much as in a real laser the distances between the frequencies of the neighboring modes are the same. Consequently, no new combination transitions, besides those indicated in Fig. 1, arise. Of course, changes take place in the values of the probabilities of the transitions because each value of the combination frequency, including also that of the fundamental tones, is realized by a large number of pairs of radiation harmonics. We can state as a result that the time of the Landau-Zener transition increases to the effective width of the nonmonochromatic radiation

$$t_{tr}^{nonmon}\sim\left|\frac{\omega_{min}-\omega_{max}}{F_a-F_b}\right|,$$

if this width exceeds the time of the Landau-Zener transition for one harmonic.

The laser-radiation spectrum, however, contains besides the superposition of the modes also a nonmonochromaticity due to the finite duration of the pulse and the stochastic phase randomization, whose influence on the character of the Landau-Zener transition calls for a separate analysis and may alter the foregoing conclusion concerning the transition time.

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Permutation symmetry of wave functions of a system of identical particles

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It is shown that if the concept of identity of particles of a system is correctly defined, the known limitation on the permutation symmetry of the wave functions of physically possible states of the system turns out to be an automatic consequence of this definition.

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1. We consider a system of N particles and assume that the quantity q_i is the complete set of dynamic variables that describe its i-th particle. Then the Hamiltonian of this system and the wave functions $|\Psi\rangle$ of its physically possible states will depend on the quantities q_i. If the particles of the system are indistinguishable, then its Hamiltonian turns out to be a symmetrical function of the quantities q_i. Analytically, this property of H is expressed by the relation

$$\hat{P}^{-1}H\hat{P}=H\rightarrow H\hat{P}=\hat{P}H, \qquad (1)$$

where $\hat{P}$ is any operator of the permutation group of the N indices of the quantities q_i. We must immediately emphasize that the explicit form of H, and in particular its property (1) is assumed to be known beforehand, since the equations of quantum mechanics become meaningful only under this condition.

Each level of the Schrödinger equation

$$H|n\rangle=E_n|n\rangle \qquad (2)$$

of a system of identical particles (SIP) corresponds as a rule to a function $|n\rangle$ which has one fully defined type of permutation symmetry (PS), i.e., to each value of E_n there corresponds only one Young pattern. However, Young patterns of different levels may not coincide: Eq. (2) obviously admits of solutions $|n\rangle$ with different types of PS. On the other hand, it is known that in nature there have been encountered so far only those $|n\rangle$ for SIP, which have a maximal PS—either fully symmetrical, or fully antisymmetrical. It can be assumed that this strong limitation on the possible type of the PS of the solutions $|n\rangle$ of the Schrödinger equation for SIP is connected with the nature of its particles—an SIP of definite nature admits also solutions only of the corresponding type PS, consequently, so far only two types of particles have been observed—bosons and fermions. This raises the question of the possibility in principle of existence in nature of particles (of course, with integer or half-integer spin in units of $\hbar$), aggregates of which would be described by functions with an intermediate type of PS—parabosons and parafermions.[1,2]

If the wave functions $|\Psi\rangle$ of the physically possible states of SIP admit of more than two types of PS, then there is apparently no unambiguous connection between the spin of the particles and the PS of these functions. It will be shown below, however, that the PS of the functions $|\Psi\rangle$ is an intrinsic property of the SIP as a

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unit (and not of the nature of its particles) and can be of only one of two types. A unique connection exists then between the spin and PS of the functions $|\Psi\rangle$ (the connection between spin and statistics). It is worthwhile noting here, also, that the PS of an arbitrary function $f(q)$ of N variables $\{q_1, q_2, \ldots, q_N\} \equiv q$ is in the general case a mixture of all possible (for a given N) types of PS, and therefore $f(q)$ cannot be represented even in the form of a combination of only fully symmetrical and fully anti-symmetrical parts

$$f(q)=\frac{1}{2}\sum_P \hat{P}f(q)+\frac{1}{2}\sum_P(-1)^P\hat{P}f(q)-\sum_{P_{\rm even}}{}' \hat{P}f(q), \qquad (3)$$

as is the case, for example, of its parity:

$$f(q)={}^1/_2[f(q)+f(-q)]+{}^1/_2[f(q)-f(-q)]. \qquad (4)$$

The primed sum in (3) does not take into account the identical permutation. It is seen from (3) that only at $N=2$ does this combination reduce to the form (4).

2. The existing limitation on the possible types of the PS of the solutions $|n\rangle$ is regarded by many physicists as an experimental fact,[3-8] whereas others, on the contrary, propose to explain this limitation by starting from the identity of the particles of the system.[9,10] In a number of rather interesting papers,[11,12] on the one hand, the available explanations of the existing limitation on the possible type of the PS of the functions $|\Psi\rangle$ are subjected to criticism, while on the other hand it is shown on the basis of a number of derived selection rules that no paraparticles should be encountered in nature. Kaplan[13] has proposed an explanation of the restriction on the PS, based on the requirement that the averages, over the functions $|\Psi\rangle$, of the operators $\hat{O}_i$ of observable quantities of individual particles of the system be independent of the number of the particle. This explanation, however, is not convincing even for a system of three particles. Namely, for the function [see p. 992 in Ref. 13 (translation)].

