Interaction of photons with a strong nonstationary electromagnetic field

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(Submitted 25 January 1979)
Zh. Eksp. Teor. Fiz. 77, 548–559 (August 1979)

An expression is found for the polarization operator in the Furry representation in e^2-order perturbation theory for the combination of a constant uniform field and a plane-wave field of a general form. The obtained result is used to investigate photon dispersion in a constant magnetic field on which is superposed a circularly polarized wave propagating along its direction.

PACS numbers: 41.10.Hv

INTRODUCTION

The most general combination, for which an exact solution to the Dirac equation can be obtained, of electromagnetic fields satisfying the Maxwell equations without a current is the super-position of a constant uniform field and a specially oriented arbitrary plane-wave field. Specifically, the wave vector, k^μ, of the plane-wave field should be the eigenvector of the tensor of the constant uniform field:

$$F_0^{\mu\nu}k_\mu=k^\nu F_0. \tag{1}$$

It follows from the condition (1) that there exists a reference system in which the spatial part, **k**, of the wave vector is parallel to the intensities, **E** and **H**, of the constant fields.

The object of the present paper is to investigate the interaction of photons with a strong classical field of the above-indicated configuration. To do this, we compute the polarization operator (PO) in the Furry representation in lowest-order perturbation theory in terms of the quantized field. In the computations we use the Green function in the proper-time representation. An expression for the latter is derived in Ref. 1, but for our purposes it turns out to be more convenient to write the Green function in a special reference system (see §1).

We consider the PO in greater detail for the case in which a circularly-polarized wave is superposed on, and propagates along the direction of, a constant magnetic field; the obtained result is used to investigate photon dispersion.

For greater compactness, the formulas are written without the sign of summation over the indices $i, j=1, 2$. Under ε_{ij} is meant the antisymmetric tensor

$$\varepsilon_{11}=\varepsilon_{22}=0,\quad \varepsilon_{12}=1,\quad \varepsilon_{21}=-1.$$

In the paper we use the relativistic system of units:

$$\hbar=c=1,\quad \alpha=e^2/4\pi.$$

1. THE GREEN FUNCTION

Let us introduce the fixed system of vectors in Minkowski space:

$$\begin{gathered} n^\mu=(1,\mathbf{n}),\quad n_+{}^\mu={}^1/_2(1,-\mathbf{n}),\quad a_i{}^\mu=(0,\mathbf{a}_i),\\ \mathbf{nn}=1,\quad \mathbf{a}_i\mathbf{a}_j=\delta_{ij},\quad \mathbf{na}_i=0,\\ n^2=n_+{}^2=(na_i)=(n_+a_i)=0,\quad (nn_+)=1,\quad (a_ia_j)=-\delta_{ij}. \end{gathered} \tag{2}$$

In such a basis, an arbitrary vector has the form

$$p^\mu=n^\mu p_++n_+{}^\mu p_-+a_i{}^\mu p_i,$$

where

$$p_-=(np),\quad p_+=(n_+p),\quad p_i=-(a_ip).$$

Let the wave vector of the plane-wave field $k^\mu=\omega n^\mu$. Then the tensor of the electromagnetic field satisfying the condition (1) can be written in the form

$$F^{\mu\nu}=E[n^\mu n_+{}^\nu-n_+{}^\mu n^\nu]+H[a_2{}^\mu a_1{}^\nu-a_1{}^\mu a_2{}^\nu]+\omega\frac{\partial\varphi_i(z)}{\partial z}[n^\mu a_i{}^\nu-a_i{}^\mu n^\nu]. \tag{3}$$

Here

$$\begin{gathered} E=\pm\{(I_1{}^2+I_2{}^2)^{1/2}-I_1\}^{1/2}=\mathbf{En},\\ H=\pm\{(I_1{}^2+I_2{}^2)^{1/2}+I_1\}^{1/2}=\mathbf{Hn}, \end{gathered} \tag{4}$$

$$\begin{gathered} I_1={}^1/_4F^{\mu\nu}F_{\mu\nu}={}^1/_2(H^2-E^2),\\ I_2=-{}^1/_8e^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}=EH \end{gathered} \tag{5}$$

are the field invariants, which, in the case under consideration, coincide with the invariants of the constant component, and the $\varphi(z)$ are arbitrary functions of the argument $z=(kx)$.

In the computations we use the potential

$$A^\mu(x)={}^1/_2E[n_+{}^\mu x_--n^\mu x_+]+{}^1/_2H[a_2{}^\mu x_1-a_1{}^\mu x_2]+a_i{}^\mu\varphi_i(z). \tag{6}$$

Let us note that the use of a potential differing from (6) leads to a change in the phase factor in the expression for the Green function.

In the basis (2), the Green function that is the solution to the equation

$$(i\hat\partial+e\hat A-m)G(x,y|A)=-\delta(x-y), \tag{7}$$

reduces to the form

$$G(x,y|A)=\frac{1}{16\pi^2}\int_0^\infty ds D[Sm+\hat Y+i\gamma^5\hat Z]e^{iW}, \tag{8}$$

$$D=\frac{e^2HE}{\sin eHs\,\mathrm{sh}\,eEs}, \tag{9}$$

$$\begin{gathered} W={}^1/_2eE(x_+y_--x_-y_+)+{}^1/_2eH(x_1y_2-x_2y_1)-m^2s\\ -{}^1/_2eE\,\mathrm{cth}\,eEs(x_--y_-)(x_+-y_+)+{}^1/_4eH\,\mathrm{ctg}\,eHs((x_1-y_1)^2+(x_2-y_2)^2)\\ -(x_i-y_i)K_i-M, \end{gathered} \tag{10}$$

$$\begin{gathered} S=\mathscr{E}(s)-L_i(\hat n\hat a_i-\hat a_i\hat n),\\ \mathscr{E}(s)=\{\mathrm{ch}\,eEs-{}^1/_2(\hat n\hat n_+-\hat n_+\hat n)\,\mathrm{sh}\,eEs\}\{\cos eHs-{}^1/_2(\hat a_2\hat a_1-\hat a_1\hat a_2)\sin eHs\},\\ L_i=\frac{\mathrm{sh}\,eEs}{eE(x_--y_-)}\{Q_i\cos eHs-\varepsilon_{ij}(K_j-Q_j{}^+)\sin eHs\}, \end{gathered} \tag{11}$$

