imum, further improvement of the technology of the multilayer monochromator-polarizer is necessary.

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Nuclear level shift and radiative transitions in a proton-antiproton atom

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A formula is obtained for the Coulomb shifts of the levels of proton-antiproton atoms in terms of the length of scattering by a strong potential V_s (without the use of perturbation theory, i.e., without assuming the shift to be small compared with the distance between the neighboring levels). The restructuring of the atomic spectrum following the formation of a bound state in the potential V_s is discussed on the basis of this formula. The connection between the level shift and the expansion of the effective radius is indicated. The average radius of the s state is calculated for an arbitrary value of the shift and for an arbitrary probability of the radiative $E1$ transition between the p and s levels. The experimental data [M. Izycki and G. Backenstoss, paper contributed to the Fourth European Antiproton Symposium, Barr, France, 1978; CERN, Geneva, 1978] on the shift of the $1s$ level of the $p\bar{p}$ atom indicate that a bound QS state (quasinuclear meson) can exist in the $p\bar{p}$ system with binding energy $\varepsilon \approx 1$ MeV and width $\Gamma \lesssim 200$ keV. Calculation shows that the probabilities of the radiative transitions $2p \to 1s$ and $2p \to Qs$ are comparable. This points to a possibility of experimentally observing the Qs level by investigating the spectrum of the γ rays produced in transitions between the levels of the $p\bar{p}$ atom.

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§ 1. INTRODUCTION

Investigations of the interaction of antinucleons with nucleons yield valuable information on the nuclear forces. This information is of particular interest in connection with the existence of bound and resonant states in the $N\bar{N}$ system (quasinuclear mesons, predicted theoretically in the papers of Shapiro and coworkers[1]; references to subsequent papers and a survey of the present status of the $N\bar{N}$ problem can be found in later papers by Shapiro[2,3]). Besides quasinuclear mesons[1)] with binding energy ≈50–300 MeV, there exist in the $p\bar{p}$ system atomic levels of the hydrogen type, due to the Coulomb interaction, with a characteristic energy on the order of several keV. These levels cannot be described by the known Balmer formula, but they experience shifts and broadening on account of strong interaction and annihilation. So long as the level shift is small, perturbation theory in terms of the scattering length is applicable[2)]:

$$\Delta E_{nl} = \frac{2(n+l)!}{(l!)^2(n-l-1)!n^{2l+4}} a_l, \qquad (1.1)$$

where a_l is the length of scattering by the nuclear potential V_s (see Refs. 4–6). When the potential V_s has a near-zero real or virtual level with angular momentum l, the scattering length a_l becomes large and the perturbation-theory equation (1.1) no longer holds. At the instant when the level is produced in the strong potential V_s, a restructuring of the atomic spectrum takes place[3,7] (a similar behavior of the s and p levels was observed[8] in the electrodynamics of strong ($Z > 137$) Coulomb fields–near the critical charge of the nucleus $Z = Z_{cr}$, corresponding to the entry of the level $1s_{1/2}$ into the lower continuum).

Several strong-interaction model potentials (square well, separable potential) were used in Ref. 7 to describe the restructuring of the level spectrum of the atom. The qualitative aspect of the problem was explained, but the solved models did not indicate that the restructuring is universal and independent of the form of the potential V_s, and made it necessary to resort to numerical calculations of the level spectrum in each concrete case. It will be shown below that the presence in the problem of the small parameter r_0/a_B (amounting to ≈1/30 for the $p\bar{p}$ atom) makes it possible to develop an analytic theory of the level shift and to describe the

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restructuring of the spectrum without using a model.

The shift of the ground level of the $p\bar{p}$ atom was recently measured[9]

$$\Delta E_{1s}=3.02\pm0.06 \text{ keV}, \ \Gamma\leqslant 200 \text{ eV}. \tag{1.2}$$

Thus, the shift is not small and amounts to $\approx 1/3$ of the distance between the neighboring levels $1s$ and $2s$. The derivation of equations not based on perturbation theory in a_l is therefore a most pressing problem.

The plan of the article is the following. In Sec. 2 we derive Eq. (2.2) for the shift of the $p\bar{p}$-atom levels by the matching method used in the well-known paper of Landau and Smorodinskiĭ[10] on pp scattering at low energies. In Sec. 3 we consider some of the properties of the scattering lengths and analyze the restructuring of the atomic spectrum in general form. In Sec. 4 we discuss the connection of Eq. (2.4) with the expansion of the effective radius and with the Bethe method.[11,12] The average radius of the s states is calculated in Sec. 5, where the limits of applicability of (2.4) are also discussed. In Sec. 6 are discussed the probabilities of $np\to\nu s$ radiative transitions in the $p\bar{p}$ atom. The details of the cumbersome mathematical calculations are relegated to the Appendix.

§ 2. NUCLEAR SHIFT OF THE LEVELS OF THE pp ATOM[3)]

We derive now a formula for the Coulomb-level shifts without assuming smallness of the shift. We use for this purpose the matching method[10] and recognize that $r_0 \ll a_B$. Here

$$r_0 \approx \frac{1}{m_\pi}\ln\left[\left(\frac{m_\pi}{2m_N}\right)^2\frac{g^2}{e^2}\right]\approx 3 \text{ F}$$

is the point at which the potential V_s becomes comparable with the Coulomb potential (we shall arbitrarily call r_0 the radius of action of the nuclear forces).

In the region $r>r_0$, the Schrödinger-equation solution that decreases at infinity is expressed in terms of the Whittaker function $W_{\nu,l+1/2}(2\lambda r)$. Expanding as $r\to 0$ we get, for example, for $l=0$

$$W_{\nu,1/2}(x)=\frac{1}{\Gamma(1-\nu)}[1+b_1 x\ln x+b_2 x+b_3 x^2\ln x+b_4 x^2+\ldots], \tag{2.1}$$

where

$$b_1=-\nu, \quad b_2=-{}^1/_2-[\psi(1-\nu)-1+2C], \quad b_3={}^1/_2\nu^2,\ldots.$$

Matching this result at the point $r=r_0$ to the logarithmic derivative of the internal wave function

$$\xi=\lim\left(r\frac{d\ln\chi(r)}{dr}\right), \quad r\to r_0-0, \quad \chi(r)=rR(r),$$

yields in the case of s-levels

$$\psi(1-\lambda^{-1})+\frac{\lambda}{2}+\ln\lambda+c_0=\frac{1}{2a}-(1-4r_0)\ln r_0, \tag{2.2}$$

where

$$\psi(z)=\frac{d}{dz}\ln\Gamma(z),$$

$c_0=2C+\ln 2=1.848$ ($C=0.5772\ldots$ is the Euler constant), $E=-\lambda^2/2$ is the level energy in units of $E_C=25.0$ keV, and $\nu=\lambda^{-1}$ is the "principal quantum number" (in general not an integer). In the derivation of (2.2) we used for the scattering length a_l by a potential of finite radius $i=i_0$ the expression

$$a_l=a_l^{(0)}\frac{\xi-(l+1)}{\xi+l}, \quad a_l^{(0)}=\frac{2^{2l}(l!)^2}{(2l)!(2l+1)!}r_0^{2l+1} \tag{2.3}$$

and discarded terms of order r_0/a_B (we note that $a_l^{(0)}$ is the length for scattering by a hard sphere of radius r_0).

Equation (2.2) contains the scattering length a for the potential $V_s(r)-r^{-1}\theta(r_0-r)$ that includes part of the Coulomb potential. The Coulomb interaction, being much weaker than the nuclear interaction at $r<r_0$, can be taken into account by perturbation theory. Using Eq. (3.5) from the next section, we obtain

$$\psi(1-\lambda^{-1})+{}^1/_2\lambda+\ln\lambda+c_0+\ln r_C=1/2a_s, \tag{2.4}$$

where a_s is the "purely nuclear" scattering length at $l=0$ (scattering by the potential V_s), while the parameter r_C is defined as follows:

$$\ln r_C=\lim_{r\to 0}\left\{\ln r+\int_r^\infty\frac{1-\chi_0^2}{r}dr\right\}. \tag{2.5}$$

Here $\chi_0(r)$ is the wave function of the instant of the onset of the s level in the potential V_s and is determined by the Schrödinger equation

$$\chi_0''-U(r)\chi_0=0, \quad U(r)=\frac{2m}{\hbar^2}V_s(r) \tag{2.6}$$

with boundary conditions

$$\chi_0(0)=0, \quad \lim_{r\to\infty}\chi_0(r)=1. \tag{2.6a}$$

Since $V_s(r)$ decreases exponentially as $r\to\infty$, it follows that the integral in (2.5) converges on the upper limit at $r\sim r_0$. As to small r, the logarithmic divergence of this integral is compensated by the term $\ln r$, therefore the limit (2.5) exists and is finite. For a number of model potentials used in nuclear physics, the value of r_C can be found without numerical calculations. Let us examine several examples. We shall compare here r_C with the effective radius

$$r_s=2\int_0^\infty(1-\chi_0^2)dr, \tag{2.7}$$

which enters in the expansion of $k\cot\delta$ as $k\to 0$ (see Refs. 14–17).

