between the refractive indices only accurate to quantities of the order of $(\Delta\varepsilon)^2\varepsilon_0^{-3/2}$, and this in turn can lead to an incorrect description of the linear transformation in cases when $|n_1-n_2|\lesssim(\Delta\varepsilon)^2\varepsilon_0^{-3/2}$ (see Ref. 6). This is precisely the case realized here, since the effect is being considered near points where $n_1=n_2$ but $\Delta\varepsilon\neq0$. We therefore must start out not with Eqs. (4)–(6), but with the more accurate geometrical-optics equations derived in Refs. 16 and 6 (a short derivation is given in Appendix 1). They differ from (4)–(6) in the following:

a) The rays are assumed in exact correspondence with the tensor $\hat{\varepsilon}(\mathbf{r})$, i.e., to correspond to an ordinary or an extraordinary wave.

b) φ_0 in formula (4) is replaced by

$$\Phi_0=\varphi_1+\frac{1}{4}k\int\varepsilon_0^{3/2}(n_1^{-2}-n_2^{-2})\,d\sigma, \tag{7}$$

where φ_1 and n_1 are the phase shift and the refractive index corresponding to the chosen type of wave, and $n_2=n_2(\mathbf{k}_1,\varepsilon)$ is the refractive index corresponding to the other type of wave, but to the wave vector $\mathbf{k}_1=\nabla\varphi_1$.

c) The unit vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ form a right-handed triad and the wave vector $\mathbf{k}_1$.

d) Equations (6) are replaced by

$$\nu_{11}=-\nu_{22}={}^1/_2\varepsilon_0^2(\varepsilon_{22}^{-1}-\varepsilon_{11}^{-1}),\quad \nu_{12}=-\varepsilon_0^2\varepsilon_{12}^{-1},\quad \nu_{21}=-\varepsilon_0^2\varepsilon_{21}^{-1}, \tag{6a}$$

where $\varepsilon_{11}^{-1},\ldots,\varepsilon_{22}^{-1}$ denote the components of the tensor $\hat{\varepsilon}^{-1}$ along the unit vectors $\mathbf{q}_1$ and $\mathbf{q}_2$.

Thus, the changes made in formulas (4)–(6) affect only quantities of second order in $\Delta\varepsilon/\varepsilon_0$, but it is precisely these quantities which guarantee here the correctness of the approximation. In certain cases (for example when describing quasitransverse wave propagation in a magnetoactive plasma, these terms make a substantial contribution to the quantities $\nu_{\alpha\beta}$, and therefore also to the transformation.[6] In our case, however, at real $\hat{\varepsilon}$, the inclusion of second-order terms does not influence the minimum of the quantity $|n_1-n_2|$, which remains equal to zero if variation with respect to k is made. Therefore the refinements in this case are small ($\sim\Delta\varepsilon/\varepsilon_0$)—they concern the directions of the optical axes and the details of the polarization picture; this makes it possible to use the quasi-isotropic approximation in its initial form (4)–(6). However, allowance for gyration (which will henceforth be assumed) calls for additional precautions in the use of Eqs. (4)–(6), and we shall therefore assume that the refined method is used, i.e., Eqs. (7), (6a), etc.

Since the tensor ε is assumed to be Hermitian, we can regard T_{eff}^{-1} as an increment to the imaginary part of ν_{12}, and will use in this connection the notation

$$\tilde{\nu}_{12}=\nu_{12}-2i\varepsilon_0^{1/2}(kT_{eff})^{-1};$$

the restrictions on $\tilde{\nu}_{12}$ will be discussed in Sec. 3. Accordingly, the singular points will be defined to be those at which the conditions

$$\nu_{11}=0, \quad \mathrm{Re}\,\nu_{12}=0, \tag{8}$$

which go over into the conditions (1) for real $\hat{\varepsilon}$, are satisfied.

