Vortex collapse

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The condition for the occurrence and the general properties of homogeneous collapse (dispersal) of a system of linear vortices are considered. Exact solutions for three, four, and five vortices are discussed in detail. The stability of the collapse and energy transfer in the spectrum are analyzed. The effect of viscosity and the relation of collapse to the loss of uniqueness of the solutions of the equations of hydrodynamics is considered.

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A system consisting of a finite number of linear vortices of the same sign in an unbounded flow of an ideal fluid is always finite.[1] In the process of evolution of the system the distances between the different vortices can vary only within limits set by internal and external scales determined by the initial configuration. If the intensities of the vortices have different signs, then the vortices can go off to infinity, as is well known.[2,3] Less well known and studied is the possibility that a group of vortices coalesces into a point, i.e., undergoes collapse.

The collapse phenomenon and its inverse—the creation of vortices—are, in our opinion, of general theoretical interest (for fluid dynamics, plasma dynamics, superfluidity, geophysics[1)] and other areas of physics) and deserve a detailed analysis.

In unbounded space two vortices cannot collapse, since the distance between them is an invariant of the motion [Eq. (1.8)]. The problem of the interaction of three vortices is considered in Ref. 1, where a detailed solution is given for the case of three identical vortices and an analogous procedure is proposed for the calculation of the interaction of three vortices of intensities having arbitrary signs and magnitudes. In a recent paper,[4] specially dedicated to the three-vortex problem, several cases of interaction of vortices with unequal intensities $\varkappa_\alpha$ have been analyzed, and it was pointed out that when the harmonic mean [Eq. (3.2)] of the intensities vanishes collapse is possible under suitable initial conditions.

In the present paper we consider the conditions for the occurrence of collapse and its general properties for a system consisting of an arbitrary number of vortices. The exact solutions of the problem of collapse of three, four, and five vortices is discussed in more detail. We consider the stability of collapse, the energy transfer through the spectrum in the collapse process, discuss the influence of viscosity and the connection between the collapse and the loss of uniqueness of the solutions of the equations of hydrodynamics.

§1. INVARIANTS OF THE MOTION AND SCALING FOR A VORTEX SYSTEM

We consider in the unbounded (infinite) plane a system of N point vortices with intensities (the vorticity of the velocity around the vortices) $\varkappa_\alpha$ ($\alpha = 1, \ldots, N$). For the sake of definiteness we assume that positive intensity corresponds to counterclockwise fluid flow. The vortex with the label β imparts to the vortex with the label α the velocity

$$v_{\alpha\beta} = \varkappa_\beta / 2\pi l_{\alpha\beta} \tag{1.1}$$

($l_{\alpha\beta}$ denotes the distance between the vortices), having a direction perpendicular to the line which joins the two vortices. The cartesian coordinates of the vortices are $x_i^{(\alpha)}(t)$ ($i = 1, 2$) and satisfy a Hamiltonian system,[2,3] which we write in the form

$$\varkappa_\alpha \frac{dx_i^{(\alpha)}}{dt} = \varepsilon_{ij}\frac{\partial H}{\partial x_j^{(\alpha)}} = -\frac{\varepsilon_{ij}}{2\pi}\sum_\beta{}' \frac{\varkappa_\alpha\varkappa_\beta (x_j^{(\alpha)} - x_j^{(\beta)})}{l_{\alpha\beta}^2}, \tag{1.2}$$

$$H = -\frac{1}{2\pi}\sum_{\alpha<\beta} \varkappa_\alpha\varkappa_\beta \ln l_{\alpha\beta}. \tag{1.3}$$

Here ε_{ij} is the antisymmetric tensor $\varepsilon_{12} = -\varepsilon_{21} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$, summation from 1 to 2 is understood over repeated indices, and the prime on the sum sign denotes the omission of the term with $\alpha = \beta$. The quantity H has the meaning of interaction energy of the vortices and is a constant of the motion. The invariance of H with respect to translations of the origin implies the existence of the following constants of the motion

$$Z_i = \sum_\alpha \varkappa_\alpha x_i^{(\alpha)}. \tag{1.4}$$

If the sum of the intensities

$$K = \sum_\alpha \varkappa_\alpha \tag{1.5}$$

does not vanish, the barycenter of the vortex system has the coordinates Z_i/K.