1) For a square well of radius r_0 and depth $k^2/2$ we have $\chi_0(r)=\sin Kr$, $r<r_0$; the appearance of the ns level corresponds to $K_n r_0=(n-\frac{1}{2})\pi$. Hence

$$r_s=r_0, \quad r_C=r_0\exp\left(-\int_0^{(n-1/2)\pi}\sin^2 x\frac{dx}{x}\right).$$

In particular, for the ground level $n=1$ we have

$$r_C/r_s=\exp\{-{}^1/_2(C+\ln\pi-\mathrm{Ci}\,\pi)\}=0.439. \tag{2.8}$$

2) In the Hulthen potential $V_s=-g/(e^{\mu r}-1)$, the first s level appears at $g=\mu^2/2$, with $\chi_0(r)=1-e^{-\mu r}$. From (2.5) and (2.7) we get

TABLE I.

$v(x)$	k_e	k_C	r_C/r_e	Remark
$\theta(1-x)$	1.00	0.439	0.439	Potential well
$(e^x-1)^{-1}$	3.00	1.123	0.374	Hulthen potential
e^{-x}/x	2.16	0.775	0.360	Yukawa potential
e^{-x}	3.55	1.399	0.394	–
$(\mathrm{ch}x)^{-2}$	2.00	0.811	0.406	–

$$r_e=3\mu^{-1}, \quad r_C/r_e=^2/_3e^{-C}=0{,}374. \tag{2.9}$$

3) Assuming

$$V_s(r)=-\frac{g}{R^2}v\left(\frac{r}{R}\right),$$

where R is the characteristic radius of the interaction while the function $v(x)$ with $x=r/R$ specifies the form of the potential, we have

$$r_e=k_eR, \quad r_C=k_CR.$$

The numerical values of the coefficients k_i are listed in Table I. Owing to the factor $1/r$, the integral in (2.5) "lands" at shorter distances than the integral (2.7) as a result of which r_C is smaller by a factor 2.5–3 than the effective radius r_e. We note that the ratio r_C/r_e depends little on the forms of the potential. The exact value of r_C is not very essential here, since it enters logarithmically in (2.4).

Equations (2.2) and (2.4) show that the shifts of the Coulomb ns levels are not independent: the measured shift of one of the levels determines uniquely the shifts of the remaining s levels without any assumptions concerning the concrete form of the potential V_s. Equation (2.4), in addition, makes it possible to calculate the scattering length a_s from the value of the shift. At $a_s=0$, according to (2.4) we have $\lambda=1/n$, and $E_n=-1/2n^2$ is a pure Coulomb spectrum. At $|a_s|\ll a_B$ we are near the pole of the function $\psi(1-\lambda^{-1})$; it is easily seen that in this case Eq. (2.4) goes over into (1.1). However, because of the large logarithm ($|\ln r_C|\gg 1$), the difference between (2.4) and the perturbation-theory formula (1.1) comes into play quite rapidly.[13]

An analysis of Eq. (2.2) was carried out in Ref. 13 (see Figs. 1 and 3 there). A more detailed discussion of the spectrum of the s levels will be presented in Sec. 4, after the introduction of corrections for the effective radius r_e.

We have considered above the case $l=0$. Because of the pecularities of the expansion of the Whittaker function $W_{\nu,l+1/2}(2\lambda r)$ as $r\to 0$, the procedures for $l=0$ and $l\geq 1$ differ somewhat. An equation similar to (2.2) for states with orbital angular momentum $l\neq 0$ is given in Ref. 13.

§ 3. PROPERTIES OF THE SCATTERING LENGTH a_l

In the analysis of (2.4) we must take into account the properties of $1/a_l$ as functions of the depth of the potential V_s. Inasmuch as in the scattering-theory texts known to us[12,15-17] this question is not discussed, we describe briefly the derivation of the necessary formulas.

Let $l=0$ and $\hbar=2m=1$. The solution of the Schrödinger equation with zero energy

$$\varphi''-U(r)\varphi=0, \quad \varphi=rR(r), \tag{3.1}$$

takes on the asymptotic forms

$$\varphi(r)\propto r, \quad r\to 0; \quad \varphi(r)\approx 1-r/a_s, \quad r\to\infty, \tag{3.2}$$

where $a_s\equiv a_0$ is the s-scattering length; it is assumed that the potential $U(r)$ is not singular at zero and decreases more rapidly than any power of r as $r\to\infty$.

We consider now Eq. (3.1) for the potentials U_1 and U_2 and calculate the Wronskian $W(\varphi_1,\varphi_2)$. The standard procedure yields

$$[\varphi_1'\varphi_2-\varphi_1\varphi_2']_0^\infty=\int_0^\infty (U_1-U_2)\varphi_1\varphi_2\,dr,$$

from which we get, taking the boundary conditions (3.2) into account,

$$\frac{1}{a_1}-\frac{1}{a_2}+\int_0^\infty (U_1-U_2)\varphi_1\varphi_2\,dr=0. \tag{3.3}$$

If $U(r)=-gv(r)$, where $v(r)$ determines the form of the potential, then we get from (3.3)

$$\frac{d}{dg}\left(\frac{1}{a_s}\right)=\int_0^\infty v(r)\varphi^2(r)\,dr. \tag{3.4}$$

Thus, $1/a_s$ is a monotonically increasing function of the coupling constant g for an arbitrary attraction potential (i.e., $v(r)\geq 0$ for all r).

The case $l\geq 1$ differs only in that it is technically more complicated. If we normalize the solution of the Schrödinger equation with $E=0$ by the condition

$$\varphi_l(r)\approx -\frac{1}{(2l-1)!!(2l+1)!!}\frac{r^{l+1}}{a_l}+r^{-l}+\ldots, \quad r\to\infty,$$

then we get in place of (3.4)

$$\delta\left(\frac{1}{a_l}\right)=-\frac{1}{[(2l-1)!!]^2}\int_0^\infty \delta U(r)\varphi_l^2\,dr. \tag{3.5}$$

The instant of the appearance of the level corresponds to $a_l=\infty$; in this case the scattering length a_l for a small variation of the potential $\delta U(r)$ can be obtained from (3.5) in explicit form:

$$a_l=-[(2l-1)!!]^2\Big/\int_0^\infty \delta U(r)\chi_l^2(r)\,dr, \tag{3.6}$$

where $\chi_l(r)$ is the wave function at the instant of the appearance of the level.[4)] If we substitute here $U(r)=-gv(r)$ and recognize that $\chi_l'\chi_l\to 0$ as $r\to 0$ for all l, we obtain

$$g\frac{d}{dg}\left(\frac{1}{a_l}\right)=\frac{1}{[(2l-1)!!]^2}\int_0^\infty dr\left\{[\chi_l']^2+\frac{l(l+1)}{r^2}[\chi_l]^2\right\}. \tag{3.7}$$

The last formula is valid only at $a_l=\infty$. It shows that at the instant of the onset of the level we have $da_l^{-1}/dg>0$ regardless of whether or not $v(r)$ is positive for all r. Therefore the dependence of a_l on g has qualitatively the form indicated in Fig. 1.

As seen from (2.3) and (2.4), the restructuring of the spectrum takes place near points at which $\xi=-l$ and a_l becomes infinite. According to (3.7), at these points

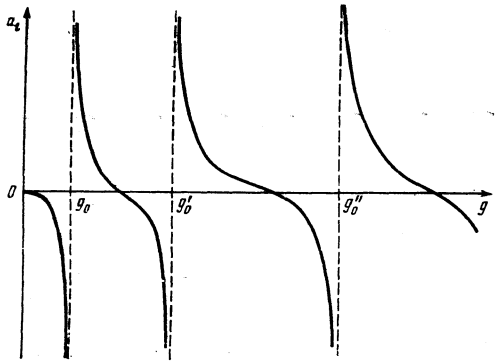

FIG. 1. Dependence of the scattering length on the coupling constant g (qualitative).

$1/a_l$ increases monotonically with increasing depth of the strong potential. This leads to the conclusion that the levels $E_n(g)$ have a universal behavior in the region of the restructuring of the atomic spectrum. This region is determined by the condition $|\xi+l| \lesssim (r_0/a_B)^{2l+1}$ and is narrower the larger the angular momentum l.

By way of illustration, we consider a number of potentials for which a_s can be calculated analytically.

1. Rectangular well with core

$$U(r)=\begin{cases} K_c^2, & 0<r<r_c, \\ -K^2, & r_c<r<r_0. \\ 0, & r>r_0 \end{cases}$$

In this case

$$a_s=r_0-\frac{1}{K}\operatorname{tg}\left\{K(r_0-r_c)+\operatorname{arctg}\left(\frac{K}{K_c}\operatorname{th}K_cr_c\right)\right\}. \tag{3.8}$$

If the core is high enough ($K_c \gg K$), then

$$a_s=r_0-(r_0-\tilde{r}_c)\beta^{-1}\operatorname{tg}\beta, \qquad \beta=K(r_0-\tilde{r}_c), \quad \tilde{r}_c=r_c(1-\operatorname{th}K_cr_c/K_cr_c)\leqslant r_c. \tag{3.9}$$

It is seen directly that (3.9) leads to Fig. 1.

2. The potential

$$U(r)=-U_0/\operatorname{ch}^2\mu r.$$

The Schrödinger equation with $E=l=0$ is solved in terms of Legendre functions[15]:

$$\chi(r)=c_1P_\varkappa(z)+c_2Q_\varkappa(z), \qquad z=\operatorname{th}\mu r, \quad \varkappa(\varkappa+1)=U_0/\mu^2.$$

The condition $\chi(0)=0$ yields the ratio

$$\frac{c_1}{c_2}=-\frac{Q_\varkappa(0)}{P_\varkappa(0)}=\frac{\pi}{2}\operatorname{tg}\frac{\pi\varkappa}{2}.$$

Taking into account the behavior of the functions $P_\varkappa(z)$ and $Q_\varkappa(z)$ as $z\to 1 (r\to\infty)$, we get

$$a_s=\mu^{-1}\left[C+\psi(\varkappa+1)-\frac{\pi}{2}\operatorname{tg}\frac{\pi\varkappa}{2}\right]. \tag{3.10}$$

At the instant of the onset of a level we have $\varkappa=1,3,5,\ldots$. In this case $c_2=0$, $\chi(r)$ satisfies the boundary condition (2.6a), and the s-scattering length has a pole.