3. LINEARIZATION OF THE COEFFICIENTS OF THE GEOMETRICAL-OPTICS EQUATIONS (CASE OF LOCALIZABILITY OF THE EFFECT)

We now assume that the condition (3) is satisfied and consider the behavior of the wave in the vicinity of some singular point Q. We assume that the size of this vicinity is much less than the inhomogeneity scale L, but much larger than the width ρ_0 of the singular zone. We introduce in this vicinity the curvilinear coordinates α, β, and σ, where σ is the linear coordinate on the ray, while α and β are the parameters of the ray family; we assume here that the points Q we have $\alpha=\beta=\sigma=0$. For example, the coordinate σ can be reckoned on all the rays from a plane passing through Q and normal to the ray; the parameters α and β can be chosen to be the Cartesian coordinates of the points where the rays cross the indicated plane; these coordinates lie on this plane with origin at the point Q and their axes are arbitrarily oriented (see Fig. 2). We assume that within the considered vicinity the contribution of the gyration and of the torsion to the wave picture is not too large:

$$|\mathrm{Im}\,\tilde{\nu}_{12}|\ll|\nu_{11}|+|\mathrm{Re}\,\nu_{12}|. \tag{9}$$

We can then use in this vicinity the expansions

$$\nu_{11}=a\alpha+b\beta+v\sigma+\ldots, \quad \tilde{\nu}_{12}=iD+A\alpha+B\beta+V\sigma+\ldots, \tag{10}$$

where a, b, v, A, B, V, D are real constants, and the dots denote terms of second order in α, β, and σ.

It is easily seen that if the two coefficients v and V in the expansions (10) are equal to zero, then Eqs. (5) lead in first order in σ to plane waves, in other words, to describe the linear transformation in this case we must consider higher terms of the expansions of ν_{11} and $\tilde{\nu}_{12}$ in powers of σ, something we do not propose to do in the present paper. We confine ourselves therefore to the case when v^2+V^2 not only is different from zero, but is nowhere comparable with the slow parameters of the problem.

Another degenerate case occurs when $a:b:v=A:B:V$, i.e., when ν_{11} and $\mathrm{Re}\,\nu_{12}$ are proportional to each other in first order in α, β, and σ. In this case, as can be easily seen from (5), at $\mathrm{Im}\,\tilde{\nu}_{12}=0$ the terms of first order in α, β, and σ likewise do not lead to linear transformations. In other words, in this case the quantities ν_{11} and $\mathrm{Re}\,\nu_{12}$ contribute to the transformation only via the quadratic and higher-order terms of their expansions in α, β, and σ.