Expressing H in terms of the polar coordinates $(\rho_\alpha, \varphi_\alpha)$ of the vortices, one can rewrite Eq. (1.2) in the form

$$\varkappa_\alpha\rho_\alpha \frac{d\rho_\alpha}{dt} = \frac{\partial H}{\partial \varphi_\alpha}, \qquad \varkappa_\alpha\rho_\alpha \frac{d\varphi_\alpha}{dt} = -\frac{\partial H}{\partial \rho_\alpha}. \tag{1.6}$$

The invariance of H with respect to rotations of the reference frame leads to the constant of the motion

$$I = \sum_\alpha \varkappa_\alpha \rho_\alpha^2. \tag{1.7}$$

For the description of the relative motion of the vortices it is convenient to make use of the following combination of the invariants I and Z_i (Ref. 1):

$$M = \sum_{\alpha,\beta} \varkappa_\alpha\varkappa_\beta l_{\alpha\beta}^2 = 2KI - 2Z_i^2. \tag{1.8}$$

It is easy to verify that the quantities Z_i, I, M do not change under rotations of any subsystem of vortices

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relative to the barycenter of that subsystem. In the sequel we shall use such rotations in order to vary the initial conditions and the energy leaving all the other constants of the motion unchanged.

We multiply the second of the equations (1.6) by ρ_α and sum over α:

$$\sum_\alpha \varkappa_\alpha \rho_\alpha^2 \frac{d\varphi_\alpha}{dt} = -\sum_\alpha \rho_\alpha \frac{\partial H}{\partial \rho_\alpha} = -\frac{\partial}{\partial \lambda} H(\lambda\rho, \varphi)|_{\lambda=1}.$$

Here and in Eq. (1.10) λ is an arbitrary constant and (ρ, φ) denotes the collection of arguments $(\rho_\alpha, \varphi_\alpha)$. Making use of Eq. (1.3), we obtain[2]

$$\sum_\alpha \varkappa_\alpha \rho_\alpha^2 \frac{d\varphi_\alpha}{dt} = \frac{1}{2\pi} \sum_{\alpha<\beta} \varkappa_\alpha \varkappa_\beta = V. \tag{1.9}$$

The quantity V, which is important for the sequel, is naturally called the virial of the vortex system. If the system rotates rigidly around its barycenter and $I \neq 0$, with the coordinates ρ_α counted from the barycenter in Eq. (1.7), we obtain for the angular velocity of rotation in (1.9) the expression $\omega = VI^{-1}$.

The equations (1.2) are invariant with respect to the scale transformation of coordinates and time:

$$x \to \lambda x', \quad t \to \lambda^2 t', \tag{1.10}$$

as is directly obvious from (1.1).

§2. HOMOGENEOUS COLLAPSE: GENERAL CONDITIONS AND PROPERTIES

We shall define as uniform collapse (dispersal) of vortices that motion for which all the distances between the vortices have the same time dependence:

$$l_{\alpha\beta}(t) = \lambda(t) l_{\alpha\beta}(0), \quad \lambda(0) = 1. \tag{2.1}$$

Substituting (2.1) into (1.3), taking account of energy conservation and Eq. (1.9), we obtain

$$V = 0. \tag{2.2}$$

In order that the virial (1.9) vanish it is, first of all, necessary that there be at least three vortices present with intensities of different signs. Further, squaring Eq. (1.5) and taking (2.2) into account we obtain

$$K^2 = \sum_\alpha \varkappa_\alpha^2 > 0. \tag{2.3}$$

Thus, collapsing, the vortices cannot disappear completely (annihilate) and the intensity of the resulting vortex has an absolute value larger than that of any of the vortices which have collapsed into it.