§4. CONNECTION BETWEEN THE LEVEL SHIFT AND THE EFFECTIVE-RADIUS APPROXIMATION

We establish now the connection between (2.2) and (2.4), on the one hand, and the effective-radius approximation,[11] on the other. This leads to a more rigorous derivation of the equations of Sec. 2, and also yields the correction of order $r_e\lambda^2$, which is difficult to obtain by the matching method.

It is known (see, e.g., Refs. 12 and 15) that the phase of the scattering by the potential $V_s(r)-\alpha r^{-1}$ is not equal to the sum of the Coulomb phase, $\sigma_l=\arg\Gamma(l+1+i\alpha/k)$, and the "purely nuclear" phase $\delta_l^{(0)}$ corresponding to the potential V_s. The total scattering phase shift is written in the form $\sigma_l+\delta_l$, where δ_l is different from $\delta_l^{(0)}$. Considering next the case $l=0$, we designate $\delta_l^{(0)}$ by δ_s and δ_0 by δ_{cs}. The energy dependence of the phase shift δ_{cs} at small momenta k is determined by the Bethe equation[11]

$$C_0^2k\operatorname{ctg}\delta_{cs}-2\alpha h(\eta)=-1/a_{cs}+{}^1/_2r_{cs}k^2-Pr_{cs}^3k^4+\ldots, \qquad C_0^2=2\pi\eta/(1-e^{-2\pi\eta}), \quad h(\eta)=\operatorname{Re}\psi(-i\eta)-{}^1/_2\ln\eta^2; \tag{4.1}$$

here $\eta=\alpha/k$ is the Coulomb parameter. For potentials customarily employed in nuclear physics, the constant P is small ($|P|<0.1$, see Ref. 18). Therefore the last term of (4.1) can be discarded in the region $kr_{cs}\lesssim 1$.

To change from (4.1) to the discrete spectrum, we make the substitution

$$k=i\lambda, \qquad \eta=-i\alpha/\lambda, \qquad \operatorname{ctg}\delta\to i. \tag{4.2}$$

Here

$$h(\eta)={}^1/_2[\psi(1-i\eta)+\psi(1+i\eta)-\ln\eta^2] \to\psi(1-\lambda^{-1})+{}^1/_2\lambda+\ln\lambda+{}^1/_2\pi[i-\operatorname{ctg}(\pi/\lambda)],$$

and $C_0^2k\operatorname{cot}\delta$ goes over into

$$2\pi i[1-\exp(2\pi i/\lambda)]^{-1}$$

and cancels out the last term in $h(\eta)$. As a result we have

$$\frac{1}{2a_{cs}}=\psi\left(1-\frac{1}{\lambda}\right)+\frac{\lambda}{2}+\ln\lambda-\frac{1}{4}r_{cs}\lambda^2+O(r_0^3\lambda^4). \tag{4.3}$$

To find the "purely nuclear" scattering length a_s we use the relation

$$C_0^2k\operatorname{ctg}\delta_{cs}-2\alpha h(\eta) =k\operatorname{ctg}\delta_s+2\alpha\lim_{r\to 0}\left\{\int_r^\infty(\bar{u}^{(0)}\bar{u}-\bar{w}^{(0)}\bar{w})\frac{dr}{r}+\ln 2\alpha r+2C\right\} \tag{4.4}$$

(see Ref. 12, Eq. (348) of Chap. 6, where the definitions of the functions $\bar{u}^{(0)}$, $\bar{u}$, etc. are also given). Putting $k\to 0$, we get

$$1/a_s-1/a_{cs}=2\alpha(\ln R_c+c_0), \tag{4.5}$$

where $c_0=2C+\ln 2$,

$$\ln R_c=\lim_{r\to 0}\left[\int_r^\infty(\bar{u}^{(0)}\bar{u}-\bar{w}^{(0)}\bar{w})\frac{dr}{r}+\ln r\right]. \tag{4.6}$$

Equations (4.4)–(4.6) are exact, but are by themselves not very useful because the functions $\bar{u}^{(0)}, \bar{u}, \ldots$ depend on a_s and a_{cs}. We note that the integral in (4.6) converges at $r\sim r_0$ (at $r>r_0$ the strong potential is turned off and $\bar{w}^{(0)}\to\bar{u}^{(0)}, \bar{w}\to\bar{u}$). Replacing $\bar{u}$ by $\bar{u}^{(0)}$ and $\bar{w}$ by $\bar{w}^{(0)}$ in the region $r\lesssim r_0$ (where the nuclear interaction is stronger by two orders of magnitude than the

Coulomb interaction), and assuming by virtue of $|E| \ll |V_s(0)|$ that $\bar{u}^{(0)}(r)=1, \bar{w}^{(0)}(r)\equiv\chi_0(r)$, we find that $R_C \to r_C$ and

$$\frac{1}{a_s}-\frac{1}{a_{cs}}=2\alpha(\ln|\alpha|r_C+c_0). \tag{4.7}$$

The scattering lengths are measured here in units of a_B, $c_0=1.848$, and r_C is defined in (2.5); $\alpha=1$ for the $p\bar{p}$ system and $\alpha=-1$ for pp scattering. From (4.7) and (4.3), discarding the correction of order r_{cs}, we arrive at Eq. (2.4).

The accuracy of (4.7) can be assessed using as an example

$$V_s(p,p')=-\frac{2g}{\pi(p^2+\beta^2)(p'^2+\beta^2)} \tag{4.8}$$

(a separable Yamaguchi potential that admits of an exact solution[19]). In such a potential, the bound level appears at $g=2\beta^3$, and $\chi_0(r)=1-e^{-\beta r}$. The integral (2.5) can be easily calculated and Eq. (4.7) yields in the case of pp scattering

$$\frac{1}{a_{cs}}=\frac{\beta}{2}\left(1-\frac{2\beta^3}{g}\right)-\frac{2}{a_B}\left[\ln\frac{\beta a_B}{4}-C\right]. \tag{4.9}$$

Table II lists the values of the parameter β and g and of the scattering lengths a_s and a_{cs}, taken from Refs. 20 and 21 (the experimental value of a_{cs}, according to the latest compilation,[22] is $a_{cs}=-7.828\pm0.008$ F for PP scattering in the $^1S_0$ state). The value of a_{cs} was calculated in Ref. 20 by numerical integration and in Ref. 21 analytically. The last column gives the values of a_{cs} calculated from Eq. (4.9). It is seen that the calculation of the Coulomb interaction in the region $r<r_0$ in accord with perturbation theory, with (4.9) and (2.4) as the result, is accurate in this case to ~1%.

We present now the scattering lengths a_s and a_{cs} for the $p\bar{p}$ system, calculated on the basis of the experimental shift of the $1s$ level.[9] Equation (4.3) yields $a_{cs}=4.64\pm0.13$ F (the error is due mainly to the experimental uncertainty of the shift ΔE_{1s}; the term $\sim r_{cs}$ in (4.3) changes a_{cs} by only ~0.1% when r_{cs} changes from 0 to 2 F). The purely nuclear a_{cs} is calculated with a greater uncertainty, since Eq. (2.4) contains $\ln r_C$. The value of r_C can in principle be easily calculated if a definite $N\bar{N}$-interaction potential is specified (for example, by choosing the OBEP one-meson exchange potential). Since the important role in the integral of (2.5) is played by $r \ll r_0$, and the properties of the NN interaction at short distances have not been sufficiently well investigated, this procedure does not seem to us reliable enough at present. We therefore

TABLE II. Parameters of Yamaguchi potential in singlet pp-scattering lengths.

β, F^{-1}	g, F^{-3}	Reference	$-a_s$, F	$-a_{cs}$, F: Exact calculation	$-a_{cs}$, F: According to (4.9)
1.074	2.25	[20]	18.40	7.72	7.73
1.075	2.25	[21]	17.84	7.70	7.63
1.074	2.24	[21]	17.55	7.65	7.57
1.074	2.25	[21]	18.40	7.80	7.73
1.074	2.26	[21]	19.33	7.95	7.89
1.095	2.40	[21]	19.41	7.92	7.86

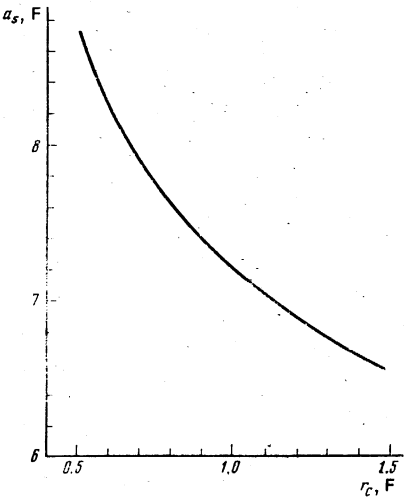

FIG. 2. The length a_s for $p\bar{p}$ scattering, corresponding to the experimental shift[9] of the level $1s$.

show in Fig. 2 a plot of a_s against the parameter r_C. Since $r_C \approx 0.4 r_e$ (see the examples in Table I), the values $r_C \approx 0.8$–1.2 F seem the most probable. The scattering length a_s then amounts to approximately 7 F.

