Stochastic self-modulation of waves in nonequilibrium media

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The paper reports the discovery and investigation, through a qualitative analysis and a numerical experiment, of a stochasticity that arises as a result of the development of modulation instability in a nonequilibrium dissipative medium with a spectrally narrow amplification increment. A detailed investigation is carried out of a three-mode model describing the modulation decay of a pair of quanta in the same state into a symmetric pair ($2\omega_0 = \omega_+ + \omega_-$) under the assumption that the ω_0 quanta are produced because the medium is in a nonequilibrium state, while the $\omega_\pm$ quanta disappear because of dissipation. The phase space of this system is found to contain an attracting set (attractor), on which the motion of the system is aperiodic, and which describes a complex dynamics. The Poincaré map corresponding to this attractor is similar to the well-known extension mapping of a segment into itself. The structure of the attractor is of the Cantor type, and the motion on it is characterized by a continuous spectrum.

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1. INTRODUCTION

Until recently, in dicussing the problem of the onset of turbulence, the question of the specific methods of eliminating an instability developing in a viscous nonlinear medium and the question of the mechanisms leading to the appearance of disordered random motion were, as a rule, considered independently. The elimination of an instability was generally attributed to systematic intraspectral transfer of the energy of diversely-scaled perturbations into the region of strongly damped small scales.[1] On the other hand, the onset of chaos was associated either with the excitation of a large number of independent perturbations,[2] or with some "conservative" mechanisms.[3] It has become clear in recent years as a result of intensive investigations of stochastic auto-oscillations in a dynamical system with a small number of modes that, in principle, the possibility of the onset of chaos in dissipative systems can itself be related to the mechanism underlying the limitation of the instability.[4-6] In particular, the onset of stationary disordered motions in systems in which the stabilization of linearly amplifiable modes is effected by the decay[7] or parametric[8] mechanism of energy transfer to damped perturbations of multiple scales has now been investigated in detail.

Natural interest attaches to the observation of stochasticity within the framework of models in which stabilization is effected through the transfer of energy to neighboring (nearly unstable) scales. The simplest model of this sort can be obtained by generalizing the well-known Landau model[2] by allowing for the broadening of the spectrum in the course of the self-modulation (or self-focusing) of the wave packet, i.e., for the excitation of close modes. The basic equation then can be the nonlinear parabolic equation with complex coefficients,

$$a_t=\gamma_0 a+(\beta+i\varkappa)a_{xx}+(i\alpha-\rho)a|a|^2, \tag{1}$$

for the complex wave amplitude $a(x, t)$ (the subscripts denote the corresponding derivatives). The present paper is devoted to the investigation of the nonlinear dynamics of the growth of perturbations within the framework of this model and the simpler, finite-dimensional model

$$\dot{x}=y(z-1+x^2)+\gamma x, \quad \dot{y}=x(3z+1-x^2)+\gamma y, \quad \dot{z}=-2z(\nu+xy), \tag{2}$$

which is obtained from (1) under certain assumptions (see below).

The complex nonlinear parabolic equation (1) describes the behavior of perturbations in nonequilibrium dissipative media near the instability threshold, where the spectrum of the unstable perturbations is narrow, and their increment is small.[9,10] This equation has now been derived for different physical systems: for the Tollmien-Schlichting waves in hydrodynamic flows,[11] wind waves on water,[10] concentration waves during chemical reactions in a medium in which diffusion occurs,[12] Langmuir waves in a plasma,[13] etc.

If the medium is slightly nonconservative, i.e., if the dimensionless parameters $\mu=\beta/\varkappa$ and $r=\rho/\alpha$ are small, then the basic equation (1) is close to the completely integrable nonlinear Schrödinger equation,[14] and some information about the nonlinear phase of the modulation instability in the model (1) can be obtained with the aid of an asymptotic method.[15] Let us consider the variation of the number, $N=\int|a|^2dx$, of quanta and the quasimomentum

$$P=\int i(a_x a^*-a_x^* a)\,dx$$

as a result of the nonconservativeness of the system ($\gamma\neq 0, \beta\neq 0, \rho\neq 0$):

$$dN/dt=2\gamma_0 N-2\beta\int |a_x|^2\,dx-2\rho\int |a|^4\,dx,$$
$$dP/dt=2\gamma_0 P-2\beta\int i(a_x^* a_{xx}-a_{xx}^* a_x)\,dx-2\rho\int i|a|^2(a_x a^*-a_x^* a)\,dx. \tag{3}$$

Assuming now that (1), like its conservative analog (for which $\gamma_0=\beta=\rho=0$), has a solution in the form of a train of solitons,

$$a(x,t)=A\exp\left[i\frac{\alpha A^2}{2}t+i\frac{V^2}{4\varkappa}t+i\frac{V}{2\varkappa}(x-Vt)\right]\mathrm{ch}^{-1}[A(x-Vt)(\alpha/2\varkappa)^{1/2}], \tag{4}$$

but with slowly varying amplitude A and velocity V, we can derive approximate equations for A and V by substituting (4) into (3) and limiting ourselves to the terms