The conservation of the constant of motion (1.8) yields the second condition for collapse:

$$M = 0, \tag{2.4}$$

which, in distinction from (2.2) depends on the distances between the vortices.

We now place the coordinate origin in the barycenter of the vortices, which is possible on account of (2.3). The polar coordinates of the vortices have now the following expressions in terms of the inter-vortex distances:

$$K\rho_\alpha^2 = \sum_\beta \varkappa_\beta l_{\alpha\beta}^2 - I, \quad 2\rho_\alpha\rho_\beta \cos(\varphi_\alpha - \varphi_\beta) = \rho_\alpha^2 + \rho_\beta^2 - l_{\alpha\beta}^2. \tag{2.5}$$

Eqs. (2.4) and (2.8) imply

$$I = 0, \tag{2.6}$$

and from (2.1), (2.5), and (2.6) it follows that

$$\rho_\alpha(t) = \lambda(t)\rho_\alpha(0), \quad \varphi_\alpha(t) - \varphi_\beta(t) = \varphi_\alpha(0) - \varphi_\beta(0). \tag{2.7}$$

The motion of the vortices in homogeneous collapse is characterized by the scale factor $\lambda(t)$ and the angular velocity $\omega(t)$. In order that the equations (1.6) should have a solution of the form (2.7) it is necessary and sufficient that the initial configuration satisfy the conditions

$$-\frac{2}{\varkappa_\alpha \rho_\alpha^2} \frac{\partial H}{\partial \varphi_\alpha}\Big|_{t=0} = \frac{1}{t_*} \qquad -\frac{1}{\varkappa_\alpha \rho_\alpha} \frac{\partial H}{\partial \rho_\alpha}\Big|_{t=0} = \omega_0 \tag{2.8}$$

[the left-hand sides of Eqs. (2.8) must not depend on the number of the vortex). Then Eq. (1.6) implies

$$\lambda^2(t) = 1 - t/t_*, \quad \omega(t) = \omega_0/\lambda^2(t). \tag{2.9}$$

Here t_* is the collapse time [the case $t_* < 0$ corresponds to dispersal (anticollapse) of the vortices], ω_0 is the initial angular velocity. The solutions (2.9) correspond to the scale invariance (1.10). The collapse parameters t_* and ω_0 will be determined below for some concrete systems.

The solution (2.9) can be continued beyond the point of collapse; when this is done the vortices jump through the barycenter. The vortex configuration is reflected in the barycenter, the quantity t_* changes its sign in (2.8), keeping its magnitude intact, and the collapse is replaced by a dispersal of the vortices.

It is interesting to estimate at least qualitatively the influence of viscosity on this picture. The competing processes are viscous diffusion and the interaction of vortices. During viscous diffusion the square of the characteristic size of the vortex is of the order νt, where ν is the kinematic viscosity. In the case of dispersal of the vortices ($t_* < 0$) we obtain from Eqs. (2.8) and (2.9) for $t > |t_*|$ the square of the characteristic distance $\sim l_0^2 t |t_*|^{-1} \sim \varkappa t$, where l_0 is the initial distance and $\varkappa$ is the characteristic intensity (vorticity) of the vortices. The relative role of these two processes is determined by a kind of Reynolds number $\mathrm{Re} = \varkappa\nu^{-1}$. For $\mathrm{Re} \gg 1$ the viscosity smoothes out the picture only slightly, whereas for $\mathrm{Re} \ll 1$ the vortices link and diffuse as a single vortex of combined intensity.

In the case of collapse ($t_* > 0$) the presence of even a small viscosity becomes quite essential. Diffusion smearing leads to a change in the form of the distribution of vorticity as the vortices get closer to each other. The problems of instability and stochastization of the vortex field which results could serve as the object of a separate investigation.