We note that the perturbation-theory equation (1.1) yields (at the same shift ΔE_{1s}) a value $a_s=3.5$ F, which is half the true value. Thus, perturbation theory in terms of the scattering amplitude is valid in a rather narrow range. The reason lies in the distinctive behavior of the Coulomb wave functions, see Ref. 13.

The large value of the scattering length $a_s \approx 7$ F points to the existence in the $p\bar{p}$ system of a near-zero level Qs of the quasinuclear type,[2,3] which in fact causes the large shifts of the Coulomb ns levels. Its binding energy $\varepsilon(Qs)$ can be obtained from the shift ΔE_{1s} by using Eq. (4.3). This yields $\nu=0.106$ and $\varepsilon(Qs)=1100$ keV at $r_{cs}=0$. With increasing radius r_{cs}, the energy ε increases to $\varepsilon \approx 1400$ keV at $r_{cs}=1.5$ F. We see therefore that the determination of the position of the Qs level is less reliable than for Coulomb ns levels, because of the high sensitivity to the correction for the effective radius. A more accurate calculation of $\varepsilon(Qs)$ must be made within the framework of the two-channel problem (see the remark in Sec. 7).

For an experimental verification of the existence of the Qs level it is important to investigate its principal properties: the average radius, the probability of radiative transitions to this level, etc. We proceed now to deal with these questions.

§5. EFFECTIVE SIZE OF $p\bar{p}$ ATOM

For the $p\bar{p}$-atom levels considered by us, the conditions $r_0 \ll a_B$ and $\lambda r_0 \ll 1$ are satisfied. In this case the average radius

$$\langle r\rangle=\int_0^\infty \chi^2 r\,dr$$

can be calculated in analytic form. In the region $r \lesssim r_0$ we can neglect the Coulomb interaction and the binding energy $\varepsilon=1/2\nu^2$ compared with the potential V_s. Therefore the wave function of the s state takes the form

$$\chi_{\nu s}(r)=\begin{cases} cW_{\nu,1/2}(2\lambda r), & r>r_0 \\ c_1\chi_0(r), & 0<r<r_0, \end{cases} \tag{5.1}$$

where $\nu=\lambda^{-1}$ and $\chi_0(r)$ is defined in (2.6).

Joining together expressions (5.1) at $r_0 \ll r \ll 1/\lambda$ and taking into account (2.1) and the normalization condition

$$\int_0^\infty \chi_{\nu s}^2(r)\,dr=1,$$

we determine the constants

$$c_1=c/\Gamma(1-\nu),$$

$$c^2=\frac{2\lambda}{J(\nu,0)}\left[1+\frac{\lambda r_e}{\Gamma^2(1-\nu)J(\nu,0)}+O(\lambda r_e^2\ln r_e)\right]. \tag{5.2}$$

Here r_e is the effective radius for the strong potential V_s [see (2.7)], while $J(\nu,0)$ is an integral calculated in Appendix A.

Mean values of the type $\langle r^q\rangle$ are determined similarly:

$$\langle r^q\rangle=(2\lambda)^{-q}\left[1+\frac{\lambda r_e}{\Gamma^2(1-\nu)J(\nu,0)}+\ldots\right]$$
$$\times\left\{\frac{J(\nu,q)}{J(\nu,0)}-\frac{(2\lambda)^{q+1}}{\Gamma^2(1-\nu)J(\nu,0)}\int_0^\infty[1-\chi_0^2(r)]r^q\,dr+\ldots\right\}. \tag{5.3}$$

The last term in the curly brackets is of the order of smallness $(r_0/a_B)^{q+1}$, so that at $q>0$ it can be neglected compared with the change of the normalization constant c (on the contrary, if $-1<q<0$ this correction is the principal one).

In particular, the average radius is

$$\langle r\rangle=\rho_C+{}^1/_2\beta r_e+\ldots,$$
$$\rho_C=\frac{\nu J(\nu,1)}{2J(\nu,0)},\quad \beta=\frac{J(\nu,1)}{[\Gamma(1-\nu)J(\nu,0)]^2}, \tag{5.4}$$

where ρ_C is the average radius of the state νs in the limit when the interaction V_s is characterized by a zero radius. With the aid of (8.8) and (8.9) we get

$$\rho_C=\frac{3}{2}\nu^2+\frac{\sin^2\pi\nu}{4\pi^2\nu}A^2(\nu), \tag{5.5}$$

$$A(\nu)=\left\{1-\frac{\sin^2\pi\nu}{\pi^2}\left[\psi'(\nu)-\frac{2\nu+1}{2\nu^2}\right]\right\}^{-1/2}$$
$$=\begin{cases} 2^{1/2}(1-\nu+{}^3/_2\nu^2+\ldots), & \nu\to 0 \\ 1+\sin^2(\pi\nu)/12\pi^2\nu^3+\ldots, & \nu\to\infty \end{cases} \tag{5.6}$$

At the integer points $\nu=n=1,2,3,\ldots$ we have $A(n)=1, \beta(n)=0$, and $\langle r\rangle=\frac{3}{2}n^2$ is the known[15] value for the s states in the field of a point charge. If ν is not an integer, then ρ_C exceeds $3\nu^2/2$. For Coulomb levels ($\nu\gtrsim 1$), the difference between these quantities is small and decreases rapidly with increasing ν. On the other hand, at $\nu\ll 1$ (the Qs level, for which $\varepsilon\gg E_C$) we have $\rho_C/(3\nu^2/2)=1/3\nu\gg 1$.

The results of the calculation of $\langle r\rangle$ are shown in Fig. 3. With increasing level binding energy, the average radius decreases monotonically. The correction for the finite radius of the nuclear forces becomes noticeable at $\nu\sim 0.5$ ($\varepsilon>50$ keV), and reaches 25% for the quasinuclear level. It is seen from Fig. 3b that $\langle r\rangle$ exceeds

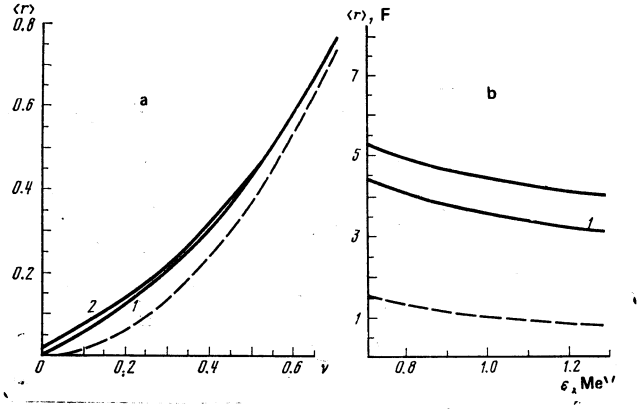

FIG. 3. Average radius of the νs state: a) as a function of the parameter $\nu(\langle r\rangle$ is given in units of a_B); b) in the region of energies ε near the quasinuclear level. Curves 1—$r_e=0.2$—$r_e=2$ F. The dashed curves are plots of $\langle r\rangle=1.5\,\nu^2$.

r_e noticeably at $\varepsilon<1.3$ MeV. Therefore the Qs state is analogous to a deuteron: p and $\bar p$ spend an appreciable fraction of the time outside the effective radius of the nuclear forces. Consequently, Eqs. (2.4) and (4.3) still remain meaningful at $\varepsilon\lesssim 1.5$ MeV. On the other hand, if $\rho_C\sim r_e$, then the properties of the bound state begin to depend substantially on the form of the potential V_s, by virtue of which such states cease to be described by Eq. (2.4) or (4.3).

§ 6. PROBABILITIES OF RADIATIVE TRANSITIONS IN THE $p\bar p$ ATOM

Measurement of the spectrum of the gamma rays emitted in transitions between the levels of the $p\bar p$ atoms provides a convenient method of verifying the theory expounded above. We have therefore calculated the probabilities of the electric dipole transitions $np\to\nu s$. Since the region of restructuring of the atomic spectrum for levels with $l=1$ is much narrower than for $l=0$, the shift of the p levels was neglected. In the calculation of the radial matrix element $\langle\nu s|r|np\rangle$ the wave function of the νs level was chosen in the form (5.1), while for np we used a solution regular at zero [see Eq. (8.11)]. As a result of the calculations, the details of which are discussed in Appendix A, the matrix element $\langle\nu s|r|np\rangle$ and the probability of the transition can be obtained in convenient analytic form:

$$w(np\to\nu s)={}^4/_9\alpha^3\omega^3|\langle\nu s|r|np\rangle|^2$$
$$=\frac{2\alpha^3}{9n^3}\left(1-\frac{1}{n^2}\right)f(n,\nu)\frac{A^2(\nu)}{\Gamma(\nu)\Gamma(\nu+1)}\{1+\beta_1 r_e+\ldots\}, \tag{6.1}$$

where $\omega=(n^2-\nu^2)/2n^2\nu^2$ is the transition frequency, $\alpha=e^2/\hbar c\approx\frac{1}{137}$,

$$f(n,\nu)=x^3(1+x)^{2\nu+2}\left[\sum_{k=2}^{n}(-1)^k\frac{(k+2)!(n-2)!}{(k-2)!(n-k)!\Gamma(k+3-\nu)}(1-x)^{k-2}\right.$$
$$\left.\times{}_2F_1\left(-\nu,1-\nu;k+3-\nu;\frac{x}{1+x}\right)\right]^2, \tag{6.2}$$

where $x=(n-\nu)/(n+\nu)$, with $0<x<1$ at $n>\nu$; the quantities $A(\nu)$ and r_e were defined in (5.6) and (2.7), and

$$\beta_1=[\nu\Gamma^2(1-\nu)J(\nu,0)]^{-1}$$
$$=\frac{1}{2\nu}\left[\frac{\sin\pi\nu}{\pi\nu}A(\nu)\right]^2$$

(the probabilities w are expressed here in units of $me^4/\hbar^3 = 3.80 \cdot 10^{19}$ sec^{-1} for the $p\bar{p}$ atom). The argument of the hypergeometric functions that enter in $f(n,\nu)$ is $x/(1+x) = (n-\nu)/2n < \frac{1}{2}$. Therefore their values can be easily calculated with the aid of the usual[23] power series for ${}_2F_1(\alpha,\beta;\gamma;z)$, which in this case converges rapidly.