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$$\hat{Y}=\frac{1}{2}(x_--y_-)\frac{eE}{\operatorname{sh} eEs}\hat{n}_+\cos eHs$$

$$+\left\{\frac{1}{2}(x_+-y_+)\frac{eE}{\operatorname{sh} eEs}\cos eHs-(x_i-y_i)P_{iY}-R_Y\right\}\hat{n}$$

$$+\left\{\frac{1}{2}(x_i-y_i)\frac{eH}{\sin eHs}\operatorname{ch} eEs+A_{iY}\right\}\hat{a}_i, \tag{12}$$

$$P_{iY}=-(n_+\partial_x)K_ie^{eEs}\cos eHs+\frac{eH}{eE}\frac{\operatorname{sh} eEs}{(x_--y_-)\sin eHs}$$

$$\times[Q_i\cos^2 eHs+(K_i-Q_i^+)\sin^2 eHs-\varepsilon_{ij}(K_j-Q_j^+-Q_j)\sin eHs\cos eHs], \tag{13}$$

$$R_Y=-(n_+\partial_x)Me^{eEs}\cos eHs-\frac{2\operatorname{sh} eEs}{eE(x_--y_-)}$$

$$\times[Q_i(K_i-Q_i^+-Q_i)\cos eHs+\varepsilon_{ij}Q_i(K_j-Q_j^+)\sin eHs], \tag{14}$$

$$A_{iY}=-\cos eHs[(K_i-Q_i^+)\operatorname{ch} eEs+Q_i\operatorname{sh} eEs]$$
$$+\varepsilon_{ij}\sin eHs[(K_j-Q_j^+)\operatorname{sh} eEs+Q_j\operatorname{ch} eEs], \tag{15}$$

$$\hat{Z}=-\frac{1}{2}(x_--y_-)\frac{eE}{\operatorname{sh} eEs}\hat{n}_+\sin eHs$$

$$+\left\{\frac{1}{2}(x_+-y_+)\frac{eE}{\operatorname{sh} eEs}\sin eHs-(x_i-y_i)P_{iZ}-R_Z\right\}\hat{n}$$

$$+\varepsilon_{ij}\left\{\frac{1}{2}(x_i-y_i)\frac{eH}{\sin eHs}\operatorname{sh} eEs+A_{iZ}\right\}\hat{a}_j, \tag{16}$$

$$P_{iZ}=-\frac{1}{(x_--y_-)}\frac{eH}{eE}\frac{\operatorname{sh} eEs}{\sin eHs}\{(K_i-Q_i^+-Q_i)\sin eHs\cos eHs$$
$$+\varepsilon_{ij}[Q_j\cos^2 eHs+(K_j-Q_j^+)\sin^2 eHs]\}-(n_+\partial_x)K_ie^{eEs}\sin eHs, \tag{17}$$

$$R_Z=-\frac{2}{(x_--y_-)}\frac{\operatorname{sh} eEs}{eE}\{\varepsilon_{ij}Q_i(K_j-Q_j^+)\cos eHs$$
$$-(K_i-Q_i^+-Q_i)(K_i-Q_i^+)\sin eHs\}-(n_+\partial_x)Me^{eEs}\sin eHs, \tag{18}$$

$$A_{iZ}=-\cos eHs[(K_i-Q_i^+)\operatorname{sh} eEs+Q_i\operatorname{ch} eEs]$$
$$+\varepsilon_{ij}\sin eHs[(K_j-Q_j^+)\operatorname{ch} eEs+Q_j\operatorname{sh} eEs]. \tag{19}$$

Here we have introduced the notation:

$$K_i\equiv K_i(x,y;s)=\frac{eH}{\sin eHs}\int_0^s ds'\{e\varphi_i(z(s'))\cos(eH(2s'-s))$$
$$+\varepsilon_{ij}e\varphi_j(z(s'))\sin(eH(2s'-s))\}, \tag{20}$$

$$Q_i\equiv Q_i(x,y)=\frac{e}{2}\{\varphi_i(kx)-\varphi_i(ky)\},$$

$$Q_i^+\equiv Q_i^+(x,y)=\frac{e}{2}\{\varphi_i(kx)+\varphi_i(ky)\},$$

$$M\equiv M(x,y;s)=\int_0^s ds'\int_0^s ds''\{e^2\varphi_i(z(s'))\varphi_i(z(s''))\{\delta(s'-s'')$$
$$-eH[\sin(2eH(s'-s''))\operatorname{sign}(s'-s'')+\cos(2eH(s'-s''))\operatorname{ctg} eHs]\}$$
$$+2e^2\varphi_1(z(s'))\varphi_2(z(s''))eH[-\cos(2eH(s'-s''))\operatorname{sign}(s'-s'')$$
$$+\sin(2eH(s'-s''))\operatorname{ctg} eHs]\}, \tag{21}$$

$$z(s')=\omega[x_-+{}^1/_2(x_--y_-)(1-\operatorname{ctg} eEs)(\exp 2eEs-\exp 2eEs')]. \tag{22}$$