The parameter $\varkappa\nu^{-1}$ was introduced earlier by one of the authors[5] for the description of the evolution of matter lines in the diffusion of a vortex. Asymptotically the matter lines take on the form of logarithmic spirals. From Eqs. (2.9) and (2.7) we obtain the logarithmic spirals for the trajectories of the vortices:

$$\rho_\alpha(t) = \rho_\alpha(0) \exp\left\{-\frac{\varphi_\alpha(t) - \varphi_\alpha(0)}{2\omega_0 t_*}\right\}.$$

§3. THE COLLAPSE OF THREE VORTICES

It is convenient to describe the dynamics of a system of vortices in terms of the relative motion.[1] The differential equation, for instance for l_{12}^2, taking into account the influence of the third vortex, has the form[1,4]

$$-l^{-2}\frac{dl_{12}^2}{dt}=\frac{2S\gamma\varkappa_3(l_{23}^2-l_{13}^2)}{\pi l_{12}^2 l_{23}^2 l_{31}^2}. \tag{3.1}$$

Here γ is the orientation of the trio of vortices, equal to 1 if the vortices are numbered counterclockwise and to -1 in the opposite case; S is the area of the triangle spanned by the vortices. The collapse conditions (2.2) and (2.4) for the case of three vortices take on the simple form[4]

$$1/\varkappa_1+1/\varkappa_2+1/\varkappa_3=0, \tag{3.2}$$

$$l_{23}^2/\varkappa_1+l_{31}^2/\varkappa_2+l_{12}^2/\varkappa_3=0. \tag{3.3}$$

From these conditions and the constancy of the energy (1.3) it is clear that the sides of the triangle can vary with time only proportionally to one another, according to (2.1). Taking into account (2.9), the right-hand side of (3.1) at $t=0$ yields the reciprocal collapse time. We also have $\operatorname{sign} t_* = \operatorname{sign}[\gamma\varkappa_3(l_{23}^2-l_{13}^2)]$. Thus, knowing the intensities of the vortices and the orientation γ it is easy to determine whether the vortices will collapse or disperse, depending on the relations between the intervortex distances. We rewrite the expression for t_* in invariant form (independent of the numbering of the vortices):

$$\frac{1}{t_*}=2S_0\left[3\pi\prod_{\alpha<\beta} l_{\alpha\beta}^2(0)\right]^{-1}\sum\nolimits_0 l_{\alpha\beta}^2(0)\,(\varkappa_\beta-\varkappa_\alpha). \tag{3.4}$$

Here the sumbol of summation $\sum_0$ denotes that the triangle is traversed counterclockwise and S_0 is the initial area of the triangle.

Taking into account (2.5), (3.2), and (3.3), equations (2.8) and (1.3) yield the invariant expression for the angular velocity

$$\omega_0=\frac{1}{4\pi}\sum_{\alpha<\beta}\frac{\varkappa_\alpha+\varkappa_\beta}{l_{\alpha\beta}^2(0)}. \tag{3.5}$$

Taking into account Eq. (3.2), we note that ω_0 cannot vanish, and that the sign is determined by the sign of the overall intensity, (1.5). Assume for definiteness that $\varkappa_1$ and $\varkappa_2$ have the same sign, and $\varkappa_3$ has the opposite sign. We use the notation: $q=\varkappa_2/(\varkappa_1+\varkappa_2)$, $0<q<1$. It is convenient to select $b_1=l_{23}^2$, $b_2=l_{31}^2$, $b_3=l_{12}^2$ as phase variables.