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of first order in the small parameters $\mu\equiv\beta/\varkappa\sim r\equiv\rho/\alpha \ll 1$ (Ref. 10):

$$\frac{dA}{dt}=\gamma_0 A-\frac{\beta\alpha}{6\varkappa}A^3-\frac{2\rho}{3}A^3-\frac{\beta}{4\varkappa^2}AV^2,\qquad \frac{dV}{dt}=-\frac{2}{3}\frac{\beta\alpha}{\varkappa}A^2V. \qquad (5)$$

From here it can, in particular, be seen that, as $t\to\infty$, all the solitons come to a stop and become equal in amplitude, i.e.,

$$V\to 0,\quad A\to A_0=\left[6\frac{\gamma_0}{\alpha}\left(\frac{\beta}{\varkappa}+4\frac{\rho}{\alpha}\right)^{-1}\right]^{1/2}.$$

It is possible that such an approach, when based on "averaging over the solitions," can be used to investigate the complex dynamics in the model (1) also. Evidently, to do this, we should take account of the interaction of a few solitons, for example, because of the overlapping of their "tails."[16] The problem of the autooscillation regimes in a quasistationary-soliton lattice, a problem which, it seems to us, is of extreme interest, has, however, thus far not been raised by anybody.[1)] And as to disordered motions in the model (1), there are now only results of direct numerical simulation. In Ref. 12 the model $a_t=a+(1-i/2)a_{xx}-(1+4i)a|a|^2$ with $a(x+L)\equiv a(x)$ is investigated, and stochastic regimes with decreasing spatial correlation functions are found.

Below we investigate in detail the finite-dimensional analog of Eq. (1)—the system (2). With the aid of the methods of qualitative theory and numerical analysis within the framework of this model, we discover the appearance of stochasticity—stable disordered motions characterized by a continuous spectrum.

2. ANALYSIS OF THE FUNDAMENTAL MODEL

The spatial spectrum of the solutions to (1) for periodic boundary conditions (in a resonant cavity) is discrete, and the equation is equivalent to an infinite chain of ordinary differential equations for the complex amplitudes of spatial harmonics (modes). However, although it is not always possible to prove this mathematically rigorously, it seems physically obvious that, because of progressive damping with increasing mode number, this infinite-dimensional system can be truncated and converted into a finite-dimensional system. Much more complex is the question of the number of modes, the interaction between which should be taken into consideration in order not to lose the qualitatively important features of the original system. The simplest situation is the one in which the spectral interval where the medium is active contains only one mode, k_0, and the satellites of this mode that arise as a result of the modulation instability are damped. Then the number of modes in the finite-dimensional model will be determined by the number of satellites that fall within the modulation-instability band. The three-mode approximation

$$\dot a_0=2\sigma a_1a_2a_0^*e^{-i\Delta\omega t}+\gamma_0a_0+\sigma a_0(|a_0|^2+2|a_1|^2+2|a_2|^2),$$
$$\dot a_{1,2}=\sigma a_{2,1}^*a_0^2e^{i\Delta\omega t}-\nu_{1,2}a_{1,2}+\sigma a_{1,2}(2|a_0|^2+|a_{1,2}|^2+2|a_{2,1}|^2) \qquad (6)$$

[where $\sigma=i\alpha-\rho, \Delta\omega=2\omega(k_0)-\omega(k_1)-\omega(k_2)$] turns out to be valid in the case illustrated in Fig. 1: The modula-

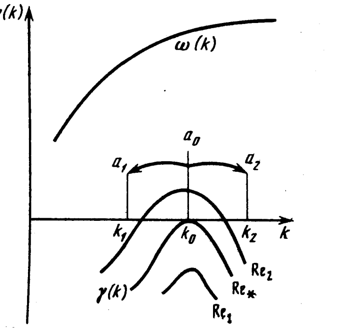

FIG. 1. Appearance of a region of instability in the wave-number spectrum for Re > Re$_*$ (Re$_2$ > Re$_*$ > Re$_1$, where Re$_*$ is the critical value of the characteristic parameter, are Reynolds numbers) and the process of spectrum broadening: decay into the damped satellites $k_{1,2}$.

tion-instability threshold has been exceeded only for the pair of satellites $k_{1,2}=k_0\pm\Delta k$. On account of the resonant character of the amplification, the damping constant $\nu_{1,2}=\beta(n\Delta k)^2-\gamma_0$ increases for distant satellites $k_n=k_0\pm n\Delta k$, and the amplitudes of these satellites in the steady-state regime decreases rapidly with increasing n.

The equations (6) can also be used directly to describe the resonance three-wave interaction of different types of waves in a medium with cubic nonlinearity if the conservation laws

$$2\omega_0=\omega_1+\omega_2,\qquad 2k_0=k_1+k_2 \qquad (7)$$

are fulfilled only for one set of three waves.