We consider now the results of the numerical calculations, the properties of the transition probabilities, as well as some limiting cases.

1. If a near-zero level with $l=0$ exists in the nuclear potential V_s, then a small change of the depth of V_s alters greatly the energies of the s levels while the np states remain practically unchanged. It is therefore of interest to determine the probabilities of the transition $np \to \nu s$ as functions of $\nu = (2\varepsilon)^{-1/2}$. As $\nu \to n$ these probabilities vanish like $(n-\nu)^3$ because of the factor ω^3; as $\nu \to 0$, they decrease because of the weak overlap of the wave function $|np\rangle$ and $|\nu s\rangle$ (see Fig. 4). Somewhat

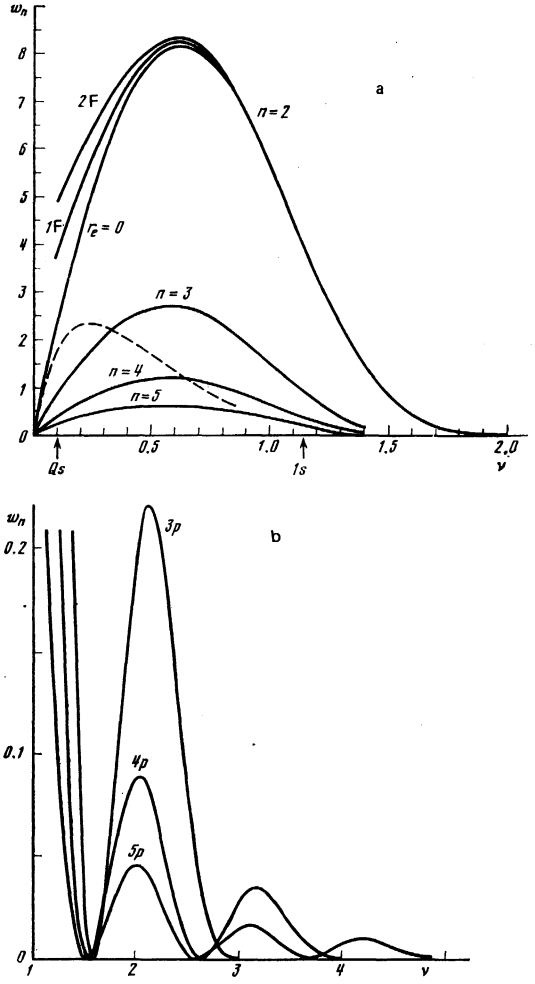

FIG. 4. Probabilities of $E1$ transitions $np \to \nu s$ as functions of the "principal quantum number" ν of the s level: a—at $\nu < 2$. b—in the region of atomic levels. The curves for $n=3$, 4, and 5 are plotted for $r_e = 0$. For the case $n=2$ (transition $2p \to \nu s$) the figure shows the plot of w_n against the effective radius r_e, as well as the approximation (6.4) (dashed curve). The probabilities $w_n \equiv w(np \to \nu s)$ are given in units of 10^{-11} sec^{-1} $= 6.58 \times 10^{-5}$ eV.

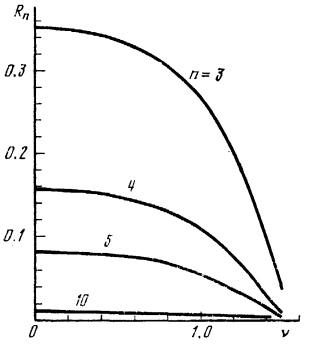

FIG. 5. Relative probabilities of the transition

$$R_n = w(np \to \nu s)/w(2p \to \nu s)$$

The values R_n at fixed ν correspond to a definite spectral series (Lyman series at $\nu = 1$, Balmer series at $\nu = 2$, etc.).

unexpected is the fact that in the interval $0 < \nu < n$ the dependence of $w(np \to \nu s)$ on the binding energy ε is not monotonic but has $n-1$ maxima and $n-2$ zeros. The zeros are due to the fact that the matrix element $\langle \nu s|r|np\rangle$ is a continuous function of ν and has different signs at $\nu = m$ and $\nu = m+1$. As a result, unique "dynamic" hindrances are imposed on the dipole transitions $np \to \nu s$ at certain (non-integer) values of the parameter ν (see Fig. 4b).

2. The relative probabilities $R_n = w(np \to \nu s)/w(2p \to \nu s)$ vary much more smoothly in the region $\nu < 1.5$. If the level is deep enough ($\nu < 0.5$, i.e., $\varepsilon \gtrsim 50$ keV), then R_n has a very weak dependence on the level energy, see Fig. 5. At small ν we have

$$R_n = A_n - B_n\nu^2 - C_n\nu^3 + O(\nu^4). \tag{6.3}$$

To calculate A_n, B_n, etc., it is convenient to start from the integral representation of $\langle \nu s|r|np\rangle$ given in Appendix B (where explicit formulas are also obtained for A_n, B_n, and C_n). The numerical values of these coefficients are given in Table III. A comparison with the results of the numerical calculation in accordance with the exact formulas (6.1) and (6.2) shows that the error of (6.3) does not exceed 1% at $\nu < 0.65$, i.e., at $\varepsilon > 30$ keV. Therefore it is convenient to use the expansion (6.3) when transitions to the quasinuclear level Qs are considered.

It should be noted that when R_n is calculated the first-order correction for the radius of the nuclear forces cancels out. Therefore the values of R_n are theoretically determined more reliably than the absolute transition probabilities w_n.

3. As $\nu \to 0$ the wave function (5.1) takes the form $\chi_{\nu s}(r) = (2/\nu)^{1/2} e^{-r/\nu}$, which corresponds to a level in a δ potential. In this limit, the calculation is elementary:

$$\langle \nu s|r|np\rangle = 2^{3/2}\left(1-\frac{1}{n^2}\right)^{1/2} n^{-3/2}\nu^{3/2},$$
$$w(np \to \nu s) = \alpha^3 \frac{16(n^2-1)}{9n^5}\nu. \tag{6.4}$$

TABLE III.

n	A_n	B_n	C_n	n	A_n	B_n	C_n
3	0.3512	0.04878	0.01301	8	0.02051	0.00481	0.00128
4	0.1563	0.02930	0.00781	9	0.01445	0.00343	0.00092
5	0.08192	0.01720	0.00459	10	0.01056	0.00253	0.00068
6	0.04801	0.01067	0.00285	$n \to \infty$	$10.7n^{-3}$	$2.67n^{-3}$	$0.711n^{-3}$
7	0.03046	0.00699	0.00187				

The corresponding curve is shown dashed in Fig. 4a.

A comparison with the exact calculation shows that this approximation has a rather narrow range of validity ($\nu \lesssim 0.1$). Because of the simplicity of the equations, however, it can be used to estimate the probabilities of the transition to the quasinuclear level. For transitions from the p states of the continuous spectrum to the νs level we obtain in this approximation

$$\frac{dw}{dk}=\frac{16}{9}\alpha^3\nu\frac{k}{1+k^2\nu^2}\frac{1+k^2}{1-e^{-2\pi/k}}\exp\left(-\frac{4}{k}\operatorname{arctg}k\nu\right),$$

where $E=k^2/2$ is the initial energy. At $E>E_C$ the value of dw/dk increases like $E^{3/2}$.

4. Another limiting case is that of transitions from highly excited p levels, when $n\gg 1$ and $n\gg\nu$. The sum (6.2) contains then many terms of the same order, thereby complicating the calculation. In the limit as $n\to\infty$, however, it is possible to obtain for the transition probability a simple asymptotic equation.

Assuming $n\to\infty$ and fixed values of l and r, we have

$$\lim_{n\to\infty}\chi_{nl}(r)=\frac{(2r)^{l+1}}{(2l+1)!n^{1/2}}\Lambda_{2l+1}(\sqrt{8r}), \tag{6.5}$$

where Λ_α is connected with the Bessel function by the relation[24]

$$J_\alpha(z)=\frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha\Lambda_\alpha(z).$$

In particular, as $r\to 0$,

$$\Lambda_{2l+1}(\sqrt{8r})=1-\frac{r}{l+1}+\frac{r^2}{(l+1)(l+3/2)}-\ldots$$

Using (6.5) and (A.4), and taking into account the value of the integral

$$\int_0^\infty e^{-\alpha x^2}J_\alpha(x)x^{\alpha+2n+1}dx=\frac{\Gamma(n+\alpha+1)}{2^{\alpha+1}a^{n+\alpha+1}\Gamma(\alpha+1)}e^{-1/4a}\Phi\left(-n,\alpha+1;\frac{1}{4a}\right),$$

we obtain

$$\lim_{n\to\infty}\langle\nu s|r|np\rangle=p(\nu)n^{-3/2},$$

where $p(\nu)$ is given by formula (A.7).