The energy surface is conical on account of the condition (3.2) and is given by the formula

$$b_3=Cb_1^q b_2^{1-q},\quad C=\exp\{-4\pi H/\varkappa_1\varkappa_2\}. \tag{3.6}$$

The condition (3.3) yields a plane through the coordinate origin:

$$b_3=qb_1+(1-q)\,b_2. \tag{3.7}$$

Generically the surfaces (3.6) and (3.7) intersect along two straight lines passing through the origin; one of these corresponds to the case of collapse and the other corresponds to dispersal of the vortices. An exclusion is the case of an equilateral triangle for which the condition (3.3) follows from (3.2) and $t_*=\infty$. In this case the energy takes on its maximal value $H=0$, and the surfaces (3.6) and (3.7) are tangent along the line $b_1=b_2=b_3$ which is spanned by such stationary configurations. It is necessary to take into account the restrictions imposed on the phase variables by the triangle inequalities defining a closed region bounded by a circular cone[1,4]

$$b_1^2+b_2^2+b_3^2=2(b_1b_2+b_2b_3+b_3b_1). \tag{3.8}$$

The axis of the cone is the bisector of the trihedral angle $b_1=b_2=b_3$, through which the plane (3.7) always passes. The part of this plane situated within the restriction cone is the angle Λ which is the phase space of the relative motion of the system of three vortices, satisfying the conditions (3.2) and (3.3). The intersection of Λ with the conical surface of constant energy (3.6) yields the phase trajectory.

All possible configurations of three vortices satisfying the collapse conditions, apart from a scale transformation and the orientation of the triangle, can be obtained from one of the configurations by rotation of two vortices relative to their barycenter. Indeed, e.g., in order to obtain all collapsing (dispersing) configurations one has to rotate the side l_{12} of the equilateral triangle in a direction having the same (opposite) sign as the orientation γ of the triangle until the vortices are collinear. This transformation in phase space corresponds to a continuous path which intersects the angle Λ from one of its bounding edges to the other. The surface of the restriction cone (3.8) contains two points which by (3.1) correspond to stationary configurations: The vortices are situated on a straight line. The corresponding energies are

$$H_{1,2}=\frac{\varkappa_3^2}{2\pi}h_{1,2}=\frac{\varkappa_3^2}{2\pi}\left(\frac{\ln[(q^2-q+1)^{1/2}\pm\gamma q]}{q}+\frac{\ln[(q^2-q+1)^{1/2}\pm\gamma(q-1)]}{1-q}\right). \tag{3.9}$$

We note that $H_1(q)=H_2(1-q)$ and that the cases q and $1-q$ differ only by the orientation of the triangle, thus it suffices to restrict one's attention to the interval $0<q\leq\frac{1}{2}$. The graphs of the functions $h_{1,2}(q)$ for $\gamma=1$ are given in Fig. 1. With the exception of the value $q=\frac{1}{2}$ the values of H_1 and H_2 are different, therefore for $\gamma=1$, $q<\frac{1}{2}$ and $H_1<H<H_2$ we only have collapse, and for $q>\frac{1}{2}$ and $H_2<H<H_1$ we only have dispersal (anticollapse). In collapse the admissible values of the energy are the following:

$$0\geqslant H\geqslant\min\{H_1, H_2\},\quad 0<q<1.$$

It is interesting to see how the distribution of kinetic energy of the fluid varies over the spectrum of motions with different wave numbers during collapse. In Ref. 1 the spectral energy density has been expressed in terms of the intervortex distances and it was shown that the average wave number varies proportionally to the quantity

$$\sigma(t)=\sum_{\alpha<\beta}\varkappa_\alpha\varkappa_\beta/l_{\alpha\beta}(t). \tag{3.10}$$

As σ is increased energy is transferred to the small-scale region, and vice versa. Differentiating (3.10) with respect to time, and taking (2.1) and (2.9) into account, we obtain

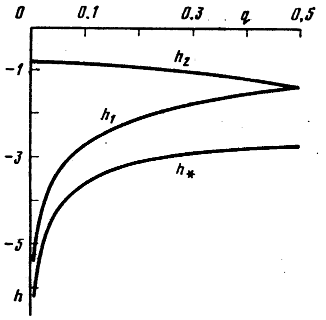

FIG. 1. Critical values of the energy, corresponding to stationary vortex configurations.