Below we shall assume that the dominant mechanism underlying the limitation of the unstable mode a_0 is the modulation interaction of this mode with the damped $a_{1,2}$ satellites, neglecting the nonlinear absorption (due to the generation of multiple harmonics, or to quasilinear effects), i.e., we shall set $\rho=0, \sigma=i\alpha$. If the system in question is near the linear-instability threshold, and to the mode a_0 corresponds the maximum increment (see Fig. 1), then the satellites can be considered to be equally damped: $\nu_1=\nu_2=\nu=\beta(\Delta k)^2-\gamma_0$. In that case it follows from (6) that

$$\frac{d}{dt}(|a_1|^2-|a_2|^2)=-2\nu(|a_1|^2-|a_2|^2), \qquad (8)$$

i.e., the amplitudes of the satellites become equal in time. Thus, considering the processes for a sufficiently long t, we can set $a_1\equiv a_2$. As a result, instead of (6), we obtain the system:

$$\dot B_0=2B_0B_1\sin\Phi+4\gamma B_0,\quad \dot B_1=-B_0B_1\sin\Phi-2\nu B_1,$$
$$\dot\Phi=s+B_1-B_0+\cos\Phi(2B_1-B_0), \qquad (9)$$

where

$$B_{0,1}=2\left|\frac{\alpha}{\Delta\omega}a_{0,1}\right|^2,\quad \Phi=\left(\Delta\omega t+2\arg\frac{a_0}{a_1}\right)\operatorname{sign}\alpha,$$
$$\gamma=\gamma_0/2\Delta\omega,\qquad s=\operatorname{sign}(\alpha\varkappa),$$

and the differentiation is carried out with respect to the dimensionless time $\tau=\Delta\omega t$. If $s=+1$, then the system (2) follows from (6), where $x=(2B_0)^{1/2}\cos(\Phi/2)$, $y=(2B_0)^{1/2}\sin(\Phi/2), z=B_1, \tau=\Delta\omega t/2$.

The processes described by the formulas (2) and (9) are similar in many respects to the processes that occur during the exchange of energy between two harmonics, ω and 2ω, in a medium with quadratic nonlinearity.[7,18,19] In particular, for $\gamma=\nu=0$, the system (9) has the integrals

$$B_0+2B_1=E,\quad B_1(2B_0\cos\Phi+2B_0+B_1-2s)=F, \tag{10}$$

owing to which the general solution of the conservative system can be expressed in an explicit form in terms of elliptic functions. Using, in the case when $\gamma=\nu=0$, the energy integral $x^2+y^2+4z=2E$, we can obtain from (2) the second-order system:

$$\dot{x}={}^1/_4 y(2E+3x^2-y^2-4s),\quad \dot{y}={}^1/_4 x(6E-7x^2-3y^2+4s), \tag{11}$$

which admits of a graphic qualitative investigation in the phase plane (see the phase portraits in Fig. 2). Since $z>0$, only the trajectories lying inside the circle $x^2+y^2=2E$ have a physical meaning. The prescription of the initial conditions on this circle is appropriate for an unmodulated wave: $B_1=z=0$, and the departure from this circle (the excitation of satellites) is possible only when $s=+1$ ($\alpha\varkappa>0$) and $E>\frac{1}{2}$, which corresponds to the Lighthill criterion for the appearance of modulation instability (see Ref. 14).

The system (2) is symmetric with respect to the substitution $x\to -x$, $y\to -y$, and has an integral plane, $z=0$, bounding the "physical" half-plane $z>0$. It corresponds to unmodulated waves ($|a_1|=0$). All the trajectories in this plane uncoil beyond all bounds, getting out of the unstable equilibrium state of the "focus" type: $x=y=0$. For $xy<-\nu$, the integral plane is unstable: the trajectories leave it, which corresponds to satellite generation–modulation instability. In the course of the development of this instability in the case when $\gamma<\nu$, the phase volume specified by the initial conditions shrinks monotonically:

$$\frac{\dot{v}}{v}=\frac{\partial\dot{x}}{\partial x}+\frac{\partial\dot{y}}{\partial y}+\frac{\partial\dot{z}}{\partial z}=2(\gamma-\nu). \tag{12}$$

Consequently, the attracting sets (attractors) corresponding to the steady-state regimes with spatial modulation have zero volume in the xyz phase space.

The nontrivial equilibrium states:

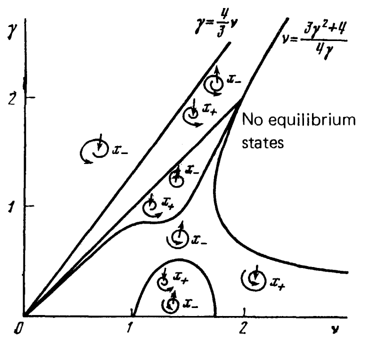

FIG. 3. The types of equilibrium states in the half-space $z>0$.

$$x_\pm^2=\frac{1\pm[1-\gamma\nu(1-3\gamma/4\nu)]^{1/2}}{2(1-3\gamma/4\nu)},\quad y_\pm=-\frac{\nu}{x_\pm},\quad z_\pm=1-\left(1-\frac{\gamma}{\nu}\right)x_\pm^2 \tag{13}$$

correspond to stationary spatial modulation. In this case the growth of the unstable mode a_0 is restricted as a result of its decay into the damped satellites. It follows from the conditions $x^2>0$, $z>0$ that two pairs of such equilibrium states ($\pm x_+$ and $\pm x_-$) exist when $3\gamma/4<\nu<(4+3\gamma^2)/4\gamma$. When $\nu=(4+3\gamma^2)/4\gamma$, the equilibrium states merge at the points $x_+=x_-=\pm[(4+3\gamma^2)/8]^{1/2}$ and disappear. Finally, when $\nu=3\gamma/4$, the equilibrium states $\pm x_+$ recede to infinity, and only one pair of equilibrium states, $\pm x_-$, symmetrically located with respect to the axis $x=y=0$, remain. In Fig. 3 we show the corresponding partition of the plane of the parameters γ and ν. It is not difficult to show that the equilibrium states at the points $\pm x_-$ are always unstable. The stability threshold for the equilibrium states $\pm x_+$ in the plane of the parameters γ and ν has been determined numerically.