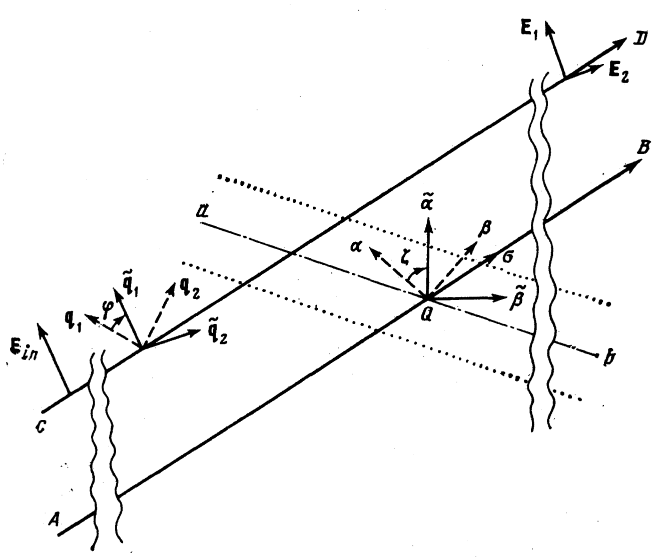

FIG. 2. System of coordinates and of polarization unit vectors in the region where the ray approaches the singular line: the dimensions of the region are much smaller than the inhomogeneity scales; Q, ab, AB, and CD are defined in Fig. 1. $\mathbf{E}_{in}$–electric vector of incident linearly polarized wave; $\mathbf{E}_1$ and $\mathbf{E}_2$–ordinary and extraordinary components of the transmitted wave ($\mathbf{E}_1 \| \mathbf{E}_{in}$, $|\mathbf{E}_1|^2+|\mathbf{E}_2|^2=|\mathbf{E}_{in}|^2$). The unit vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are the initial (arbitrarily chosen) polarization unit vectors; $\tilde{\mathbf{q}}_1$ and $\tilde{\mathbf{q}}_2$ are the unit vectors of the plane waves $\mathbf{E}_1$ and $\mathbf{E}_2$; φ is the angle of rotation from $\mathbf{q}_1$ to $\tilde{\mathbf{q}}_1$. All the unit vectors $\mathbf{q}_1,\ldots,\tilde{\mathbf{q}}_3$ are perpendicular to the ray. σ, α, β, $\tilde{\alpha}$, and $\tilde{\beta}$ are spatial (locally Cartesian) components: σ—coordinate along the ray, α,β—initial (arbitrary) transverse coordinates; the coordinates $\tilde{\alpha}$ and $\tilde{\beta}$ are chosen such that the singular line lies in the (σ,β) plane; ζ is the angle of rotation from the α axis to the $\tilde{\alpha}$ axis.

It is obvious that the two degenerate cases can be eliminated by imposing the condition

$$d=(aV-Av)^2+(bV-Bv)^2\neq 0 \tag{11}$$

and by stipulating also that d not be a small parameter anywhere. It is useful to note that the quantity d is invariant to arbitrary rotations of the unit vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ and of the axes α and β.

We now choose the directions of the unit vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ and of the axes α and β (which have so far been arbitrary) such as to simplify to the utmost the expressions (10), and consequently Eqs. (5). Namely, we rotate the unit vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in such a way that V becomes equal to zero, and we rotate the axes α and β in such a way that the coefficient B becomes equal to zero:

$$\tilde{\mathbf{q}}_1=\mathbf{q}_1\cos\varphi+\mathbf{q}_2\sin\varphi,\ldots, \quad \tilde{\Gamma}_1=\Gamma_1\cos\varphi+\Gamma_2\sin\varphi,\ldots,$$
$$\varphi={}^1/_2\arg(v+iV),$$
$$\tilde{\alpha}=\alpha\cos\zeta+\beta\sin\zeta,\ldots, \quad \zeta=\arg[(-aV+Av)+i(-bV+Bv)]. \tag{12}$$

We denote the new values of the coefficients $a, \ldots, V$ by $\tilde{a}, \ldots, \tilde{V}$:

$$\tilde{V}=\tilde{B}=0, \quad \tilde{v}=(v^2+V^2)^{1/2}, \quad \tilde{A}=d^{1/2}\tilde{v}^{-1},$$
$$\tilde{a}\tilde{\alpha}+\tilde{b}\tilde{\beta}=[(av+AV)\alpha+(bv+BV)\beta]\tilde{v}^{-1}. \tag{13}$$

These transformations have a simple physical meaning: $\mathbf{q}_1$ and $\mathbf{q}_2$ are chosen such that at sufficiently large σ, i.e., at sufficiently large distances from the singular point, the main contribution to the coefficients of Eq. (5) is made by the diagonal component ν_{11}. This in turn makes it possible to specify the initial condition for the normal wave in a very simple form, namely, for the initially extraordinary wave this condition takes the form

$$|\Gamma_1(-\infty)|=1, \quad \Gamma_2(-\infty)=0; \tag{14}$$

and in the case of the initially ordinary wave the subscripts 1 and 2 in (14) should be interchanged.

The rotation of the axes α and β likewise have a simple physical meaning. The point is that in terms of the new variables the equations of the singular line (8) take the form

$$\tilde{a}\tilde{\alpha}+\tilde{b}\tilde{\beta}+\tilde{v}\sigma=0, \quad \tilde{A}\tilde{\alpha}=0,$$

i.e., the line lies in the coordinate plane $\tilde{\alpha}=0$. At $\tilde{\alpha}\neq 0$ there are no singular points at all (within the limits of the considered singular region).

