Kinetics of a system of phonons with an almost linear dispersion law

V. L. Gurevich and B. D. Laĭkhtman

A. F. Ioffe Institute of Engineering Physics, Academy of Sciences of the USSR
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Methods are developed for describing kinetically a system of waves with an almost linear dispersion law. As has been shown before by Zakharov, L'vov, Gurevich, and Laikhtman [Zh. Prik. Mat. Tekh. Fiz. **4**, 35 (1965); Izv. vuzov Radiofizika **18**, 1470 (1975); Ann. Phys. (N.Y.) **106**, 444 (1977)], for the description of phonon-phonon interaction in such a system by the methods of standard perturbation theory it must be required that the ratio of the phonon damping (owing to anharmonicity) to the primarily assumed dispersion (i.e., the contribution to the frequency that is nonlinear in the wave vector) be small. The case in which the primary dispersion is small and the ratio in question is large is analyzed. It is shown that the anharmonic interaction of the phonons introduces an additional dispersion. It is negative and of larger absolute value than the damping. This permits the use of an "improved" perturbation theory (which employs a renormalized phonon spectrum) for the calculation of the probabilities of phonon-phonon collisions. For solids, in particular, this makes it possible to determine when the well known assumption of Simons, that the uncertainty in the phonon energies owing to collisions makes "almost allowed" processes completely allowed, is justified. Generally speaking it is correct if the interaction with phonons of "other" vibrational branches [L. E. Gurevich and B. I. Shklovskii, Sov. Phys. Solid State **9**, 401 (1967)] predominates, and is not justified if the interaction inside a given branch predominates. The "improved" perturbation theory turns out to be always applicable for the description of phonon-phonon collisions in helium II.

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In the present paper we wish to analyze the difficulties that arise in the description of the kinetics of a quantum wave system with an almost linear dispersion law, and to show ways to overcome them. The difficulties in question are not in general of a specifically quantum nature. They have been noted repeatedly in the literature, both that devoted to quantum problems and that on classical problems, and arise in attempts to apply perturbation theory to the description of interactions between such waves. The simplest approach is to assume in zeroth approximation that the waves are noninteracting and are characterized by a certain primary dispersion law. The interaction is dealt with in the lowest order in which it appears, and in order to neglect higher-order corrections it is assumed that the anharmonicity is sufficiently small in comparison with the characteristic frequency of the waves.

The situation is much more complicated, however, if the primarily assumed dispersion law $\omega_k = c_0 k[1+\xi(k)]$ ($\mathbf{k}$ is the wave vector and ω_k is the frequency) is nearly linear, i.e., if $|\xi(k)| \ll 1$. In this case the criterion for applicability of perturbation theory is that the anharmonicity be small in comparison with the dispersion. For $\xi > 0$ we can take as a measure of the anharmonicity the damping Γ associated with three-wave interaction, and the criterion takes the form

$$\Gamma/c_0 k|\xi| \ll 1. \tag{1}$$

If $\xi < 0$, three-wave processes are forbidden. This sort of spectrum is nondecaying, or a spectrum with negative dispersion. For a nondecaying spectrum the criterion (1) can be written in the same form, taking Γ as the same combination of the various quantities (anharmonicity coefficient, speed of sound, and so on) as for $\xi > 0$, but no longer with the physical meaning of a damping. Accordingly, we have the problem of constructing a kinetics when the inequality (1) is violated; this is the problem we shall solve.

The physical cause of the appearance of the parameter (1) is that when the dispersion is small, waves that propagate in almost the same direction are in resonance. For clarity let us consider the waves propagating within a cone of directions with aperture angle $\vartheta \ll 1$, and change to a reference system in which one of the waves is at rest. In this reference system the wave patterns of the other waves change not in times of order ω_k^{-1}, as would be the case for strong dispersion (ω is the characteristic frequency), but in times of order $\omega_k^{-1}|\vartheta^2+\xi|^{-1}$. The result is that the effective coupling constant of the waves is proportional to $|\vartheta^2+\xi|^{-1}$.

It is well known that an analogous parameter, the ratio of the anharmonicity to the dispersion, appears in the theory of the propagation of a monochromatic wave.[4-6] In a medium with strong dispersion the amplitudes of the higher harmonics in such a wave remain small in proportion to the parameter (1). On the other hand, in a medium with weak dispersion the amplitudes of higher harmonics increase during the propagation (which with zero dispersion leads at last to the formation of shock waves). However, the problem of the propagation of a monochromatic wave admits an exact solution in a number of cases. In studying systems of large numbers of waves, practically the only method at our disposal is perturbation theory.[1)] As Kadomtsev and Petviashvili have shown,[8] the evolution of such a system in the classical case can lead to the formation of a large number of shocks.