When the equilibrium state x_+ loses its stability, a stable limit cycle is produced from it. The motion of a phase point along this limit cycle corresponds to the periodic variation in time of the spatial modulation. The disposition and shape of the stable limit cycles have been investigated with the aid of numerical calculations on a computer, as well as with the aid of an analog-digital simulation of the system (2).

Figure 4 shows the regions in the (γ, ν)-parameter plane that correspond to the various spatial-modulation

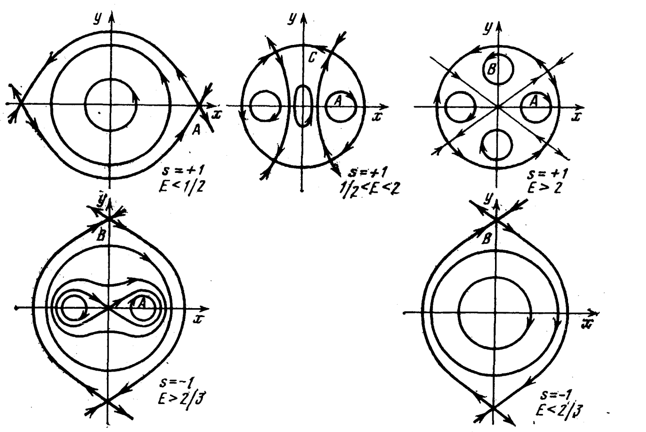

FIG. 2. The phase plane of the conservative system describing the modulation decay of the fundamental harmonic into a pair of identical satellites. The equilibrium-state coordinates are: A) $x=[(6E+4s)/7]^{1/2}$, $y=0$; B) $x=0$, $y=(2E-4s)^{1/2}$; C) $x=s^{1/2}$, $y=(2E-s)^{1/2}$.

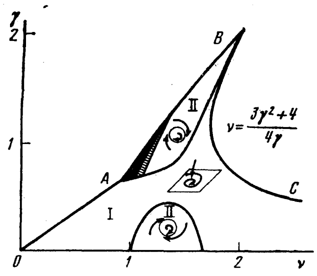

FIG. 4. The possible regimes of stabilization of the unstable mode a_0: I) the static-modulation region (the stable equilibrium state occurs at the point x_+); II) the region of periodic variation of the modulation (limit cycle). The hatched modulation region oscillates with a doubled period. The region of complex motions is the solidly painted region. Stabilization is impossible above the curve ABC.

regimes: the stationary regime, the periodic regime, and then the regime in which the period of the modulation (limit cycle) is doubled. These regimes appear by turns as the ratio γ/ν is increased. As a result of a series of transformations, the limit cycle corresponding to the simple modulation regime is transformed into an extremely complex attracting set that densely fills a region of phase space.

3. NUMERICAL INVESTIGATION OF THE STOCHASTICITY

As the direct numerical solution of the complex nonlinear parabolic equation (1) shows, for $r \equiv \rho/\alpha \ll 1$, $\mu \equiv \beta/\varkappa \sim 1$, the development of the modulation instability in an active medium can lead to the establishment of a stationary turbulent regime.[12] The observation and investigation of such a regime within the framework of the model (2) form the content of the present section.[2)]

Let us note that the investigation of the qualitative characteristics of the complex stationary regimes is based on the numerical solution of the system on a computer. In this case the results of the investigation of the qualitative structure (topology) of the attracting sets (attractors) in phase space do not depend on the accuracy of the calculation (up to 10^{-8}). The correlation function, the spectrum, and the other statistical characteristics obtained by averaging over the series (ensemble) of realizations, also turn out to be just as "crude."[3)]

The region of complex aperiodic regimes occupies in the complex plane of the parameters γ and ν a relatively narrow area in the shape of a "tongue" stretched along the ray $\gamma/\nu = \text{const}$ (see Fig. 4). At smaller values of the ratio γ/ν, there appear unstable limit cycles, whose shape may, however, be fairly complicated.

At large values of the ratio γ/ν, the complex set in phase space becomes unstable: the phase trajectory leaves it, uncoiling without restriction in the process. Thus, when the increment γ greatly exceeds the magnitude of the dissipation ν, limitation as a result of modulation instability is impossible: The fraction of the ω_0 quanta that is lost as a result of decay into satellites is small, and the fundamental mode a_0 grows without restriction as a result of the linear instability.