Substituting now the transformed expressions for ν_{11} and ν_{12} in (5) and discarding the terms nonlinear in α, β, and σ, we obtain Eqs. (5) in the form

$$\frac{d\Gamma_1}{d\sigma}=\frac{ik}{2\varepsilon_0^{1/2}}[(\tilde{a}\tilde{\alpha}+\tilde{b}\tilde{\beta}+\tilde{v}\sigma)\Gamma_1+(iD+\tilde{A}\tilde{\alpha})\Gamma_2],$$
$$\frac{d\Gamma_2}{d\sigma}=\frac{ik}{2\varepsilon_0^{1/2}}[(-iD+\tilde{A}\tilde{\alpha})\Gamma_1-(\tilde{a}\tilde{\alpha}+\tilde{b}\tilde{\beta}+\tilde{v}\sigma)\Gamma_2]. \tag{15}$$

4. FIELDS AND TRANSFORMATION COEFFICIENTS WHEN THE EFFECT IS LOCALIZED

If we introduce in (15) the dimensionless variable

$$\xi=(k\tilde{v}/2\sqrt{\varepsilon_0})^{1/2}[\sigma+\tilde{v}^{-1}(\tilde{a}\tilde{\alpha}+\tilde{b}\tilde{\beta})]$$

and put

$$\Gamma_2'=\Gamma_2 \arg(iD+\tilde{A}\tilde{\alpha}),$$

then Eqs. (15) are transformed into the system

$$\frac{d\Gamma_1}{d\xi}=i\xi\Gamma_1+i\frac{\sqrt{p}}{2}\Gamma_2', \quad \frac{d\Gamma_2'}{d\xi}=i\frac{\sqrt{p}}{2}\Gamma_1-i\xi\Gamma_2', \tag{16}$$

which contains only one parameter

$$p=2\varepsilon_0^{-1/2}k\tilde{v}^{-1}(D^2+\tilde{A}^2\tilde{\alpha}^2)=2\varepsilon_0^{-1/2}k\tilde{v}^{-1}(D^2+d\tilde{v}^{-2}\tilde{\alpha}^2), \tag{17}$$

whose physical meaning is that it determines the change $\Delta\varphi$ of the phase difference of the normal waves in the section of rotation of the polarization directions of these waves (i.e., of the principal axes of the tensor $\eta_{\alpha\beta}$). In fact, inasmuch as in terms of the unit vectors $\tilde{\mathbf{q}}_1$ and $\tilde{\mathbf{q}}_2$ the component $\tilde{\nu}_{12}$ is constant in σ (and its modulus is $(D^2+\tilde{A}^2\tilde{\alpha}^2)^{1/2}$], and the component ν_{11} varies linearly with σ, the length l of the rotation section (where $|\tilde{v}_{12}|\lesssim|v_{11}|$) is of the order of

$$\tilde{v}^{-1}(D^2+\tilde{A}^2\tilde{\alpha}^2)^{1/2}.$$

The difference Δn between the refractive indices in this section is $\sim\varepsilon_0^{-1/2}|\tilde{v}_{12}|$. Thus, the quantity $\Delta\varphi=kl\Delta n$ is actually of the order of p.

We note that the reason why p becomes infinite at $\tilde{v}=0$ and vanishes at $d=0$ is (as indicated in Sec. 3) the neglect of second-order quantities in α, β, and σ in the expansions (10).

Equations (16) and the initial condition (14) coincide (subject to the notation change $\tilde{\Gamma}_1\to\gamma_2$, $\Gamma_2'\to\gamma_1$) with the system of equations and the initial condition considered in Ref. 5 (and the meaning of the parameter p is essentially the same). The solution of this boundary-value problem is (accurate to an arbitrary common phase factor):

$$\Gamma_1=C(8/p)^{1/2}e^{i\pi/4}D_{ip/8}(\sqrt{2}e^{3i\pi/4}\xi),$$
$$\Gamma_2'=CD_{ip/8-1}(\sqrt{2}e^{3i\pi/4}\xi), \quad C=(p/8)^{1/2}e^{-\pi p/32},$$
$$|\Gamma_1(+\infty)|^2=e^{-\pi p/4}, \quad |\Gamma_2(+\infty)|^2=1-e^{-\pi p/4}, \tag{18}$$

where $D_\nu(z)$ is a parabolic-cylinder function.[17]