$$A_1\Psi_1^{[21]}+A_2\Psi_2^{[21]}\equiv|\rangle,$$

which obviously has an intermediate type of PS, one obtains

$$\langle|f_i|\rangle={}^2/_3(2f_{aa}+f_{bb}),\quad i=1,\ 2,\ 3,$$

if

$$|A_2|^2=|A_1|^2,\quad A_1^*A_2=-A_1A_2^*.$$

The purpose of the present paper is to call attention to the fact that if the concept of identity of the system particles is correctly defined, then the limitation on the possible type of the PS of the functions $|\Psi\rangle$ turns out to be automatically its consequence. Initially, however, it is useful to note the following. At first glance it may seem that an analysis of bosons should begin not with an attempt to describe their gathering (assumed in an unknown manner), but with the quantization of the classical electromagnetic field. Then the states of the quantum field will now already be described in terms of bosons (photons), while the commutation relations for the photon creation and annihilation operators will follow automatically from the classical equations of motion (to be sure, there is apparently no such variant for fermions). It may therefore turn out that the question of at least parabosons should not arise at all. In fact, however, this is not so: as shown by Wigner,[14] correct commutation relations for bosons are not an unambiguous consequence of the classical equations of motion. We can now turn to the definition of the concept of identity of particles of a system.

3. If the mean value of the operator $\hat{O}$ of any observable quantity of a system of particles remains unchanged for arbitrary permutations of the indices of the quantities q_i in the function $|\Psi\rangle$, over which this mean value is calculated, then the particles of this system should be regarded as identical (indistinguishable).

This condition takes the analytic form

$$\langle\Psi'|\hat{O}|\Psi'\rangle=\langle\Psi|\hat{O}|\Psi\rangle,\quad |\Psi'\rangle=\hat{P}|\Psi\rangle, \qquad (5)$$

or, taking into account the unitarity $\hat{P}^+=\hat{P}^{-1}$ of the permutation-group operators,

$$\langle\Psi|\hat{P}^+\hat{O}\hat{P}|\Psi\rangle=\langle\Psi|\hat{P}^{-1}\hat{O}\hat{P}|\Psi\rangle=\langle\Psi|\hat{O}|\Psi\rangle. \qquad (6)$$

By virtue of the supervision principle and the arbitrariness of $|\Psi\rangle$ in the condition (6), the latter reduces to the stronger condition

$$\langle 2|\hat{P}^{-1}\hat{O}\hat{P}|1\rangle=\langle 2|\hat{O}|1\rangle, \qquad (7)$$

where $|1\rangle$ and $|2\rangle$ are any of the number of the functions corresponding to the physically possible states of the SIP.

Thus, with respect to this space of the functions of the SIP, we have the following operator equation

$$\hat{P}^{-1}\hat{O}\hat{P}=\hat{O}, \qquad (8)$$

which reduces at $\hat{O}=H$ to the condition (1). The condition (1), however, differs qualitatively from the condition (8): first, the latter concerns not only the Hamiltonian of the SIP, as does (1), but all the operators of the observable quantities; second, the condition (8) is satisfied only for a special function space—a space corresponding only to the physically possible states of the SIP, whereas (1) should be satisfied with respect to any function space. Of course, the condition (8) together with (6) can be regarded as a new definition of the identity of particles of this system, in place of the customarily employed definition (1).[4,9] The advantage of (8) over (1) lies in the fact that (8) leads directly to a striking result: for SIP there are actually realized only one-dimensional representations of the permutation group–something that could in no way be obtained from the definition (1).

In fact, condition (8) states that in the space of the functions $|\Psi\rangle$ all (precisely all!) operators $\hat{O}$ of the observable quantities of the SIP should commute with each operator P of the permutation group. In matrix notation, condition (8) reads

$$\sum_j O_{ij}P_{jk}=\sum_j P_{ij}O_{jk}. \qquad (9)$$

By way of one of the matrices $\hat{O}$ of an SIP we can choose the matrix

$$O_{ij}=\lambda\delta_{ii_0}\delta_{jj_0}+\lambda^*\delta_{ij_0}\delta_{ji_0}. \qquad (10)$$

Then the condition (9) takes the form

$$\lambda(P_{ii_0}\delta_{kj_0}-P_{j_0k}\delta_{ii_0})=\lambda^*(P_{i_0k}\delta_{ij_0}-P_{ij_0}\delta_{ki_0}), \qquad (11)$$

from which it follows directly that the matrix P_{ij} is a multiple of the unit matrix, i.e., a result similar to Schur's lemma is obtained.[15]

To each observable quantity there corresponds a Hermitian matrix, but the converse need not necessarily be true. Therefore it is also necessary to show that the matrix (10) is observable. As a rule, this question turns out to be very difficult, but a simple criterion is sometimes helpful (Ref. 4, Sec. 9): "$\hat{O}$ is observable if it satisfies an algebraic equation of a polynomial type." The matrix (10) precisely satisfies this criterion, since $\hat{O}^3-|\lambda|^2\hat{O}=0$. This completes the proof.

From the foregoing proof follows also another important result: the type of the PS of the functions $|\Psi\rangle$ corresponding to the physically possible space of an SIP does not depend on the presence of interaction between the particles of the system–this is a purely quantum mechanical effect, and stems only from the indistinguishability of the particles, which, in turn, can be postulated only at $N>1$.

Thus, the fact that in nature there are encountered, or more accurately realized, only such states of a quantum-mechanical SIP whose wave functions are either fully symmetrical or fully antisymmetrical relative to arbitrary permutation as the indices of their particle coordinates q_i is the consequence only of their identity, and not of the nature of the particles of the systems, i.e., it is an intrinsic property of the SIP as a whole. Then it follows directly from the Pauli theorem[16-18] that "the wave functions of any quantum-mechanical SIP having an integer spin are symmetrical, and those of SIP with half-integer spin are antisymmetrical," that in nature there can be no other particles except bosons and fermions. Consequently the question of the possibility of parastatistics as a physical question should be regarded as aimless.

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