Although the obtained expression is somewhat noncompact, it is nonetheless quite suitable for use. Thus, with the aid of the Green function, written in the form (8), it is not difficult to derive the Lagrangian of the electromagnetic field with the radiative corrections. As is well known,[2] the correction, $\mathscr{L}'(x|A)$, to the classical Lagrangian of an electromagnetic field with the potential $A(x)$ is connected with the Green function by the relation

$$\mathscr{L}'(x|A)=\int dy F^{\mu\nu}(x)F_{\mu\nu}(y)\Pi_0(x-y)$$
$$+i\int_m^\infty dm\,\mathrm{Sp}\left\{G(x,x|A)-G(x,x|A)|_{e=0}-\frac{e^2}{2}\frac{\partial^2}{\partial e^2}G(x,x|A)|_{e=0}\right\}. \tag{23}$$

Here

$$\Pi_0(x-y)=\frac{1}{(2\pi)^4}\int dp e^{ip(x-y)}\frac{\alpha}{4\pi}\int_0^1(1-\beta^2)\ln\left(1-\frac{p^2}{4m^2}(1-\beta^2)\right)d\beta. \tag{24}$$

Substituting (8) into (23), we find

$$\mathscr{L}'(x|A)=\frac{1}{8\pi^2}\int_0^\infty\frac{ds}{s^3}\exp(-im^2s)\{(es)^2I_2\operatorname{ctg} eHs\operatorname{cth} eEs-1+{}^2/_3(es)^2I_1\}. \tag{25}$$

It can be seen from this formula that the correction to the Lagrangian does not depend on the parameters of the plane-wave field entering into the configuration, but is determined only by the value of the constant component. This fact has already been pointed out.[1]

Finally, writing down the Green function in the form (8) allows us to easily verify that, in the field of the configuration in question, the mean value of the induced local current,

$$\langle j^\mu\rangle=ie\,\mathrm{Sp}\,\gamma^\mu G(x,x|A) \tag{26}$$

is equal to zero.

2. THE POLARIZATION OPERATOR

Let us compute the PO of a photon in the presence of the classical field (6) in lowest-order perturbation theory. In the coordinate representation, the PO is given by the formula

$$\Pi^{\mu\nu}(x,y)=-ie^2\,\mathrm{Sp}\{\gamma^\mu G(x,y|A)\gamma^\nu G(y,x|A)\}. \tag{27}$$

Substituting the Green function (8) into (27), and going over to the momentum representation, we obtain

$$\Pi^{\mu\nu}(p,q)=-\frac{i\alpha}{4\pi}\int\frac{dx\,dy}{(2\pi)^4}\int_0^\infty ds_1\,ds_2 D(s_1)D(s_2)\{T_1^{\mu\nu}m^2+T_2^{\mu\nu}\}$$
$$\times\exp\{i[W(x,y;s_1)+W(y,x;s_2)+(px-qy)]\}, \tag{28}$$

where

$$T_1^{\mu\nu}={}^1/_4\mathrm{Sp}\{\gamma^\mu S(x,y;s_1)\gamma^\nu S(y,x;s_2)\}, \tag{29}$$

$$T_2^{\mu\nu}={}^1/_4\mathrm{Sp}\{\gamma^\mu[\hat{Y}(x,y;s_1)+i\gamma^5\hat{Z}(x,y;s_1)]\gamma^\nu[\hat{Y}(y,x;s_2)+i\gamma^5\hat{Z}(y,x;s_2)]\}. \tag{30}$$

For the integration over x and y, it is convenient to go over to the new variables

$$r_{+1}={}^1/_2(x_1+y_1),\quad r_{+2}={}^1/_2(x_2+y_2),\quad r_{+3}={}^1/_2(x_++y_+),$$
$$r_1={}^1/_2(x_1-y_1)-\Delta_1,\quad r_2={}^1/_2(x_2-y_2)-\Delta_2,\quad t=(x_+-y_+),$$
$$f_+={}^1/_2\omega(x_-+y_-),\quad f_-={}^1/_2\omega(x_--y_-), \tag{31}$$
$$\Delta_i=\frac{1}{b_2}\{p_i+[K_i(x,y;s_1)-K_i(y,x;s_2)]\}.$$

The functions entering into the expression for $\Pi^{\mu\nu}(x,y)$ do not depend on r_{+k}; therefore, the integration over these variables yields the δ functions of the conservation laws.

Using the formulas

$$\int dr_1\,dr_2(r_i)^n\exp\{ib_2[(r_1)^2+(r_2)^2]\}=\begin{cases}i\pi/b_2, & n=0\\ 0, & n=1,\\ -\pi/2b_2^2, & n=2\end{cases}$$

$$\int dt t^n\exp\left\{it\left[p_--\frac{b_1}{\omega}f_-\right]\right\}=2\pi(i)^n\delta^{(n)}\left[p_--\frac{b_1}{\omega}f_-\right], \tag{32}$$

$$\int df_-\delta^{(n)}\left[p_--\frac{b_1}{\omega}f_-\right]F(f_-)=-\frac{\partial^nF(f_-)}{\partial f_-^n}\left(-\frac{\omega}{b_1}\right)^{n+1}\Bigg|_{f_-=\omega p_-/b_1},$$

we carry out the integration over the remaining variables, except f_+. As a result, we have

$$\Pi^{\mu\nu}(p,q)=\frac{\alpha}{\pi}\int\frac{df_+}{2\pi\omega}\Big\{\delta(p_1-q_1)\delta(p_2-q_2)\delta(p_--q_-)$$

$$\times\int_0^\infty ds_1\,ds_2\frac{D(s_1)D(s_2)}{b_1b_2}[m^2T_1^{\mu\nu}(f_-,f_+,t,r_i)+T_2^{\mu\nu}(f_-,f_+,t,r_i)]$$

$$\times\exp\Big\{i\Big[\tilde W+\frac{f_+}{\omega}(p_+-q_+)\Big]\Big\}\Big\}. \qquad (33)$$