Thus, in the limit of large n the dependence of the transition probabilities on the quantum numbers of the initial and final states factors out:

$$w(np\to\nu s)_{n\to\infty}=\omega_0P(\nu)n^{-3}, \tag{6.6}$$

where

$$P(\nu)=\frac{2^{2\nu+3}A^2(\nu)}{9\Gamma(\nu)\Gamma(\nu+1)}\left[\sum_{k=0}^{\infty}\frac{\Gamma(k+5)}{k!\Gamma(k+5-\nu)}(-2\nu)^k\right.$$
$$\left.\times{}_2F_1\left(-\nu,1-\nu;k+5-\nu;\frac{1}{2}\right)\right]^2, \tag{6.7}$$

$\omega_0=\alpha^3me^4/\hbar^3=1.47\cdot10^{13}$ sec^{-1}, and $A(\nu)$ is defined in (5.6).

We note the limiting cases

$$P(\nu)=^{16}/_9\nu+O(\nu^3),\quad \nu\to 0, \tag{6.8}$$

and for integer $\nu=1,2,3,\ldots$

$$P(\nu)=\frac{128}{9}e^{-i\pi\nu}\nu!(\nu-1)!\left[\sum_{k=0}^{\nu-1}\frac{(2\nu-k)(-4\nu)^k}{(\nu-k-1)!k!(k+2)!}\right]^2. \tag{6.9}$$

A numerical calculation yields for $P(\nu)$ the curve shown in Fig. 6. We note that the zeros of the function $P(\nu)$ lie at $\nu=1.52, 2.56, \ldots$. These points correspond to the energy values $E(\nu s)=-12.5\nu^{-2}$[keV] at which the probabilities of $E1$ transitions to the level νs from the high-lying p states vanish. The largest values of the probability of the transition $np\to\nu s$ are reached at $\nu=0.55$, when $P(\nu)=0.53$. With increasing ν, the value of $P(\nu)$ decreases: $P(1)=0.2605$, $P(2)=0.0382$, $P(3)=0.0128$, etc.

5. It follows from (6.1) (with account taken of the experimental shift[6] of the level $1s$), that the probabilities of the dipole transitions from the $2p$ level to the atomic level $1s$ and to the quasinuclear level Qs are comparable. This follows also from a simple dimensional estimate of the ratio $R=w(2p\to Qs)/w(2p\to 1s)$, which was obtained by I. S. Shapiro and stimulated the exact calculations described above. The probability of the transition from the $2p$ state to the s state is

$$w\propto\omega^3\left|\int\psi_{2p}r\psi_s\,d^3r\right|^2.$$

Recognizing that at small distances we have $\psi_{2p}\sim r$ and $\psi_{Qs}/\psi_{1s}\sim(\langle r\rangle/a_B)^{-3/2}$, we obtain

$$R\approx\left(\frac{\omega_Q}{\omega_1}\right)^3\left(\frac{\langle r\rangle}{a_B}\right)^7, \tag{6.10}$$

where ω_Q and ω_1 are the frequencies of the transitions $2p\to Qs$ and $2p\to 1s$, $\langle r\rangle$ is the average radius of the Qs state. At $\omega_Q/\omega_1\sim10^2$ and $\langle r\rangle/a_B\approx0.10$–$0.15$ the value of R changes from 0.1 to 1.

6. When ν becomes an integer, the shift $\Delta E_{\nu s}$ vanishes. Equation (6.1) becomes simpler and coincides with Gordon's formula[25] for the transition probabilities in a hydrogenlike atom. We demonstrate this with $\nu=1$ as an example. In this case the hypergeometric function in (6.2) becomes equal to unity, and the sum can be calculated explicitly:

$$\sum_{k=2}^{n}(-1)^k\frac{k+2}{(k-2)!(n-k)!}(1-x)^{k-2}=\frac{x^{n-3}}{(n-2)!}[(n+2)x-(n-2)],$$

where $x+(n-1/n+1)$. Hence

$$w(np\to 1s)=\frac{128n(n-1)^{2n-2}}{9(n+1)^{2n+2}}\omega_0, \tag{6.11}$$

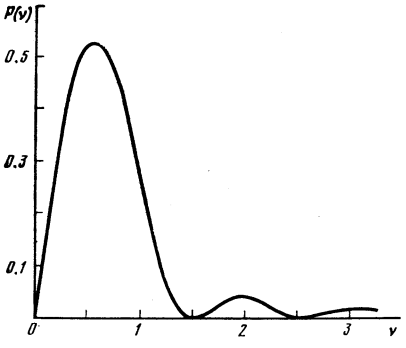

FIG. 6. Dependence of the probability of the transition $np\to\nu s$ on the position of the s level in the limit $n\gg\nu$.

which coincides with the known result[25,26] for the Lyman series in the hydrogen atom.

§7. CONCLUSION

We make now a few concluding remarks.

1. Equations (2.4) and (4.3) enable us, by measuring the shift of one of the s levels of the $p\bar{p}$ atom, to calculate the shifts of the remaining Coulomb levels, and also the scattering lengths a_s and a_{cs}. From the calculations described in Sec. 6, it follows that the probabilities of the radiative transitions from the $2p$ level to the levels $1s$ and Qs are comparable in magnitude (see Fig. 4a). Therefore the conclusion that a quasinuclear level Qs exists can be verified by direct experiment: we must seek in the gamma-ray spectrum a line with energy ~1 MeV having approximately the same intensity as for the $2p \to 1s$ transition.

2. We used everywhere above the single-channel approximation. At distances $r < r_0$, however, the transition $p\bar{p} \to n\bar{n}$ is possible, so that the Coulomb spectrum of the $p\bar{p}$ atom must be determined, strictly speaking, with account taken of both channels. Under certain assumptions concerning the $N\bar{N}$-scattering lengths in the states with isospin $I=0$ and $I=1$, the binding energy of the Qs level decreases when account is taken of the $n\bar{n}$ channel, by an amount on the order of 100–300 keV (private communication from B. O. Kerbitov). This estimate, however, did not take into account the Coulomb interaction at $r < r_0$. Calculations of the two-channel problem with complete allowance for the Coulomb interaction and for the mass difference $m_n - m_p$ are presently in progress.

3. The properties of the discrete spectrum in a Coulomb field distorted at short distances were first investigated apparently by Zel'dovich[27] (in connection with the question of the energy levels of an electron in an impurity semiconductor). In this case

$$V(r) = -e^2/\varepsilon r, \quad r > r_0, \tag{7.1}$$

and the potential in the region $r < r_0$ can differ quite strongly from the Coulomb potential. The electron detached from the donor moves at $r \gg r_0$ in the field of a positive ion in a medium with dielectric constant ε. At $\varepsilon \gg 1$ the radius of the first Bohr orbit increases, and the energy of the Coulomb level decreases:

$$a_B = \varepsilon/\mu, \quad E_n = -\mu/2\varepsilon^2 n^2 \tag{7.2}$$

(in atomic units). Here $\mu = m_{\text{eff}}/m$, m is the electron mass, m_{eff} is its effective mass in the lattice, and $r_0 \sim \hbar^2/me^2 = 1$ is the radius of the ion. In a medium with a large ε, the condition $r_0 \ll a_B$, which is similar to that used in the theory of the $p\bar{p}$ atom, is satisfied.

Zel'dovich[27] obtained for the s-level shifts an equation equivalent to (1.1), indicated the connection between the shift and the scattering length a_s, and showed that the well ($r < r_0$) can exert a substantial influence on the Coulomb levels only in the presence of a bound state (real or virtual) with low binding energy. He indicated that rearrangement of the atomic spectrum sets in at the instant when a bound level appears in the well (in the absence of a Coulomb field), and estimated the width of the restructuring region at $r_0/a_B - m_{\text{eff}}/m\varepsilon \ll 1$. Thus, Zel'dovich's paper[27] anticipates some of the later results.[7,8] It does not contain, however, equations such as (2.2) or (2.4), which describe the level spectrum at all values of the well depth (including the restructuring region).

4. We wish to note finally that in a different physical situation[5)] Eq. (4.3) with $r_{cs} = 0$ was obtained by Nikishov and Ritus.[28] They have qualitatively investigated the influence of a short-range potential on a system of Coulomb levels, but neglected the difference between the scattering lengths a_s and a_{cs}, which can be substantial at a low binding energy ε (see Eq. (4.7) and Table II above).

The authors consider it their pleasant duty to thank O. D. Dal'karov, B. O. Kerbikov, V. E. Markushin, and Yu. A. Simonov for interesting discussions in the course of the work and for a number of helpful remarks, as well as Ya. B. Zel'dovich and K. A. Ter-Martirosyan for a discussion of the results. We are particularly grateful to I. S. Shapiro for constant interest and support, and also N. I. Borisova and V. L. Eletskiĭ for help with the numerical calculations.