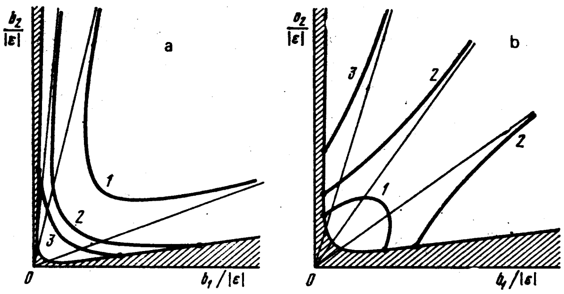

FIG. 2. Typical form of perturbed trajectories in the cases: a) $M>0$, $1-0>H\geqslant H_2$, $2-H_2>H\geqslant H_1$, $3-H_1>H>H^*$; b) $M<0$, $1-H>0$, $2-0\geqslant H>H_2$, $3-H_2\geqslant H>H_1$.

$$\frac{d\sigma(t)}{dt}=\frac{\sigma(t)}{2(t_*-t)}=\frac{\sigma(0)t_*}{2(t_*-t)^2}. \tag{3.11}$$

Making use of the conditions (3.2) and (3.3) one can show that $\sigma<0$. We see from Eq. (3.11) that for collapse ($t_*<0$) energy is systematically transferred into the region of larger scales, and for vortex dispersal, into the region of smaller scales.

We now consider the influence exerted on collapse by small perturbations of the initial values of the variables b_1, b_2, b_3, to which correspond small variations of the integrals H and M. Let $M=\varkappa_1\varkappa_2\varepsilon, |\varepsilon|\ll b_3(0)$, and let H vary over an interval somewhat larger than the one admissible for collapse. In distinction from the unperturbed case the integral $M=\text{const}$ defines a plane

$$b_3=qb_1+(1-q)b_2+\varepsilon, \tag{3.12}$$

parallel to the plane (3.7), therefore intersecting the restriction cone along a hyperbola which bounds the admissible region for the phase variables (b_1, b_2, b_3), and given in the projection on the $b_3=0$ plane by the polar-coordinate expression

$$r=\varepsilon\frac{qp+1-q\pm2\sqrt{p}}{[q^2p^2-2(q^2-q+2)p+(1-q)^2]\cos\varphi}. \tag{3.13}$$

Here $r^2=b_1^2+b_2^2$, $\varphi=\arctan p$, $p=b_2/b_1$. The intersection of the plane (3.12) with the conical energy surface (3.6) is given in the same projection by the expression:

$$r=\varepsilon/[Cp^{1-q}-(1-q)p-q]\cos\varphi. \tag{3.14}$$

The mutual position of the curves (3.13) and (3.14) determines the character of the behavior of the phase trajectories, which depends on the sign of the perturbation ε. In the unperturbed case, $\varepsilon=0$, the expression (3.14) yields two rays, $b_2=p_{1,2}b_1$, which are asymptotes to the curves (3.14) with the same value of C. If $\varepsilon>0$ the curve (3.14) is completely situated inside the angle formed by the rays $b_2=p_{1,2}b_1$, and outside this angle if $\varepsilon<0$. Depending on the energy, a part of the curve (3.14) may not end up in the region admitted by the restrictions. The points of intersection of the trajectory with the restriction cone corresponding to the vortices reaching a straight line for $V=0$ and $M\neq0$ can be stationary only in the case when the position of the vortex $\varkappa_3$ divides the segment l_{12} in the ratio $\varkappa_1/\varkappa_2$; then the mutual influence of the two vortices cancels and they remain at rest. In this case (cf. Fig. 1) $M=4\varkappa_3^2b_3$ and

$$H_*=\frac{\varkappa_3^2}{2\pi}h_*=\frac{\varkappa_3^2}{2\pi}\ln q^{1/(1-q)}(1-q)^{1/q}.$$

One can show that $H_*<H_1$, and therefore for sufficiently small perturbations the phase point reaches the restriction cone when the orientation changes and the motion is reversed.