Here

$$\tilde W=\frac{f_-}{\omega}(p_++q_+)-b_2[\Delta_1^2+\Delta_2^2]-\{M(x,y;s_1)+M(y,x;s_2)\}-m^2(s_1+s_2),$$

$$b_1=eE\frac{\operatorname{sh}(eE(s_1+s_2))}{\operatorname{sh}eEs_1\,\operatorname{sh}eEs_2},\qquad b_2=eH\frac{\sin(eH(s_1+s_2))}{\sin eHs_1\sin eHs_2}. \qquad (34)$$

We should, in accordance with the formulas (32), make in (33) the substitutions:

$$(r_i)^n=\begin{cases}0, & n=1\\ i/2b_2, & n=2\end{cases},\qquad t=-i\Big(\frac{\omega}{b_1}\Big)\frac{\partial}{\partial f_-}, \qquad (35)$$

and set $f_-=\omega p_-/b_1$ after the differentiation.

The computation of the traces in (29) and (30) are easy to perform. After carrying out the regularization procedure, which amounts to the substraction from the expression obtained for $\Pi^{\mu\nu}(p,q)$ the value of this expression for zero external field and the addition of the regularized PO in a vacuum,

$$\Pi^{\mu\nu}_{Reg}(p,q)=\Pi^{\mu\nu}(p,q)-\Pi^{\mu\nu}_{F=0}(p,q)+\Pi_0{}^{\mu\nu}_{Reg}(p,q), \qquad (36)$$

$$\Pi_0{}^{\mu\nu}_{Reg}(p,q)=\delta(p-q)[q^\mu p^\nu-g^{\mu\nu}(p,q)]\frac{\alpha}{4\pi}\int_0^1(1-\beta^2)\ln\Big\{1-\frac{p^2}{4m^2}(1-\beta^2)\Big\}d\beta,$$

we obtain the sought result.

Because of gauge invariance, the tensor $\Pi^{\mu\nu}_{\rm Reg}(p,q)$ should satisfy the transversality condition

$$p_\mu\Pi^{\mu\nu}_{Reg}(p,g)=\Pi^{\mu\nu}_{Reg}(p,q)q_\nu=0. \qquad (37)$$

In order to represent the result in an explicitly transverse form, let us introduce the two sets of vectors $\Lambda_k(p)$, $\Lambda_k(q)$:

$$\Lambda_0^\mu(p)=\frac{p^\mu}{(p^2)^{1/2}},\qquad \Lambda_3^\mu(p)=\frac{k^\mu p^2-p^\mu(kp)}{(p^2)^{1/2}(kp)},$$

$$\Lambda_i^\mu(p)=\frac{p_\nu}{(kp)}[k^\nu a_i^\mu-a_i^\nu k^\mu],\qquad i=1,2. \qquad (38)$$

The vectors $\Lambda_k(p)$, $\Lambda_k(q)$ form complete orthonormalized systems

$$g_{\mu\nu}\Lambda_k^\mu(p)\Lambda_l^\nu=g_{kl},\qquad \sum_{k,l}g^{kl}\Lambda_k^\mu\Lambda_l^\nu=g^{\mu\nu}. \qquad (39)$$

On account of the conservation laws $p_1=q_1$, $p_2=q_2$, and $p_-=q_-$, we have the relations

$$\Lambda_1^\mu(p)=\Lambda_1^\mu(q)=\Lambda_1^\mu,\qquad \Lambda_2^\mu(p)=\Lambda_2^\mu(q)=\Lambda_2^\mu,$$

$$\Lambda_3^\mu(p)=[\Lambda_3^\mu(q)\Lambda_+-\Lambda_0^\mu(q)\Lambda_-],$$

$$\Lambda_0^\mu(p)=[\Lambda_0^\mu(q)\Lambda_+-\Lambda_3^\mu(q)\Lambda_-], \qquad (40)$$

$$\Lambda_\pm=\frac{(q^2\pm p^2)}{2(p^2)^{1/2}(q^2)^{1/2}}.$$

It is clear that

$$\{q^\mu p^\nu-g^{\mu\nu}(pq)\}=(pq)[\Lambda_1^\mu\Lambda_1^\nu+\Lambda_2^\mu\Lambda_2^\nu]+(p^2)^{1/2}(q^2)^{1/2}\Lambda_3^\mu(p)\Lambda_3^\nu(q). \qquad (41)$$