APPENDIX A

Calculation of integrals with coulomb wave functions

As seen from (2.1), the function $W_{\nu, l+1/2}(2\lambda r)$ with $l=0$ is finite at zero for any energy. Therefore the contribution of the region $r < r_0$ is small, and in the normalization and calculation of the matrix elements of the type $\langle \nu s | r | np \rangle$ we can use the approximation with a zero radius of the nuclear forces

$$\chi_{\nu s}(r) = rR_{\nu s}(r) = c_\nu W_{\nu, 1/2}(2\lambda r). \tag{A.1}$$

Determining the constant c_ν from the condition

$$\int_0^\infty \chi_{\nu s}^2(r)\, dr = 1,$$

we obtain

$$c_\nu = [2\lambda/J(\nu, 0)]^{1/2} = \frac{A(\nu)}{\nu^{1/2}\Gamma(\nu+1)}, \qquad \langle r^q \rangle = (\nu/2)^q J(\nu, q)/J(\nu, 0), \quad q > -1, \tag{A.2}$$

where $\nu = \lambda^{-1}$, $A(\nu)$ is given in (5.6), and

$$J(\nu, q) = \int_0^\infty [W_{\nu, 1/2}(x)]^2 x^q\, dx. \tag{A.3}$$

To calculate this integral, assume that $\nu < 1$ (the results for arbitrary ν will be obtained by analytic continuation). We replace in (A.3) each of the Whittaker functions by an integral representation[23]

$$W_{\nu, 1/2}(x) = \frac{x}{\Gamma(1-\nu)} \int_0^\infty e^{-x(1+2t)/2} t^{-\nu} (1+t)^\nu\, dt, \tag{A.4}$$

after which the integral with respect to x is calculated in elementary fashion:

$$J(\nu, q) = \frac{\Gamma(q+3)}{[\Gamma(1-\nu)]^2} \int_0^\infty dt\, d\tau \frac{[(1+t^{-1})(1+\tau^{-1})]^\nu}{(1+t+\tau)^{q+3}}.$$

We make the change of variables $t/(1+t)=x$, $\tau/(1+\tau)=y$; then

$$1+t+\tau=(1-xy)/(1-x)(1-y),$$

$$J(\nu,q)=\frac{\Gamma(q+3)}{[\Gamma(1-\nu)]^2}\int_0^1 dx\,dy\,\frac{[(1-x)(1-y)]^{q+1}}{(xy)^{\nu}(1-xy)^{q+3}}.$$

The rest of the integration is carried out with the aid of Eqs. (9.111) and (7.512.5) of the handbook[23] and yields $J(\nu, q)$ in terms of a generalized hypergeometric series[23]:

$$J(\nu,q)=\frac{\Gamma(q+3)[\Gamma(q+2)]^2}{[\Gamma(q+3-\nu)]^2}\,{}_3F_2(q+3,1-\nu,1-\nu;\,q+3-\nu,q+3-\nu;1).$$

With the aid of the identity[29]

$${}_3F_2(a,b,c;e,f;1)=\frac{\Gamma(e)\Gamma(f)\Gamma(p)}{\Gamma(a)\Gamma(b+p)\Gamma(c+p)}\,{}_3F_2(e-a,f-a,p;\,b+p,c+p;1),$$

where $p=e+f-a-b-c$, we obtain the final results for the integral (A.3):

$$J(\nu,q)=\left[\frac{\Gamma(q+2)}{\Gamma(q-\nu+2)}\right]^2\Gamma(q+1)\,{}_3F_2(-\nu,-\nu,q+1;\,q-\nu+2,q-\nu+2;1). \tag{A.5}$$

At $q=0, 1, 2, \ldots$ the hypergeometric function contained in this expression reduces to simpler special functions. For example, at $q=0$

$${}_3F_2(-\nu,-\nu,1;2-\nu,2-\nu;1)=\nu^2(\nu-1)^2\sum_{n=0}^{\infty}[(n-\nu)(n+1-\nu)]^{-2}.$$

Differentiating the equation

$$\sum_{n=0}^{\infty}\frac{1}{(n+\varkappa)(n+\lambda)}=\frac{\psi(\varkappa)-\psi(\lambda)}{\varkappa-\lambda} \tag{A.6}$$

with respect to the parameters $\varkappa$ and λ, and putting $\varkappa=-\nu$ and $\lambda=1-\nu$, we get

$$\sum_{n=0}^{\infty}[(n-\nu)(n+1-\nu)]^{-2}=2\psi'(-\nu)+\frac{2\nu-1}{\nu^2}, \tag{A.7}$$

whence

$$J(\nu,0)=2[\Gamma(\nu+1)]^2\left\{1-\frac{\sin^2\pi\nu}{\pi^2}\left[\psi'(\nu)-\frac{2\nu+1}{2\nu^2}\right]\right\}. \tag{A.8}$$

Similarly at $q=1$

$$J(\nu,1)=6\nu[\Gamma(\nu+1)]^2\left\{1-\frac{\sin^2\pi\nu}{\pi^2}\left[\psi'(\nu)-\left(\frac{1}{\nu}+\frac{1}{2\nu^2}+\frac{1}{6\nu^3}\right)\right]\right\}. \tag{A.9}$$

This yields Eq. (5.5) for the average radius of the state νs.

In the calculation of the magnetic element of the electric dipole transition, we take into account the fact that[15]

$$\chi_{np}(r)=\frac{2}{3}n^{-3/2}\left(1-\frac{1}{n^2}\right)^{1/2}e^{-r/n}r^2\Phi\left(2-n,4;\frac{2r}{n}\right). \tag{A.10}$$

Here $n=2, 3, \ldots$, so that the confluent hypergeometric function reduces to a polynomial:

$$\chi_{np}(r)=\left(\frac{n^2-1}{n}\right)^{1/2}e^{-r/n}\sum_{k=2}^{n}c_{nk}r^k,$$

$$c_{nk}=\frac{(n-2)!}{(k-2)!(k+1)!(n-k)!}\left(-\frac{2}{n}\right)^k. \tag{A.11}$$

The matrix element is represented as a sum of integrals of the type

$$\int_0^{\infty}e^{-r/n}r^{k+1}W^2_{\nu,1/2}(2r/\nu)\,dr,$$

which can be expressed by using Eq. (7.62.1.3) from Ref. 23 in terms of the function

$${}_2F_1\left(1-\nu,k+3;\,k+3-\nu;-\frac{n-\nu}{n+\nu}\right).$$

Converting the hypergeometric function from the argument $-z$ to the argument $z/(1+z)$, we obtain

$$\langle\nu s|r|np\rangle=4^{-1}c_{\nu}n^{3/2}(n^2-1)^{1/2}(1+z)^{\nu}$$

$$\times\sum_{k=2}^{n}(-1)^k\frac{(n-2)!(k+2)!}{(n-k)!(k-2)!\Gamma(k+3-\nu)}(1-z)^{k+2}\,{}_2F_1\left(-\nu,1-\nu;\,k+3-\gamma;\frac{z}{1+z}\right), \tag{A.12}$$

where $z=(n-\nu)/(n+\nu)$. This leads directly to (6.1).

APPENDIX B

Dipole-transition matrix element as $n\to\infty$

We obtain an integral representation for the matrix element of the electric dipole transition

$$\langle\nu s|r|np\rangle=C_1\int_0^{\infty}e^{-r/n}r^3\Phi\left(2-n,4;\frac{2r}{n}\right)W_{\nu,1/2}(2\lambda r)\,dr. \tag{B.1}$$

Replacing the Whittaker function by means of Eq. (8.4), we integrate with respect to r using the equality

$$\int_0^{\infty}e^{-pr}\Phi(\alpha,\beta;qr)r^{\beta}\,dr=\Gamma(\beta)\frac{\beta p+(\alpha-\beta)q}{p^{\beta+2}(1-q/p)^{\alpha+1}}, \tag{B.2}$$

and obtain

$$\langle\nu s|r|np\rangle=C_2\int_0^{\infty}dt(1+t^{-1})^{\nu}[2\lambda(1+2t)-1]\frac{[n\lambda(1+2t)-1]^{n-3}}{[n\lambda(1+2t)+1]^{n+3}}. \tag{B.3}$$

The change of variable $x=(1+2t)^{-1}$ leads to the sought representation

$$\langle\nu s|r|np\rangle=\frac{C_3}{\Gamma(1-\nu)}\int_0^1\left(\frac{1-x}{1+x}\right)^{-\nu}\frac{(1-\nu x/n)^{n-3}}{(1+\nu x/n)^{n+3}}\left(x^3-\frac{\nu}{2}x^4\right)dx, \tag{B.4}$$

which contains only elementary functions. From this we obtain as $n\to\infty$

$$\lim_{n\to\infty}\langle\nu s|r|np\rangle=p(\nu)n^{-3/2}, \tag{B.5}$$

$$p(\nu)=\frac{8\nu^4c_{\nu}}{\Gamma(1-\nu)}\int_0^1\left(\frac{1-x}{1+x}\right)^{-\nu}e^{-2\nu x}(2x^3-\nu x^4)\,dx \tag{B.6}$$

(the dependence of the matrix element on the numbers n and ν factors out). Since we used (A.4) in the derivation, this integral converges only at $\nu<1$. An analytic continuation to the region $\mathrm{Re}\,\nu>1$ can be obtained by expanding the exponential in (B.6) in a series an integrating term by term with account taken of the formula

$$\frac{1}{\Gamma(1-\nu)}\int_0^1\left(\frac{1+x}{1-x}\right)^{\nu}x^q\,dx=2^{\nu}\frac{\Gamma(\nu+1)}{\Gamma(q+2-\nu)}\,{}_2F_1(-\nu,1-\nu;\,q+2-\nu;\,1/2).$$

As a result we get

$$p(\nu)=2^{\nu-2}c_\nu\sum_{k=4}^{\infty}\frac{k(k-1)(k-2)(k-3)}{\Gamma(k+1-\nu)}(-2\nu)^k{}_2F_1(-\nu,1-\nu;k+1-\nu;{}^1/_2). \tag{B.7}$$

Since $w(np\to\nu s)=\alpha^3p^2(\nu)/18\nu^6n^3$ as $n\to\infty$, we obtain from this equation (6.7).