Thus, the wave linearly polarized in the direction of the unit vector $\mathbf{q}_1$ is transformed into a wave linearly polarized in a perpendicular direction. The coefficient of this transformation is

$$k_{tr}=1-e^{-\pi p/4}. \tag{19}$$

On the rays passing through the singular line, this coefficient becomes minimal—it is equal to zero if there is no gyration or torsion. On the other hand the coefficient of transformation of the ordinary wave into the extraordinary one is maximal in this case, as expected from general considerations (see Sec. 1). Conversely, with increasing distance from the singular line we have $k_{tr}\to 1$, but the mutual transformation of the ordinary waves into extraordinary ones vanishes.

The parameter D contained in (17) and describing the gyrotropy of the medium and of the degree of "twist" of both the ray and of the anisotropic properties of the medium, always leads to a smoothing of the effect, since the minimum of the parameter p increases (see Appendix 2).

Equation (17) yields also a more accurate estimate of the width ρ_0 of the singular band in a direction perpendicular to the rays; it suffices for this purpose to set $p=0$. In the simplest case, when

$$D\leqslant(\varepsilon_0^{1/2}\tilde{v}/k)^{1/2},$$

we obtain

$$\rho_0\sim\tilde{\alpha}\sim\varepsilon_0^{1/4}k^{-1/2}\tilde{v}^{1/2}d^{-1/2}.$$

The width of the singular band in the direction of the rays can be obtained by choosing for ξ the variation interval $(-1, 1)$:

$$\rho_0\sim|\sigma_+-\sigma_-|\sim\varepsilon_0^{1/4}(k\tilde{v})^{-1/2}.$$

It is clear that both estimates agree with each other and with the estimate (2) in the simplest cases (far from degeneracy), when

$$\tilde{v} \sim d^{1/4} \sim \max_{i,j,k} |\partial \Delta\varepsilon_{ij}/\partial x_k|.$$

5. "IMPLICIT" RAY PICTURE OF "TANGENTIAL" CONICAL REFRACTION

We consider the Hamilton equations

$$\frac{d\mathbf{r}}{\partial t} = c\frac{\partial n^{-1}|\mathbf{k}|}{\partial \mathbf{k}}, \quad \frac{d\mathbf{k}}{dt} = -c\frac{\partial n^{-1}|\mathbf{k}|}{\partial \mathbf{r}}, \tag{20}$$

using the invariant form of the expression for $n(\mathbf{k},\hat{\varepsilon})$:

$$n^{-2} = {}^1/_2\{(\eta_{ii} - \tau_m\tau_n\eta_{mn}) \pm [(\eta_{ii} - \tau_m\tau_n\eta_{mn})^2 - 2\tau_i\tau_j e_{imn}e_{jpq}\eta_{mp}\eta_{nq}]^{1/2}\},$$

where the upper sign corresponds to the extraordinary wave and the lower to the ordinary wave. The first equation of (20) then yields the following expressions for the components of the group velocity along the unit vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3 = \mathbf{k}/|\mathbf{k}|$:

$$\begin{aligned} v_1 &= \frac{1}{2}nc\left\{-\eta_{13}' \pm \frac{(\eta_{22}-\eta_{11})\eta_{31}' - 2(\eta_{12}'\eta_{23}' - \eta_{12}''\eta_{23}'')}{[(\eta_{11}-\eta_{22})^2 + 4\eta_{12}\eta_{21}]^{1/2}}\right\}, \\ v_2 &= \frac{1}{2}nc\left\{-\eta_{23}' \pm \frac{(\eta_{11}-\eta_{22})\eta_{23}' - 2(\eta_{12}'\eta_{13}' + \eta_{12}''\eta_{13}'')}{[(\eta_{11}-\eta_{22})^2 + 4\eta_{12}\eta_{21}]^{1/2}}\right\}, \\ v_3 &= n^{-1}c, \quad n^{-2} = {}^1/_2\{(\eta_{11}+\eta_{22}) \pm [(\eta_{11}-\eta_{22}) + 4\eta_{12}\eta_{21}]^{1/2}\}, \end{aligned} \tag{21}$$