With the aid of the introduced sets of vectors, we can write the expressions for $T_k^{\mu\nu}$ in the form (without the longitudinal part)

$$T_1^{\mu\nu}=-(\Lambda_1^\mu\Lambda_1^\nu+\Lambda_2^\mu\Lambda_2^\nu)\operatorname{ch}(eE(s_1+s_2))\cos(eH(s_2-s_1))$$
$$-(\Lambda_1^\mu\Lambda_2^\nu-\Lambda_2^\mu\Lambda_1^\nu)\operatorname{ch}(eE(s_1+s_2))\sin(eH(s_2-s_1)), \qquad (42)$$

$$T_2^{\mu\nu}=H^{\mu\nu\lambda\rho}\{Y_\lambda(x,y;s_1)Y_\rho(y,x;s_2)-Z_\lambda(x,y;s_1)Z_\rho(y,x;s_2)\}$$
$$-G^{\mu\nu\lambda\rho}\{Y_\lambda(x,y;s_1)Z_\rho(y,x;s_2)+Z_\lambda(x,y;s_1)Y_\rho(y,x;s_2)\}, \qquad (43)$$

where

$$G^{\mu\nu\lambda\rho}=[\Lambda_1^\mu\Lambda_2^\nu-\Lambda_2^\mu\Lambda_1^\nu]\Big\{[n_+^\lambda n^\rho-n^\lambda n_+^\rho]-\frac{p_i}{p_-}[n^\lambda a_i^\rho-a_i^\lambda n^\rho]\Big\}$$
$$-\varepsilon_{ij}\Big\{\Lambda_i^\mu\Lambda_3^\nu(q)\frac{(q^2)^{1/2}}{p_-}-\Lambda_3^\mu(p)\Lambda_i^\nu\frac{(p^2)^{1/2}}{p_-}\Big\}[n^\lambda a_j^\rho-a_j^\lambda n^\rho], \qquad (44)$$

$$H^{\mu\nu\lambda\rho}=\Lambda_i^\mu\Lambda_j^\nu\Big\{\Big[\Big(a_i^\lambda+\frac{p_i}{p_-}n^\lambda\Big)\Big(a_j^\rho+\frac{p_j}{p_-}n^\rho\Big)$$
$$+\Big(a_j^\lambda+\frac{p_j}{p_-}n^\lambda\Big)\Big(a_i^\rho+\frac{p_i}{p_-}n^\rho\Big)\Big]+\delta_{ij}[n^\lambda n_+^\rho+n_+^\lambda n^\rho-a_1^\lambda a_1^\rho-a_2^\lambda a_2^\rho]\Big\}$$
$$+2\Lambda_3^\mu(p)\Lambda_3^\nu(q)\frac{(p^2)^{1/2}(q^2)^{1/2}}{(p_-)^2}n^\lambda n^\rho+\Big\{\Lambda_i^\mu\Lambda_3^\nu(q)\frac{(q^2)^{1/2}}{p_-}+\Lambda_3^\mu(p)\Lambda_i^\nu\frac{(p^2)^{1/2}}{p_-}\Big\}$$
$$\times\Big\{n^\lambda\Big(a_i^\rho+\frac{p_i}{p_-}n^\rho\Big)+\Big(a_i^\lambda+\frac{p_i}{p_-}n^\lambda\Big)n^\rho\Big\}. \qquad (45)$$

It is understood that the arguments x, y of the functions Y, Z in the formula (43) should be expressed in terms of the variables (31).

The formulas (33), (34), (36), (42)–(45) are the sought expression for the PO in the field (3). The obtained result can be transformed by summing over λ and ρ and differentiating with respect to f in accordance with (35). But the formulas then lose their compactness; therefore, it is more expedient to carry out the subsequent transformations for a specific form of the plane-wave field.

In the present paper we restrict ourselves to the investigation of the interesting particular case when there is no constant electric field, and the plane-wave field is a circularly-polarized monochromatic wave. The tensor of such a field can be written in the form

$$F^{\mu\nu}=H[a_2^\mu a_1^\nu-a_1^\mu a_2^\nu]+\omega\frac{\partial\varphi_i(kx)}{\partial(kx)}[n^\mu a_i^\nu-a_i^\mu n^\nu]; \qquad (46)$$

$$\varphi_1=(m/e)\xi\cos(kx),\qquad \varphi_2=(m/e)\xi\sin(kx),\qquad \xi=eE_0/m\omega,$$

where ω and E_0 are the frequency and strength of the wave field.

In the case under consideration the integrals (20) and (21) are expressible in terms of elementary functions. After carrying out a number of simple transformations (see above), and going over to dimensionless integration parameters, we obtain for the PO the expression

$$\Pi^{\mu\nu}_{Reg}(p,q)=\delta(p_1-q_1)\delta(p_2-q_2)\delta(p_--q_-)\sum_{k,l=1}^{3}\tilde\Lambda_k^\mu(p)\tilde\Lambda_l^\nu(q)N_{kl}(p,q), \qquad (47)$$

$$N_{kl}(p,q)=\int\frac{d\theta}{2\pi\omega}\exp\Big(\frac{i(\theta-\Psi)}{\omega}\Big)(p_+-q_+)$$
$$\times\frac{\alpha}{2\pi}m^2\int_{-1}^{1}d\beta\int_0^\infty d\rho\Big\{e^{iU}\Phi_{kl}+\frac{1-\beta^2}{2\rho}e^{-i\rho}\tilde\Phi_{kl}\Big\}, \qquad (48)$$

$$\Phi_{12}=\Phi_{21}=(pq)/m^2,\qquad \Phi_{33}=1,$$

the rest of the components of the $\bar\Phi_{kl}$ tensor are equal to zero.