A realization, i.e., a solution with prescribed initial conditions, that is typical of the complex regimes (see Fig. 5), does not exhibit any noticeable periodicity (see also the representation of the trajectory in phase space in Fig. 6). As a consequence, the autocorrelation function of the process is a decay function (see Fig. 7a), and the spectrum for the given resolution ($\Delta\omega = 0.03$) is broad, with no strongly pronounced discrete components

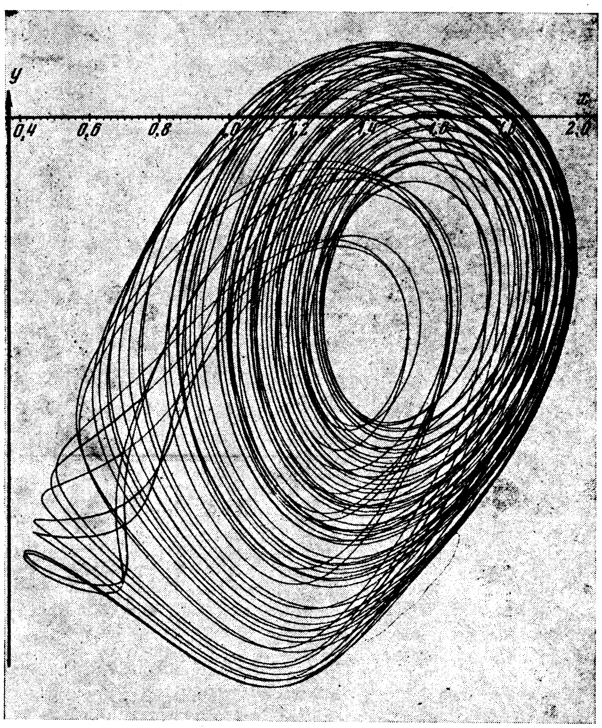

FIG. 6. Three-dimensional phase portrait of the complex motion (the xy-projection), $\nu = 1.1$, $\gamma = 0.87$.

(Fig. 7b). It is only in the vicinity of the boundaries of the region of complex regimes in the parameter space that "traces" of limit cycles in the form of some bunching of the phase trajectories are noticeable; "peaks" then appear in the spectrum.

We carry out the investigation of the structure of the "complex" attractor with the aid of the Poincaré mapping. For this purpose, let us select in phase space the plane $z = z_+$, which intersects this attractor and passes through the equilibrium state (13). We shall now consider the mapping of this plane into itself by the phase trajectories that originate from the plane and subsequently again intersect it. If we take the successive points of intersection of the plane by one phase curve, we obtain the intersection of the attractor by this plane (Fig. 8). To the limit cycles in the intersecting plane corresponds a finite number of points, which are mapped into each other. The "complex" attractor, on the other hand, looks significantly different in the intersection (Fig. 8). Let us note its principal properties found in the numerical experiment.

1. *The divergence of the trajectories.* The successive maps of two neighboring points in the intersecting plane find themselves farther and farther away from each other with each period—the trajectories on the

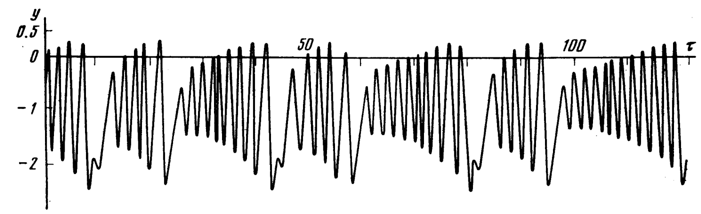

FIG. 5. Oscillogram of the complex motion: $\nu = 1.1$, $\gamma = 0.87$.

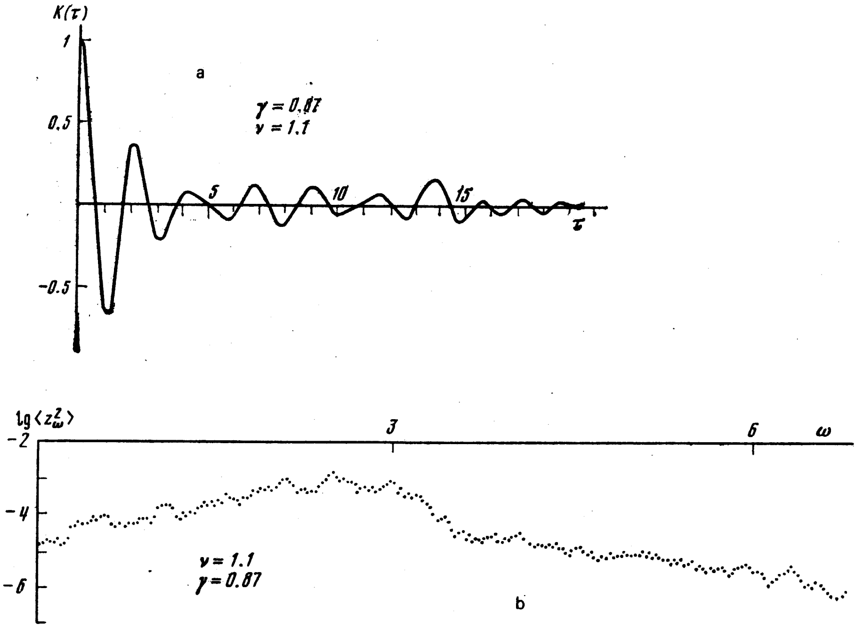

FIG. 7. Statistical characteristics of the complex motions, obtained by averaging over the statistically independent realizations: a) correlation function, b) spectrum.

attractor are unstable.