where

$$\hat{\eta}' = \mathrm{Re}\,\hat{\eta}, \quad \hat{\eta}'' = \mathrm{Im}\,\hat{\eta}, \quad \hat{\eta} = \hat{\eta}^+$$

(since $\hat{\varepsilon} = \hat{\varepsilon}^+$).

If the tensor $\hat{\eta}'$ corresponds to a uniaxial medium and $\eta_{23}'' = \eta_{13}'' = 0$, then on approaching the singular point (where $\eta_{11} - \eta_{22} = \eta_{12}' = 0$) we have $\eta_{13}', \eta_{23}' \to 0$ and $v_1, v_2 \to 0$, i.e., the vectors $\mathbf{v}$ and $\mathbf{k}$ become parallel. On the other hand, if the tensor η' corresponds to a biaxial medium, then when the singular point is approached the vector $\mathbf{v}$ begins to change rapidly its direction, moving along the cone corresponding to the internal conical refraction. Similar jumps occur in the vector $d\mathbf{k}/dt$, but not in the vector $\mathbf{k}$, as is verified by the second equation of (20). It is easily seen from (21) that the integral break in the vector $\mathbf{v}$ is independent, both in magnitude and direction, of the impact parameter of the ray relative to the singular point and of the side on which the ray passes near this line. Consequently, all the rays "break" in like fashion (by an angle equal to the apex angle of the internal conversion refraction cone) and there is no scattering of the rays.

We note that allowance for these breaks does not affect adversely the estimates of the accuracy of the employed geometrical-optics method ($\sim\Delta\varepsilon/\varepsilon_0$), as can be easily verified by investigating the discrepancy Eqs. (5), i.e., their discarded terms. The structure of this error is indicated in the Appendix 1.

In conclusion, the author considers it his pleasant duty to thank S. N. Stolyarov and A. G. Prudkovskiĭ for a discussion of the work and for useful remarks.

APPENDIX 1

DERIVATION OF THE GEOMETRICAL-OPTICS EQUATIONS

We write down Maxwell's equations for the electric induction

$$k^2\mathbf{D} - \mathrm{rot\,rot}\,(\hat{\eta}\mathbf{D}) = 0 \tag{A.1}$$

and seek the solutions in the form

$$\mathbf{D} = \varepsilon_0\Phi_0(\Gamma_1\mathbf{q}_1 + \Gamma_2\mathbf{q}_2)\exp(i\varphi_0), \tag{A.2}$$

where the notation is the same as in Sec. 2. Substituting (A.2) in (A.1) and projecting it on the unit vectors $\mathbf{q}_1$ and $\mathbf{q}_2$, we obtain Eqs. (5) and (6a) if we discard the error—the terms of order of

$$|\mathbf{k}|^{-1}\frac{\partial^2\Gamma_\alpha}{\partial x_m\,\partial x_n}, \quad \frac{\Gamma_\alpha}{|\mathbf{k}|}\frac{\partial^2 q_\beta}{\partial x_m\,\partial x_n},$$
$$\frac{\Delta\varepsilon_{mn}}{\varepsilon_0}\frac{\partial\Gamma_\alpha}{\partial x_k}, \quad \frac{\Delta\varepsilon_{mn}}{\varepsilon_0}\Gamma_\alpha\frac{\partial q_\beta}{\partial x_n}, \quad \frac{\Gamma_\alpha}{\varepsilon_0}\frac{\partial\Delta\varepsilon_{mn}}{\partial x_i}$$

which lead to corrections not larger than

$$\sim L^{-1}|\mathbf{k}|^{-1} + \varepsilon_0^{-1}\Delta\varepsilon.$$

Finally, without substantially affecting the approximation accuracy, we can replace $\mathbf{D}$ in the left-hand side of (A.2) by $\mathbf{E}\varepsilon_0$.