In the quantum system the situation is very dif-

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ferent. Let us consider a wave excited in a system with small dispersion. It would seem that during its propagation anharmonic effects must lead to an increase of the higher harmonics. However, as is known from quantum mechanics, to excite a wave with frequency Ω an energy not smaller than $\hbar\Omega$ must be expended. But in a wave with energy of the order of $\hbar\Omega$ there is not enough energy to form higher harmonics of appreciable amplitude. This means that spontaneous processes with formation of higher harmonics are forbidden.

Decay processes, however, are allowed in both the quantum system and the classical system. Let us consider a quantum system in which a phonon with wave number k can decay, owing to interaction with the zero-point vibrations, into phonons with wave vectors k′ and k − k′. We are thinking of virtual processes, in which these phonons again merge into a phonon with wave number k, which can decay again, and so on. The succession of such processes leads to a renormalization of the dispersion law, which as a result takes the form

$$\Omega_k = ck + \Delta_k, \quad \Delta_k = c_0 k \xi(k) + \lambda_k. \tag{2}$$

Here Δ_k characterizes the total dispersion, which is the sum of the primary dispersion $c_0 k\xi(k)$ and a correction λ_k which is due to the interaction; $c = c_0 + \delta c$ is the renormalized speed of sound; λ_k is proportional to the square of the effective coupling constant, which, in turn, is proportional to $(\vartheta^2 + \Delta)^{-1}$. In calculating it one must take into account all possible directions of the intermediate phonons, i.e., integrate over ϑ^2. The result is that λ_k is proportional to the large logarithm $\ln(1/|\Delta|)$. As for the damping Γ, in the case of a decaying spectrum it differs from λ by the absence of the large logarithm, since it is due to real processes only. The result is that the ratio of the anharmonicity to the renormalized dispersion is always small.

The main contents of this paper reduce to the proof of two assertions. First, we can "improve" perturbation theory so that it will be constructed not in terms of the parameter (1), but in terms of the ratio of the anharmonicity (of which Γ is a measure) to the renormalized dispersion Δ_k. Second, the ratio of Γ to Δ_k is logarithmically small, and this conclusion is not altered by including higher approximations of perturbation theory.

We conduct the analysis with the example of a specific system, namely a system of phonons in superfluid helium (at such low temperatures that rotons are not excited). We at once note that our results can also bear on elastic vibrations in solids. The main complication that arises in this case in comparison with helium II is by no means due to anisotropy of the phonon spectrum—along directions of sufficiently high symmetry the problem of phonon-phonon collisions is formulated almost exactly as for an isotropic medium—it is due to the presence of three vibrational branches instead of one.

It is well known that at normal pressure there is a decay region $0 < k < k^{(\infty)}$ in the phonon spectrum of helium II (see Fig. 1).[9] In this range decay of a phonon with wave vector k into a number of phonons with smaller wave vectors is allowed. The vector $k^{(\infty)}$ is determined by the condition $\Omega(k^{(\infty)}) = ck^{(\infty)}$, where $\Omega(k)$ is the frequency of a phonon with wave vector k, ϑ and $c = (\partial\Omega/\partial k)_{k=0}$. In the range $0 < k < k^{(2)} < k^{(\infty)}$ the decay of a phonon with wave number k into two phonons is allowed.[2)] Experiment shows that with increasing pressure the value of $k^{(2)}$ decreases and goes to zero at about 17 atm. This means that the dispersion Δ_k, which is positive at low pressures, decreases with increasing pressure and even changes sign.

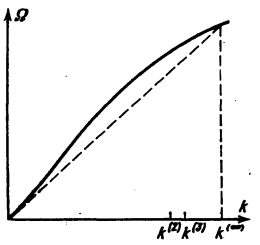

FIG. 1. Initial section of the spectrum of excitations in helium II. The deviations from linearity are exaggerated in the figure. According to Maris[9] at normal pressure $\hbar\Omega^{(2)} = 7.9$ K, $\hbar\Omega^{(3)} = 8.7$ K, $\hbar\Omega^{(4)} = 9.1$ K, and $\hbar\Omega^{(\infty)} = 9.9$ K.

From these statements one gets the impression that there must exist a range of pressures where the characteristic value of the dispersion is so small that the parameter (1) is larger than or of the order of unity. Then the standard perturbation theory does not apply, and the question arises as to how to describe the kinetics of a system of phonons in this case.