2. *Finiteness*. If the initial point is chosen near the attractor, then its subsequent maps "wander" over the attractor without going outside its limits. This, coupled with the instability, leads to a complex "entanglement" of the trajectories—to intermixing.

3. *"Horseshoe."* A rectangle in the intersecting plane is mapped into a curved figure reminiscent of a horseshoe. The intermixing of the phase trajectories is determined by the successive application of such a transformation.

4. *"Cantor property."* The successive points of intersection of the intersecting plane form strongly pronounced layers. The points densely fill each of these layers. In the case of a sufficiently large number of points (long realizations) we can observe a finer lamination of each coarse layer, so that the structure of their intersection turns out to be similar to that of a Cantor set.

5. *"Hyperbolicity."* Points taken in the intersecting plane near the attractor are very quickly attracted to the attractor, finding themselves after a small number of mappings in one of its layers. At the same time there occurs along each of the layers a stretching of an elemental area because of the above-discovered divergence of the trajectories.

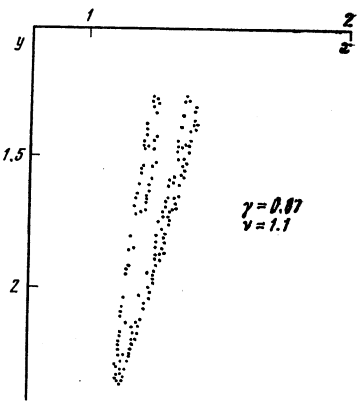

FIG. 8. Two-dimensional point map for the complex motion.

The enumerated properties of the Poincaré mapping, which were found in the numerical experiment, allow us to assume[21] that the "complex" modulation regimes correspond within the framework of our model to a strange attractor in the (x, y, z) phase space.[22,4)] The transition from the stable limit cycle to the strange attractor can be followed, as is usually done in finite-dimensional models, in a one-dimensional (model) point transformation. Indeed, let us consider the mapping of the y axis in the intersecting plane, ignoring the x coordinate. The dependence of y_{n+1} on y_n will then be nonunique. However, because of the "foliation" of the two-dimensional mapping, the points in the (y_{n+1}, y_n) plane bunch around narrow strips (Fig. 9a). If we "roughen" this nonuniqueness by identifying the points across a strip, then we obtain a Lamerey diagram for the one-dimensional point transformation (Fig. 9b).

The point of intersection of the one-dimensional map with the bisectrix $y_{n+1} = y_n$ corresponds to a limit cycle. Since the slope of the map at this point is greater than unity, this cycle is unstable. As the parameter γ/ν is decreased, the slope of the map decreases until a stable limit cycle appears. Thus, there occurs a "cycle-strange attractor" bifurcation, i.e., a transition from simple modulation to a turbulent regime.

One important distinctive feature of the obtained one-dimensional map shoud be noted. It has a smooth "hump," so that the existence of stable limit cycles is possible, these being the cycles that rest on this "hump"[24] (see Fig. 9b). These cycles have a complex shape and a long period, so that, because of the limited accuracy of the numerical solution, it is extremely difficult to detect them. Thus, the question whether the attractor found here is strange in the strict sense, or there are within it stable limit cycles with a small attraction region remains at present an open question.

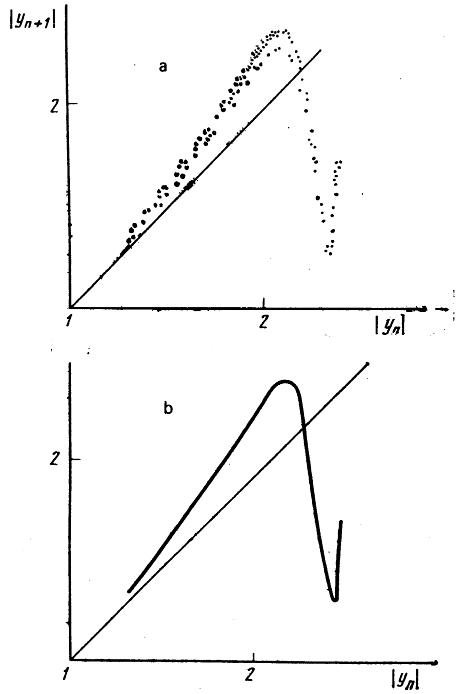

FIG. 9. The one-dimensional point map $y_{n+1}=f(y_n)$: a) obtained by a numerical calculation, b) "roughly approximated variant" ($\nu=1.1$; $\gamma=0/87$).