APPENDIX 2

ESTIMATES OF THE INFLUENCE OF Im $\hat{\varepsilon}$

We estimate now the critical value D_{cr} of the quantity $D = \mathrm{Im}\,\tilde{v}_{12}$, the approach to which increases strongly the influence of gyration on the transformation process (in the case when the effect is localized). It is clear from (17) that this takes place at

$$|D| \gtrsim D_{cr} = \varepsilon_0^{1/4}(\tilde{v}/k)^{1/2}. \tag{A.3}$$

At $D \gg D_{cr}$ the effect vanishes almost completely, since the mutual transformation of the ordinary and extraordinary waves ceases everywhere. At $D \sim D_{cr}$ this transformation is preserved in the singular zone, but even on passing through the singular line the coefficient of this transformation is much less than unity.

Comparing (A.3) with the estimate obtained in Sec. 4 for the width ρ_0 of the singular zone in the direction of the rays, we obtain

$$D_{cr} \sim \varepsilon_0^{1/2}(k\rho_0)^{-1}. \tag{A.4}$$

In other words, the effects of gyration are significant in the case when the gyration manages, over the length of the singular zone, to make a contribution of at least of the order of a radian to the phase shift between the ordinary and extraordinary waves.

It is clear that so radical an influence on the transformation process can be expected also from the anti-Hermitian (dissipative) terms of the tensor $\hat{\varepsilon}$, and their critical values should in this case satisfy estimates similar to the estimates (A.3) and (A.4). It follows therefore, in particular, that the influence of the dissipation on the transformation is small if the integral absorption on passing through the singular zone is small. A more detailed allowance of the dissipation is not a simple task, since the geometrical-optics equations used here (which are connected with Eqs. (7) and (6a) are essentially based on the assumption that the tensor $\hat{\varepsilon}$ is Hermitian. On the other hand, the quasi-isotropic approximation [formulas (4) and (6)], although not using this assumption, likewise do not guarantee a correct description of the effect (see Sec. 2).

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Radiative transitions in collisions of atoms and the photodissociation of vibrationally excited molecules

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We have developed in the quasiclassical approximation a theory which permits determination of the probability of radiative electronic transitions in the case when the extremum of the difference in the potential energies occurs close to the turning points on the potential curves. This theory contains previous results as limiting cases and, together with them, solves the problem of determining the spectral characteristics of the considered transitions over the entire frequency region.

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Radiation and absorption of photons in collisions of atoms is due mainly to transitions between the electronic terms of the quasimolecule formed in the collision process. The basis of the classical and quasiclassical theory of processes of this type was set forth in the work of Kramers and ter-Haar,[1] Bates,[2] and Jablonski.[3] Subsequently this theory was extended to the photodissociation of molecules with high vibrational levels.[4,5] The classical approach, which assumes that the nuclei are moving along classical trajectories, leads to a dependence of the cross sections for these processes on the difference of the potentials of the two electronic states. This approach is not valid, however, in the region of internuclear distances where the potential difference has an extremum, and for determination of the cross sections it is already impossible to use the concept of a classical trajectory of the motion of the nuclei. In this case an applicable method is that which uses quasiclassical wave functions of the nuclear motion for calculation of the probability of an electronic radiative transition. Both approaches are incorrect if the radiative transitions occur near the turning points on the potential curves. Use of quantum-mechanical nuclear wave functions[6] in this region of distances permits the transition probability to be obtained if the difference of the slopes of the potential curves near the turning points is sufficiently great.[1)]

In the present work we have developed a quantum-mechanical theory which permits determination of the radiative transition probability even in the case when the extremum of the difference of the potential curves occurs close to the turning points (the slopes of the potential curves at the turning points differ insignificantly). This theory contains the previous results as limiting cases. By combining this theory with existing the-