$$U=U_0-\xi\Big(\frac{p_\perp^2}{m^2}\Big)^{1/2}U_1\sin\theta+\xi^2U_2, \qquad (49)$$

$$U_0=\Big(\frac{(pq)}{m^2}+\frac{p_\perp^2}{m^2}\Big)\frac{1-\beta^2}{4}\rho-\frac{p_\perp^2}{bm^2}-\rho, \qquad (50)$$

$$U_1=2b^{-1}(C_+-C_-), \qquad (51)$$

$$U_2=\frac{\eta^2\beta_+\beta_--\rho}{(1-\beta_+\eta)(1+\beta_-\eta)}\Big(1-\frac{C_+C_-}{\rho\beta_+\beta_-b}\Big), \qquad (52)$$

$$\Phi_{22}=\Phi_{11}^{*}=\mu\,\mathrm{ctg}\,\mu\rho\left\{\frac{\partial U_0}{\partial\rho}+\left(1-\frac{(pq)}{4m^2}(1-\beta^2)\right)+i\xi\left(\frac{p_\perp^2}{m^2}\right)^{1/2}\frac{\partial U_1}{\partial\rho}e^{i\theta}-\xi^2\frac{\partial U_2}{\partial\rho}e^{2i\theta}\right\}, \tag{53}$$

$$\Phi_{12}=\Phi_{21}^{*}=e^{i\mu\rho\beta}\left\{-\frac{\mu}{\sin\mu\rho}\left[1+\frac{(pq)}{4m^2}(1+\beta^2)+\frac{\partial U}{\partial\rho}\right]+i\frac{\mu}{\rho\sin\mu\rho}\mathrm{ctg}\,\mu\rho\frac{\partial U}{\partial\beta}+\frac{\mu^2}{2b}\left\{\frac{1}{2}(1-\beta^2)\left(T_+\frac{\sin\mu\rho\beta_-}{\sin\mu\rho\beta_+}+T_-\frac{\sin\mu\rho\beta_+}{\sin\mu\rho\beta_-}\right)+i\left[\beta\xi^2\eta^2\frac{(C_+-C_-)^2}{\sin\mu\rho}-\frac{\beta}{\sin\mu\rho}(T_++T_-)+\frac{1}{2}(1-\beta^2)\mathrm{ctg}\,\mu\rho\left(T_-\frac{\sin\mu\rho\beta_+}{\sin\mu\rho\beta_-}-T_+\frac{\sin\mu\rho\beta_-}{\sin\eta\rho\beta_+}\right)\right]\right\}\right\}, \tag{54}$$

$$T_\pm=\left\{\frac{p_\perp^2}{m^2}+2\xi\eta\left(\frac{p_\perp^2}{m^2}\right)^{1/2}C_\pm\sin\theta+\xi^2\eta^2C_\pm^2\right\}, \tag{55}$$

$$\Phi_{33}=-{}^1/_2(1-\beta^2)\mu\,\mathrm{ctg}\,\mu\rho, \tag{56}$$

$$\Phi_{23}=\Phi_{13}^{*}=2^{-1/2}\left\{\frac{\mu}{2}\left(\frac{p_\perp^2}{m^2}\right)^{1/2}A+\xi e^{i\theta}R\right\}, \tag{57}$$

$$\Phi_{31}=\Phi_{32}^{*}=2^{-1/2}\left\{-\frac{\mu}{2}\left(\frac{p_\perp^2}{m^2}\right)^{1/2}A+\xi e^{-i\theta}R\right\}, \tag{58}$$

$$A=\frac{1}{\sin\mu\rho}\exp(-i\mu\rho\beta)\{\beta\,\mathrm{ctg}\,\mu\rho+i\}-\frac{\beta}{\sin^2\mu\rho}-i(1-\beta^2)\mathrm{ctg}\,\mu\rho, \tag{59}$$

$$R=\frac{\mu^2\eta}{b}(\beta\,\mathrm{ctg}\,\mu\rho+i)\left\{\frac{\beta_+C_+}{\sin\mu\rho\beta_+}\exp(-i\mu\rho\beta_+)-\frac{\beta_-C_-}{\sin\mu\rho\beta_-}\exp(i\mu\rho\beta_-)\right\}. \tag{60}$$

Here we have introduced the notation:

$$C_\pm=\frac{\sin(\mu\rho\beta_\mp(1\mp\beta_\pm\eta))}{(1\mp\beta_\pm\eta)\sin\mu\rho\beta_\mp}$$
$$\mu=eH/m^2,\quad \eta=\omega p_-/eH,\quad p_\perp^2=p_1^2+p_2^2, \tag{61}$$
$$\Psi=\mathrm{arctg}\frac{p_1}{p_2},\quad b=\frac{\mu\sin\mu\rho}{\sin\mu\rho\beta_-\sin\mu\rho\beta_+}$$
$$\beta_\pm={}^1/_2(1\pm\beta).$$

The vectors $\tilde{\Lambda}_i(p)$ and $\tilde{\Lambda}_i(q)$ are connected with the vectors $\Lambda_i(p)$ and $\Lambda_i(q)$ in the following manner:

$$\tilde{\Lambda}_1=\frac{(p_2+ip_1)}{(2p_\perp^2)^{1/2}}(\Lambda_1+i\Lambda_2),$$
$$\tilde{\Lambda}_2=\frac{(p_2-ip_1)}{(2p_\perp^2)^{1/2}}(\Lambda_1-i\Lambda_2), \tag{62}$$
$$\tilde{\Lambda}_3(p)=(p^2/m^2)^{1/2}\Lambda_3(p),\quad \tilde{\Lambda}_3(q)=(q^2/m^2)^{1/2}\Lambda_3(q).$$

The expression for the PO can, after the integration over θ has been performed, be represented in the form of a double integral and a sum. After the integration, each term of the sum will contain $\delta(p-q\pm nk)$, where $n=0, 1, 2, \ldots$. Thus, the n-th term can be interpreted as the amplitude of the process in which the absorption or emission of n photons of the wave occurs. The term with $n=0$ describes the elastic scattering of the photon.