Depending on the sign of M and the magnitude of H we have six types of trajectory (Fig. 2). For $M>0$ and $H_2\leqslant H<0$ the phase point, moving in from infinity, approaches the coordinate origin to within certain limits determined by the magnitude of ε, and then goes off to infinity. If $H_1\leqslant H<H_2$ (for $M<0$, $H_1\leqslant H\leqslant0$) then the going off to infinity is accompanied by the vortices going onto a straight line and a change in the orientation. For $H_*<H<H_1$ (and $M<0$, $H>0$) the motion occurs within finite limits. In the case $H=H_*$ the phase point is asymptotically attracted to the indicated stationary configuration.

Thus, in spite of the instability of the collapse, the tendency of vortices to get closer to each other to small distances determined by the magnitudes of the perturbations is preserved for a large class of perturbations.

§4. COLLAPSE OF FOUR AND FIVE VORTICES

Besides the three-vortex system, other integrable systems are those composed of four vortices with pairwise equal intensities $\varkappa_1$ and $\varkappa_2$ situated at the vertices of a parallelogram, and a system of five vortices, if the fifth vortex of intensity $\varkappa_0$ is located at the intersection of the diagonals of that parallelogram.[1] The necessary conditions (2.2) and (2.4) of homogeneous collapse take in these cases the form

$$V=\varkappa_1^2+4\varkappa_1\varkappa_2+\varkappa_2^2+2\varkappa_0(\varkappa_1+\varkappa_2)=0, \tag{4.1}$$

$$M=K(\varkappa_1d_1^2+\varkappa_2d_2^2)=0. \tag{4.2}$$

Here d_α are the diagonals of the parallelogram joining the symmetric vortices of intensities $\varkappa_\alpha$ ($\alpha=1,2$). In the case of four votrices one must set $\alpha_0=0$ in Eq. (4.1). It follows from (4.2) that in the case of collapse the intensities $\varkappa_1$ and $\varkappa_2$ must be of opposite sign, and (4.1) $|\varkappa_1|\neq|\varkappa_2|$ if one takes into account. This implies, in particular, that vortices situated at the vertices of a rectangle cannot collapse. Setting $\varkappa_0=0$ in (4.1) we obtain for four vortices [taking (4.2) into account] $-\varkappa_2/\varkappa_1=2\pm\sqrt{3}=d_1^2/d_2^2$. The two values differ by the order of numbering of the vortices.

Making use of the differential equations which describe the dynamics of the vortices in the form (1.6), we obtain the parameters determining the collapse:

$$t_* = -\frac{2\pi a_1^2 a_2^2}{\varkappa_1\varkappa_2 d_1 d_2 \sin 2\psi}, \tag{4.3}$$

$$\omega_0 = \frac{d_1^2+d_2^2}{2\pi d_1^2 d_2^2}\left[\varkappa_1+\varkappa_2+2\varkappa_0+\frac{(d_1^2-d_2^2)(\varkappa_2 d_2^2-\varkappa_1 d_1^2)}{8a_1^2a_2^2}\right]. \tag{4.4}$$

Here a_1 and a_2 are the sides, and ψ is the angle between the diagonals of the parallelogram, measured from the first vortex to the second in the positive sense ($0\leq\psi<2\pi$). The expression for the collapse time is the same for the case of four and five vortices. Collapse occurs if $0<\psi<\pi/2$ and $\pi<\psi<3\pi/2$. For $\psi=0$, π we have a stationary configuration—all the vortices lie on a straight line and $\psi=\pi/2$ or $\psi=3\pi/2$ corresponds to uniform rotation of a rhombus. For other values of ψ the vortices disperse. As in the case of three vortices all possible parallelograms satisfying the collapse conditions can be obtained by means of the transformations described above, rotating a pair of symmetric vortices. The expression (4.4) for the angular velocity ω_0 is valid both for four and for five vortices. One can show that

$$\operatorname{sign}\omega_0=\operatorname{sign}(\varkappa_1+\varkappa_2).$$

In distinction from the case of three vortices, for four and five vortices the quantity $\sigma(t)$, (3.10), which characterizes the direction of energy transfer, does not preserve its sign. Figure 3 illustrates the region where the function σ has constant sign. The shaded region corresponds to $\sigma<0$, i.e., to energy transfer into the region of large scales (of length).