Equation (B.7) yields an analytic continuation of the function $p(\nu)$ to the entire complex ν plane. Indeed, as $k\to\infty$ we have

$${}_2F_1(-\nu,1-\nu;k+1-\nu;{}^1/_2)=1+\frac{\nu(\nu-1)}{2k}+O\left(\frac{1}{k^2}\right),$$

Therefore a k-th term of the series behaves like $(-2\nu)^k k^{\nu+4}/k!$ and the series (B.7) converges absolutely at all values of $|\nu|$. Thus, the restriction $\mathrm{Re}\,\nu<1$ can be disregarded in the final formulas.

We present the explicit form of the constants C_i contained in the equations of this appendix:

$$\begin{aligned}&C_1=\frac{2}{3n^{3/2}}\left(1-\frac{1}{n^2}\right)^{1/2}c_\nu, \qquad C_2=\frac{24n^6}{\nu\Gamma(1-\nu)}C_1,\\ &C_3=16n^{-3/2}\left(1-\frac{1}{n^2}\right)^{1/2}\nu^4c_\nu, \qquad c_\nu=\frac{A(\nu)}{\nu^{1/2}\Gamma(\nu+1)},\end{aligned} \tag{B.8}$$

where $A(\nu)$ was defined in (5.6).

The integral representation (B.4) is convenient in the limiting case $\nu\to 0$, which corresponds to a deep s level. Substituting the expansions

$$\left(\frac{1-x}{1+x}\right)^{-\nu}=1+2\nu t+2\nu^2t^2+\frac{4}{3}\nu^3t^3+\dots, \quad t=\mathrm{Arth}\,x,$$

$$\left(1-\frac{\nu x}{2}\right)\frac{(1-\nu x/n)^{n-3}}{(1+\nu x/n)^{n+3}}=1-\frac{5}{2}\nu x+\nu^2\frac{3(n^2+1)}{n^2}x^2-\nu^3\left(\frac{7}{3}+\frac{49}{6n^2}\right)x^3+\dots,$$

in the integral of (B.4), we get

$$\langle vs|r|np\rangle=\frac{C_3}{\Gamma(1-\nu)}[K+L\nu+M_n\nu^2+N_n\nu^3+\dots],$$

where

$$K=\frac{1}{4}, \quad L=\int_0^1\left(2t(x)-\frac{5}{2}x\right)x^3\,dx=\frac{1}{6}.$$

The subsequent coefficients already depend on n and have a rather unwieldy form, but the most complicated integrals cancel out in the differences M_n-M_2 and N_n-N_2. We obtain

$$\begin{aligned}\frac{\langle vs|r|np\rangle}{\langle vs|r|2p\rangle}&=\frac{C_3(n,\nu)}{C_3(2,\nu)}\left\{1+\frac{M_n-M_2}{K}\nu^2\right.\\ &\left.+\left[\frac{N_n-N_2}{K}-\frac{L}{K^2}(M_n-M_2)\right]\nu^3+\dots\right\}\\ &=\frac{2^{5/2}}{3^{1/2}n^{3/2}}\left(1-\frac{1}{n^2}\right)^{1/2}\left[1-\frac{n^2-4}{2n^2}\nu^2-\frac{n^2-4}{30n^3}\nu^3+\dots\right].\end{aligned} \tag{B.9}$$

In this calculation we used the value of the integral

$$I_\rho=\int_0^1 x^\rho\,\mathrm{Arth}\,x\,dx=\frac{1}{2(\rho+1)}[\psi(\rho+2)-\beta(\rho+2)+C+\ln 2],$$

where $\psi(z)$ is the logarithmic derivative of the gamma function and

$$\beta(z)=\frac{1}{2}\left[\psi\left(\frac{z+1}{2}\right)-\psi\left(\frac{z}{2}\right)\right],$$

see Ref. 23. I_ρ becomes particularly simple if ρ is odd ($\rho=2m+1$),

$$I_\rho=\frac{1}{2m+2}\sum_{k=0}^{m}\frac{1}{2k+1} \qquad (m=0,1,2,\dots).$$

From (B.9) follows the expansion (6.3), in which

$$A_n=\frac{32(n^2-1)}{3n^5}, \quad B_n=\frac{8}{3n^7}(n^2-1)(n^2-4), \quad C_n=\frac{4}{15}B_n. \tag{B.10}$$

[1)] The spectrum of which is much richer than the bound-state spectrum of the NN system (where there is only one bound state—the deuteron). The reason is that the attraction between N and $\bar N$ is stronger than between two nucleons at approximately the same effective radius of the nuclear forces.[2]

[2)] We use in this article the Coulomb units $\hbar=m=e=1$. For the $p\bar p$ system the energy unit is $E_C=me^4/\hbar^2=25.0$ keV, the unit of length is the Bohr radius $a_B=\hbar^2/me^2=57.6$, where $m=M_p/2$ is the reduced mass.

[3)] A brief exposition of the results of this section was published earlier.[13]

[4)] We note that at the instant when the level is produced (i.e., $g=g_0$, $a_l=\infty$), a substantial change takes place in the asymptotic form of the zero-energy wave function $\varphi_l(r,g)$ as $r\to\infty$. To emphasize this, we have designated $\varphi_l(r,g_0)$ by $\chi_l(r)$. Since $\chi_l\sim r^{-l}$ as $r\to\infty$, this function is normalized if $l\geqslant 1$ (the rapid decrease of χ_l at infinity is due to the centrifugal barrier). At $l=0$ it satisfies the boundary condition (2.6a).

[5)] Namely, in the problem where a level bound by δ forces is ionized by the field of an electromagnetic wave.

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Contribution to the statistics of an ensemble of nonlinear quantum oscillators excited by an external periodic force

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The level population of an ensemble of oscillators is considered in the case when the resonant quantum number n_r determined from the condition $E_{n_r+1} - E_{n_r} \approx \hbar\omega_l$ is satisfied (ω_l is the frequency of the external force). It is shown that on levels with $n \geq n_r$ there can appear an excess population that can be relatively large even if the relaxation time is much shorter than the time of excitation by the external force. It is noted that laser radiation can exert an appreciable stimulating action on chemical reactions with large activation energy, whereas reactions with low activation energy are not subject to this influence.

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1. Experiments[1-5] have shown that even relatively weak laser radiation acting continuously on a dense gas mixture[1-4] or on a solid[5] is capable of exciting the above-thermal impurity resonant molecules contained in the medium, and consequently stimulate chemical reactions in which these molecules participate. A theoretical interpretation of these experiments calls for the study of the statistics of an ensemble of particles excited by an external force (i. e., by a laser radiation) and located in a thermostat (i. e., in an ambient of nonresonant molecules).

Similar problems were considered recently in a number of studies.[6-9] The present note deals with an ensemble of nonlinear one-dimensional oscillators. After simplifying the initial equations by the averaging method (sec. 2), we obtain their solution in the case of oscillators with rapidly relaxing phase (Sec. 3). The last model admits of a simple analytic solution and at the same time contains enough physical substance for numerical estimates (Sec. 4) and some qualitative conclusions (Sec. 5).

2. The initial equation for the density matrix ρ of a linear oscillator can be assumed in the form[10,11]

$$i\hbar\dot{\rho}-[H-F(t)x,\rho]=\frac{\hbar\nu}{2}((1+N)(2a\rho a^+-a^+a\rho-\rho a^+a) +N(2a^+\rho a-aa^+\rho-\rho aa^+)), \quad (1)$$

where H is the Hamiltonian, with $H|n\rangle=E_n|n\rangle$ and

$$E_n=\hbar\omega_v(n+1/2), \quad (2)$$

ω_v is the natural frequency, $F(t)$ is the external force,

$$x=(\hbar/2m\omega_v)^{1/2}(a+a^+), \quad (3)$$

$$\langle n|a^+|n-1\rangle=\langle n-1|a|n\rangle=n^{1/2}, \quad (4)$$

$$N=[\exp(\hbar\omega_v/T)-1]^{-1} \quad (5)$$

is the thermal quantum number, T is the temperature of the thermostat, and ν^{-1} is the characteristic relaxation time of the oscillator in the thermostat. Equation (1) admits under conditions (2)–(5) of an exact solution.[11]

Proceeding to consideration of a nonlinear oscillator, we confine ourselves to a harmonic external force $F(t)=F\cos\omega_l t$ and to the resonant case

$$|E_n-\hbar\omega_l(n+1/2)|\ll E_n. \quad (6)$$

Expressions (1) and (3)–(5) can then be left unchanged, apart from the substitution $\omega_v\to\omega_l$ in (3) and in (5). Thus, the nonlinearity of the oscillator manifests itself only in that its energy spectrum E_n differs from (2). That this approximation is admissible was proved in Ref. 12.

We introduce the matrix β:

$$\rho_{mn}=\beta_{mn}\exp[-i\omega_l(m-n)t].$$

The matrix β varies little over the period of the external force; now, discarding the rapidly oscillating terms, we get from (1)

$$\dot{\beta}_{mn}+i\beta_{mn}[(E_m-E_n)/\hbar-\omega_l(m-n)]+1/2i(\beta_{m\,n-1}n^{1/2}f^*-\beta_{m-1\,n}m^{1/2}f +\beta_{m\,n+1}(n+1)^{1/2}f-\beta_{m+1\,n}(m+1)^{1/2}f^*) =\nu N(mn)^{1/2}\beta_{m-1\,n-1}+\nu(1+N)[(m+1)(n+1)]^{1/2}\beta_{m+1\,n+1} -\nu[(m+n)(N+1/2)+N]\beta_{mn}, \quad (7)$$

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