Notice that if there is no magnetic field, then the number n can assume only even values. This circumstance is a direct consequence of the Furry theorem. It is significant that inelastic processes involving real photons cannot occur. Indeed, using the conservation laws, we can easily verify that the relations

$$p^2=q^2=k^2=0,\quad (pq)=(pk)=(qk)=0,$$

should be fulfilled in this case, i.e., the initial and final photons should move along the direction of propagation of the external wave. However, the PO vanishes when these relations are fulfilled.

It is easy to obtain the PO for particular cases from the formulas (47)–(61). Thus, from our result follows the expression for the PO in a constant magnetic field[3] when the wave field is switched off (i.e., for $\xi\to0$). Switching off the magnetic field (i.e., letting $\mu\to0$, $\eta\to\infty$, with $\mu\eta=\omega p_-/m^2$), we obtain the PO in a plane monochromatic wave.[4,5] It should be noted that other sets of basis vectors are used in the cited papers, and therefore complete coincidence of the results is attained after some transformations.

§3. PROPAGATION OF A PHOTON IN A WAVE FIELD AND A CONSTANT MAGNETIC FIELD

For a weak wave (a low-energy photon), the external field can be regarded as some material medium that is in general nonlinear. In the linear approximation, the propagation of a photon in such a medium is described by the solutions to the Maxwell equation

$$p^2A^\mu(p)+\int dq\Pi^{\mu\nu}(p,q)A_\nu(q)=0. \tag{63}$$

The solution to this equation can be written in the form

$$A^\mu(p)=\frac{1}{p^2}\sum_{k=1}^{3}\int dp' e_k^\mu(p,p')\delta(1+\chi_k(p'))h_k(p'). \tag{64}$$

Here $e_k^\mu(p,p')$ and $\chi_k(p')$ are the eigenvectors and eigenvalues of the operator $\Pi^{\mu\nu}(p,p')$:

$$\int\frac{dp'}{(p')^2}\Pi^{\mu\nu}(p,p')e_{k\nu}(p',q)=\chi_k(q)e_k^\mu(p,q), \tag{65}$$

while the $h_k(p')$ are the amplitudes of the normal modes.

Let us consider the propagation of a photon in the field (46), using the polarization operator (47) as the kernel of Eq. (63). To find out the principal laws governing this process, it is sufficient to investigate the propagation of a photon along the lines of force of the constant magnetic field. The problem then gets simplified significantly, since (65) reduces to a system of algebraic equations. Indeed, if $p_\perp^2=0$, then the PO is given by the expression

$$\Pi^{\mu\nu}_{Reg}(p,q)|_{p_\perp^2=0}=\tilde{\Lambda}_1^\mu\tilde{\Lambda}_1^\nu\tilde{N}_{11}\delta(p-q-2k)+\tilde{\Lambda}_2^\mu\tilde{\Lambda}_2^\nu\tilde{N}_{22}\delta(p-q+2k)$$
$$+\delta(p-q)[\tilde{\Lambda}_3^\mu(p)\tilde{\Lambda}_3^\nu(q)\tilde{N}_{33}+\tilde{\Lambda}_1^\mu\tilde{\Lambda}_2^\nu\tilde{N}_{12}+\tilde{\Lambda}_2^\mu\tilde{\Lambda}_1^\nu\tilde{N}_{21}]+\delta(p-q-k)$$
$$\times[\tilde{\Lambda}_1^\mu\tilde{\Lambda}_3^\nu(q)\tilde{N}_{13}+\tilde{\Lambda}_3^\mu(p)\tilde{\Lambda}_1^\nu\tilde{N}_{31}]+\delta(p-q+k)[\tilde{\Lambda}_2^\mu\tilde{\Lambda}_3^\nu(q)\tilde{N}_{23}+\tilde{\Lambda}_3^\mu(p)\tilde{\Lambda}_2^\nu\tilde{N}_{32}], \tag{66}$$

where

$$\tilde{N}_{kl}\equiv\tilde{N}_{kl}(p)=\int dqN_{kl}(p,q)|_{p_\perp^2=0}. \tag{67}$$

Substituting (66) into (65), we obtain the eigenvectors

$$e_k^\mu(p,q)=\tilde{\Lambda}_1^\mu(p)f_{1,k}(q)\delta(p-q-k)+\tilde{\Lambda}_2^\mu(p)f_{2,k}(q)\delta(p-q+k)+\tilde{\Lambda}_3^\mu(p)f_{3,k}(q)\delta(p-q) \tag{68}$$

and a system of equations for the coefficients $f_{i,k}$:

$$\left[\chi(q)+\frac{N_+}{(q+k)^2}\right]f_1+\frac{N_1}{(q-k)^2}f_2+\frac{S_+}{m^2}f_3=0,$$
$$\frac{N_1}{(q+k)^2}f_1+\left[\chi(q)+\frac{N_-}{(q-k)^2}\right]f_2+\frac{S_-}{m^2}f_3=0, \tag{69}$$
$$\frac{S_+}{(q+k)^2}f_1+\frac{S_-}{(q-k)^2}f_2+\left[\chi(q)+\frac{N_3}{m^2}\right]f_3=0.$$