4. CONCLUSION

Let us give estimates illustrating the applicability of the basic parabolic equation and the ensuing results to waves in a plasma and, in particular, to Langmuir ion waves. A nonlinear-limitation mechanism for Langmuir ion waves excited near the instability threshold in a current-carrying nonisothermal plasma as a result of stimulated (nonresonant) generation of damped harmonics is discussed in Ref. 25. It is well-known, however, that waves of this type are characterized by self-modulation, and therefore the nonlinear phase of their development should be described by a parabolic equation (for the amplitude of the oscillations of the ion velocity):

$$\frac{dV}{dt}=\gamma V-2i\frac{k_0^2}{\omega_{pi}}V|V|^2-\frac{\nu_i k_0^2}{\omega_{pi}}V|V|^2-3i\frac{v_{Ti}^2}{\omega_{pi}}\frac{\partial^2 V}{\partial x^2}+\beta\frac{\partial^2 V}{\partial x^2}. \quad (14)$$

Here γ is the linear increment, ω_{pi} is the ion Langmuir frequency, k_0 is the threshold wave number, $v_{Ti}=(T_i/M)^{1/2}$ is the ion thermal velocity, and ν_i is the ion collision rate. The dispersion term has been written on the basis of the wave-dispersion law: $\omega^2=\omega_{pi}^2+3k^2v_{Ti}^2$, while the coefficient

$$\beta=\frac{16}{81}\left(\frac{\pi m T_i^3}{2MT_e^3}\right)^{1/2}\frac{v_{Ti}^2}{\omega_{pi}}\left(\frac{V_0}{v_{Ti}}\right)^5,$$

which determines the spectral width of the increment, has been found directly from the expression (see Ref. 23)

$$\gamma(k)=\left(\frac{\pi m}{8M}\right)^{1/2}\frac{\omega_{pi}}{k^3 r_{De}^3}\left(\frac{kV}{\omega}-1\right)-\frac{\nu_i}{2}.$$

As a result, we find the dimensionless coefficients:

$$r=\frac{\nu_i}{2\omega_{pi}},\qquad \mu=\left(\frac{8M}{\pi m}\right)^{3/4}\frac{T_e}{T_i}\left(\frac{\nu_i}{\omega_{pi}}\right)^{3/2},$$

determining the dynamics of the solutions to Eq. (14).

Let us note that both terms in the expression for the total increment—the proper increment, due to the Cerenkov effect on electrons, and the decrement, which is connected with the ion collisions—are of the same order of magnitude [otherwise the system will be far from the instability threshold, and the parabolic approximation for $\gamma(k)$ cannot be used]. Then the coefficient $\mu\sim(T_e/T_i)(m/M)^{1/2}\gtrsim 10^{-1}$, and can be sufficiently high. Consequently, in a highly nonisothermal plasma, it is precisely the modulation mechanism of limitation (decay into damped satellites) that is dominant for the current instability near the threshold: it limits the amplitude of the ion Langmuir waves at lower levels than does the mechanism proposed in Ref. 25. Furthermore, since $r\ll 1$, it can be expected that a nonstationary, possibly, stochastic regime of stabilization will be established.

The authors are grateful to A. S. Pikovskiĭ for fruitful discussions.

[1] Even though the appearance of stochasticity during the interaction of solitons in conservative media was discussed fairly long ago.[17]

[2] The numerical solution to (6) with allowance for the fact that $0<\beta<<\alpha$ did not exhibit any qualitative deviations from the $\rho=0$ case. If, on the other hand, $\rho\sim\alpha$, then the complex attracting sets in phase space lose their stability, or are "smoothed" over, being converted into simple limit cycles, i.e., when the nonlinear dissipation is sufficiently high, only a simple regime of periodic modulation is possible.

[3] Allowance for the small fifth-order nonlinear correction to the frequency (i.e., for the additional term proportional to $a|a|^4$ in (1)) also does not alter the qualitative properties of the complex attractor.[20] Evidently, its statistical characteristics—the spectrum, correlation function—as well as the qualitative characteristics of the point transformation (see below) are coarse, i.e., they do not depend on small changes in the right members of the system (2).

[4] Similar to the Henon attractor,[23] which also possesses the enumerated properties.

1. A. S. Monin and A. M. Yaglom, Statisticheskaya gidromekhanika (Statistical Fluid Mechanics), Nauka, 1965 (Eng. Transl., MIT Press, Cambridge, Mass., 1971.
2. L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, 1954 (Eng. Transl., Pergamon, 1959).
3. G. M. Zaslavskiĭ, Statisticheskaya neobratimost' v nelineĭnykh sistemakh (Statistical Irreversibility in Nonlinear Systems), Nauka, 1970.
4. M. I. Rabinovich, Usp. Fiz. Nauk 125, 123 (1978).
5. A. S. Pikovskiĭ and M. I. Rabinovich, in: Nelineĭnye volny (Nonlinear Waves), ed. by A. V. Gaponov, Nauka, 1979, p. 176.
6. M. I. Rabinovich and A. L. Fabrikant, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 19, 721 (1976) [Radiophys. Quantum Electron. 19, 508 (1976)].
7. S. Ya. Byshkind and M. I. Rabinovich, Zh. Eksp. Teor. Fiz. 71, 557 (1976) [Sov. Phys. JETP 44, 292 (1976)].
8. A. B. Ezerskiĭ, M. I. Rabinovich, Yu. A. Stepanyants, and M. F. Shapiro, Zh. Eksp. Teor. Fiz. 76, 991 (1979).
9. R. G. Kenneth, Stud. Appl. Math. 53, 317 (1974).
10. A. A. Andronov and A. L. Fabrikant, in: Nelineĭnye volny (Nonlinear Waves), ed. by A. V. Gaponov, Nauka, 1979, p. 68.