In the case of four vertices a numerical integration on an electronic computer has shown that if one selects initial conditions close to the conditions of collapse (4.3), then the general tendency of vortices to approach each other is preserved. If the condition (4.2) holds, then the ratio of the distances between vortices $l_{\alpha\beta}(t)/l_{\alpha\beta}(0)$ got down to 10^{-3} with an accuracy of eight decimal places.

The necessary collapse conditions (2.2) and (2.4) are, of course, not equivalent to the sufficient ones (2.8). Even for four vortices computer calculations have shown that different configurations obtained from collapsing parallelograms by rotation of two unequal vortices relative to their barycenter do not collapse. It will be of interest to search for other collapsing configurations of vortices.

§5. LOSS OF UNIQUENESS OF SOLUTIONS OF EULER'S EQUATIONS

The phenomenon of vortex collapse discussed above is related to a widely discussed problem: the unique solvability of the equations of hydrodynamics. For the case of two-dimensional flows of an ideal fluid, a proof of the theorem of uniqueness of solutions in the large (over a large time interval) was given in Ref. 6. However, the conditions of this theorem impose definite restrictions on the smoothness of the initial fields and do not encompass the case of delta-function-like singularities of the vortex field (the presence of discrete vortices.

The phenomenon of vortex collapse indicates that there is loss of uniqueness. Indeed, in view of the time-reversal invariance of the equations of motion

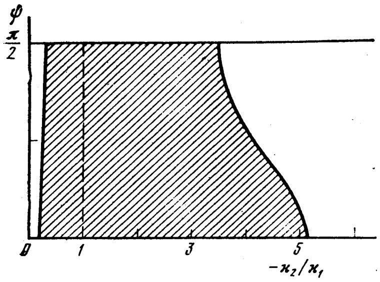

FIG. 3. Energy transfer over the spectrum.

of an ideal fluid, in addition to collapse, the creation of vortices, i.e., the decay of one vortex into a group of three of more vortices, must also be possible. From a mathematical point of view this is none other than the fundamental fact that the uniqueness of solutions of Euler's equations is lost in a function space that admits of delta-like singularities of the vortex field.

It follows from a previous paper of the authors,[7] that a system of four or more vortices of the same sign is not completely integrable: Quasiperiodicity is absent in certain intervals of the values of the invariants of the motion.[2)] We now see that, in the presence of three or more vortices of different signs, collapse is possible in the general case and the uniqueness of solutions of Euler's equations is lost. These facts should stimulate new directions of investigation of fluid and plasma dynamics.

Added in proof (June 22, 1979). The contents of the first half of Sec. 3 of the present paper overlaps the recently published paper by Hassan Aref, Phys. Fluids 23, 393 (1979).

[1)]Collapse may turn out to be the mechanism responsible for the formation of some large atmospheric vortices.

[2)]We indicate here some critical values for the configuration temperature Θ introduced in Ref. 1 for the case of four identical vortices. The minimal value $\Theta_0=2^{10}\times3^{-6}\simeq1.40$ corresponds to a stable rotation of the square formed by the vertices. The unstable rotation of an equilateral triangle formed by three vortices around the fourth located at its center[7] corresponds to $\Theta_1=2^6\times3^{-3}\simeq2.37$. Absence of quasiperiodicity has been observed in the interval $\Theta_1\ll\Theta<\Theta_2$, where $\Theta_2=4\Theta_1$, corresponding to four vortices becoming aligned on one straight line. Outside this range some transitions between convex and nonconvex configurations are forbidden.[7] For $\Theta>\Theta_2$ direct transitions between two different convex configurations of between nonconvex configurations may occur, a fact which was not noted in Ref. 7.

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