Here we have introduced the notation:

$$N_+=N_{12}(q+k),\quad N_-=N_{21}(q-k),\quad N_1=N_{11}(q+k),$$
$$N_3=N_{33}(q),\quad S_+=N_{13}(q+k),\quad S_-=N_{23}(q-k). \tag{70}$$

The system (69) has solutions provided that its determinant vanishes. As a result we have the dispersion equation

$$\begin{vmatrix} N_+/(q+k)^2-1 & N_1/(q-k)^2 & S_+/m^2 \\ N_1/(q+k)^2 & N_-/(q-k)^2-1 & S_-/m^2 \\ S_+/(q+k)^2 & S_-/(q-k)^2 & N_3/m^2-1 \end{vmatrix} = 0. \qquad (71)$$

The eigenvectors (68) correspond to elliptically polarized waves, the intensity of the electric field of the wave having a longitudinal component in the general case. Indeed, for $p_\perp^2=0$ the polarization vectors are

$$\tilde{\Lambda}_1^\mu(p)=2^{-1/2}(0,1,i,0), \quad \tilde{\Lambda}_2^\mu(p)=2^{-1/2}(0,1,-i,0), \qquad (72)$$
$$\tilde{\Lambda}_3^\mu(p)=(p_\parallel/m,0,0,p_0/m),$$

and the corresponding field intensities are

$$\mathbf{E}_1=2^{-1/2}p_0(\mathbf{a}_1+i\mathbf{a}_2), \quad \mathbf{H}_1=2^{-1/2}p_\parallel(\mathbf{a}_2-i\mathbf{a}_1),$$
$$\mathbf{E}_2=2^{-1/2}p_0(\mathbf{a}_1-i\mathbf{a}_2), \quad \mathbf{H}_2=2^{-1/2}p_\parallel(\mathbf{a}_2+i\mathbf{a}_1), \qquad (73)$$
$$\mathbf{E}_3=m^{-1}(p_0^2-p_\parallel^2)\mathbf{n}, \quad \mathbf{H}_3=0.$$

It should be noted that, since the functions (70) vanish at $p_-=0$, the propagation of transverse photons with the vacuum dispersion law along the external wave is always possible.

In weak fields the square of the photon momentum differs little from zero for an arbitrary direction of propagation of the photon.[1)] Therefore, it is of interest to analyze Eq. (71) in the region of asymptotically strong fields, where we can expect significant deviations from the vacuum dispersion law.

Let us derive the dispersion equation in the approximation of a superstrong constant magnetic field. Assuming that

$$\mu>1, \quad \eta\ll 1, \quad \xi^2\eta^2/\mu\ll 1, \quad q^2/\mu m^2\ll 1,$$

but not assuming the wave-field intensity to be small compared to the magnetic-field intensity, we have up to terms that are constants with respect to μ

$$\frac{N_\pm}{m^2}=\frac{\alpha}{3\pi}(t\pm 2\lambda-\xi^2\eta^2)\ln 2\mu+\xi^2\eta^2\Pi(t),$$
$$\frac{N_3}{m^2}=\frac{\alpha}{3\pi}\ln 2\mu+2\Pi(t), \qquad (74)$$
$$\frac{N_1}{m^2}=\xi^2\eta^2\Pi(t), \quad \frac{S_\pm}{m^2}=2^{1/2}\xi\eta\Pi(t),$$

where

$$\Pi(t)=\frac{\alpha\mu}{\pi t}\left\{1-\frac{1}{\sqrt{t(t-4)}}\ln\frac{[-1-t^{-1}(t(t-4))^{1/2}][1+t^{-1}(t(t-4))^{1/2}]}{[1-t^{-1}(t(t-4))^{1/2}][-1+t^{-1}(t(t-4))^{1/2}]}\right. \qquad (75)$$
$$t=q^2/m^2+\xi^2\eta^2, \quad \lambda=\omega p_-/m^2.$$

The substitution of (74) into (71) yields the dispersion equation[2)]

$$[2\Pi(t)-1][(t+2\lambda)/\xi^2\eta^2-1-\Pi(t)][(t-2\lambda)/\xi^2\eta^2-1-\Pi(t)]$$
$$+2\Pi^2(t)[(t+2\lambda)/\xi^2\eta^2-1-\Pi(t)]$$
$$+2\Pi^2(t)[(t-2\lambda)/\xi^2\eta^2-1-\Pi(t)]$$
$$+\Pi(t)\{4\Pi(t)\Pi(t)-\Pi(t)[2\Pi(t)-1]\}=0. \qquad (76)$$

This equation can easily be solved for $\xi^2\eta^2$, and the qualitative behavior of the dispersion curves $t=t(\xi^2\eta^2,\lambda)$ can be obtained, using the method of Refs. 7.

The presence in (76) of shifted arguments $(t\pm\lambda)$ and $(t\pm 2\lambda)$ is due to the absorption or emission of photons of the external wave. If such processes can be neglected, then the equation gets simplified considerably, and assumes the form

$$2\Pi(t)=1-\xi^2\eta^2/t. \qquad (77)$$

The expression (77) coincides in form with the well-studied dispersion equation in a strong constant magnetic field,[8] the dynamical variable $\xi^2\eta^2$ being the analog of the square of the transverse photon momentum $p_\perp^2$.

In conclusion, the authors express their gratitude to I. M. Ternov and V. N. Rodionov for useful discussions.

[1)]For fields of the configuration in question, corrections to the dispersion law in the case of weak fields are obtained in Ref. 6, in which the calculations are carried out by a different method.

[2)]In the formula (76), we have neglected the term $(\alpha/3\pi)\ln 2\mu$ in comparison with unity.

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Translated by A. K. Agyei