[11]L. M. Hocking and K. Stewartson, Proc. R. Soc. London Ser. A 326, 289 (1972).
[12]Y. Kuramoto and T. Yamada, Prog. Theor. Phys. 56, 679 (1976).
[13]N. R. Pereira and L. Stenflo, Phys. Fluids 20, 1733 (1977).
[14]V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. 61, 118 (1971) [Sov. Phys. JETP 34, 62 (1972)].
[15]L. A. Ostrovskiĭ, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 17, 454 (1974) [Radiophys. Quantum Electron. 17, 344 (1974)].
[16]K. A. Gorshkov, L. A. Ostrovskiĭ, and V. V. Papko, Zh. Eksp. Teor. Fiz. 71, 585 (1976) [Sov. Phys. JETP 44, 306 (1976)].
[17]G. M. Zaslavskiĭ and N. N. Filonenko, Zh. Eksp. Teor. Fiz. 57, 1240 (1969) [Sov. Phys. JETP 30, 676 (1970)].
[18]N. Bloembergen, Nonlinear Optics, Benjamin-Cummings, Reading, Mass., 1965 (Russ. Transl., Mir, 1966).
[19]S. A. Akhmanov and R. V. Khokhlov, Problemy nelineĭnoĭ optiki (Problems of Nonlinear Optics), Nauka, 1964 (Eng. Transl. publ. under title "Nonlinear Optics," Gordon Press, New York, 1972).
[20]M. I. Rabinovich and A. L. Fabrikant, in: Fizika kosmicheskoĭ plazmy (The Physics of the Cosmic Plasma), Nauka, 1979, p. 147.
[21]Ya. G. Sinaĭ, in: Nelineĭnye volny (Nonlinear Waves), ed. by A. V. Gaponov, Nauka, 1979, p. 216.
[22]D. Reille and F. Takens, Commun. Math. Phys. 20, 167 (1971).
[23]M. Henon, Commun. Math. Phys. 50, 69 (1976).
[24]R. M. May, Nature, 261, No. 5560, 459 (1976).
[25]E. Abu-Asali, B. A. Al'terkop, and A. A. Rukhadze, Zh. Eksp. Teor. Fiz. 63, 1293 (1972) [Sov. Phys. JETP 36, 682 (1973)].

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Concentrational ferro-antiferromagnetic transitions in systems based on Fe

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The method of small-angle magnetic scattering of neutrons is used to study the magnetic states that occur during concentrational ferro-antiferromagnetic transitions. Subcritical neutron scattering, caused by the fluctuations of spin density that accompany the concentrational transitions, is observed. The concentration dependences of the magnetic transition temperatures and of the scattering cross sections are found, and the critical concentrations of the transitions are also determined. A cluster model of the transition is proposed; it enables one to calculate the concentration and the mean total value of the fluctuations of the spin density. It is shown that the production of an antiferromagnetic state in a ferromagnetic matrix is due to the γ-Fe atoms.

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INTRODUCTION

The deviation of the values of the mean magnetic moment $\bar{\mu}$, in systems based on Fe, from the Slater-Pauling concentration dependence is interpreted by many investigators[1-3] as a decrease of $\bar{\mu}$ caused by formation of an antiferromagnetic state in a ferromagnetic matrix. This implies the existence of a concentrational ferro-antiferromagnetic transition at some critical concentration c_0, where the Curie temperature $T_c = 0$ K. A complete concentrational ferro-antiferromagnetic transition, with replacement of long-range ferromagnetic order by long-range antiferromagnetic, is observed in the system $Fe_{65}(Ni_{1-x}Mn_x)_{35}$.[1] Here the magnetic state near c_0 is found to be two-phase: antiferromagnetic clusters in a ferromagnetic matrix for $c_{Ni} > c_0$ (Ref. 4) and ferromagnetic clusters in an antiferromagnetic matrix for $c_{Ni} < c_0$.[5] But the reasons for production of an antiferromagnetic state and the mechanism of the change of magnetic order still remain unclear. Furthermore, near c_0 and on the periphery of the clusters, where a change of sign of the exchange interaction occurs, the conditions arise for formation of a "spin glass," which has recently been the object of intensive study.[6]

In the system Fe–Ni, investigation of the magnetic properties near the postulated values c_0 is made difficult by the martensitic transformation, which occurs below 77 K when $c_{Ni} < 34\%$. Nevertheless, the method of small-angle critical scattering of neutrons enables one to study the magnetic structure of the γ phase, which remains after the martensitic transformation. In the present paper, this method is used to investigate the magnetic states of the systems Fe–Ni and $Fe_{65}(Ni_{1-x}Me_x)_{35}$ (Me = Mn, Cr, V) and to determine the parameters of the concentrational transitions and the mechanism and causes of the formation of an antiferromagnetic state.

Earlier,[7,8] diagrams of the magnetic states were constructed for the trinary systems Fe–Ni–Mn and Fe–Ni–Cr over the whole range of concentrations of the γ phase. In these papers, the concentration dependence of the small-angle neutron scattering was studied, but its temperature dependence was not studied, and therefore the origin of the subcritical scattering[4] during formation of the new magnetic phase and the basic physical parameters of the concentrational transitions remained unclarified.