This question primarily involves three-phonon processes, which are allowed for positive dispersion and forbidden for negative dispersion. This question was first posed by Kawasaki.[10] He suggested that to describe three-phonon processes with small negative dispersion ("almost allowed" processes) it suffices to consider the finite lifetime of the phonons, by replacing the δ function which expresses the law of conservation of energy with a Lorentz function. Such a procedure partly allows three-phonon processes, and in addition permits us to describe their gradually becoming forbidden as the change from positive to negative dispersion goes on. A similar way out of the situation was proposed by Simons,[11] who also had in mind phonons in solids.

Leggett and ter Haar[12] examined critically the results found by Simons,[11] using a graphical technique. They showed that a consistent treatment of the finite lifetime of phonon states is equivalent to considering many-phonon processes, and for this there must be no restriction to a finite order of perturbation theory. Gurevich and Shklovskii[13] showed that in the special case when the finite lifetime of states in a given branch is mainly due to interaction with phonons of other branches, Simons' procedure is well founded.

Our result here is that if the interaction within one branch predominates the situation is quite different. Namely, the interaction between photons brings in a

negative dispersion, which in the three-dimensional case overcomes the damping; the ratio of the two quantities is at least proportional to a large logarithm. From this we conclude that in the quantum case perturbation theory can always be used to describe phonon-phonon collisions. If the primary dispersion is comparatively large, this means that the condition (1) is satisfied, i.e., standard perturbation theory is valid. In the general case one can use an "improved" perturbation theory; the condition for this to be applicable is

$$\Gamma/|\Delta| \ll 1 \tag{3}$$

and is satisfied for small primary dispersion owing to the large logarithmic factor.

It might seem that there is not sufficient foundation for this conclusion, since in the expression (2) for Δ_k the first and second terms could almost exactly cancel each other, so that Δ_k would be of the order of Γ. We shall show, however, that this is not so. Strictly speaking, λ_k is a functional of Δ_k; what we actually find is an integral equation for the function Δ_k [Eq. (31), see below]. The study of this equation shows that whatever the primary or bare function $\xi(k)$ may be, Δ_k can go to zero only at particular points. Its characteristic value is at least logarithmically large, so that the condition (3) is always satisfied.

In final analysis, perturbation theory can be used with a small bare dispersion because the anharmonicity is small:

$$\hbar k^4/\rho c \ll 1 \tag{4}$$

(ρ is the density of the helium). The expansion, however, is not in powers of the parameter (4), as would be the case for $\xi \sim 1$, but in inverse powers of its logarithm. An essential point is that in quantum theory there is a scale factor $(\hbar/\rho c)^{1/4}$ of the dimension of length, which is not directly connected with the value of the dispersion, and this makes the argument of the logarithm dimensionless.

It must be emphasized that this theory is essentially quantic, and does not work in the classical region, i.e., when the phonon occupation numbers are large. Formally this is manifested in the fact that in the latter case the anharmonicity Γ increases along with the numbers of phonons, and in proportion to them, so that the inequality (3) is no longer satisfied.

An immediate consequence of the applicability of perturbation theory is that one can use the ordinary kinetic equation with a three-phonon collision integral to describe the kinetics of phonons in the region $k < k^{(2)}$. Beyond the three-phonon threshold, i.e., for $k > k^{(2)}$, the three-phonon collision integral becomes equal to zero, and it is necessary to take into account four-phonon processes of types $2-2$ and $3 \rightleftarrows 1$. Processes of the type $2-2$ are allowed for arbitrary k, and those of the type $3 \rightleftarrows 1$, only up to the four-phonon threshold, i.e., for $k < k^{(3)}$, where $k^{(3)}$ is determined from the equation $\Omega(k^{(3)}) = 3\Omega(k^{(3)}/3)$.

Which values of k are important in a given kinetic problem is determined by the characteristic quantity $k_T = T/\hbar c$, where T is the temperature of the helium.[3)] As Landau and Khalatnikov[14,15] pointed out, under definite conditions there can exist in helium II a state of incomplete equilibrium, in which the temperature T is a function of the direction in k-space. In this case the characteristic values k_T can also be different for different directions. If $k_T \ll k^{(2)} \sim k^{(3)}$, the main role in the kinetics is played by three-photon collisions. They determine both the rate of longitudinal relaxation along a given direction in k-space, and the form of the kinetic coefficient that appears in the operator for transverse relaxation. The latter characterizes the rate of establishment of complete equilibrium, i.e., of the equalizing of the temperature among different directions, and, as has been shown in papers by Gurevich and Laĭkhman,[3,16] it is a fourth-order differential operator acting on the temperature as a function of direction.

If now $k_T \gtrsim k^{(3)}$, along with three-phonon processes four-phonon processes will also be important. But processes in which larger numbers of phonons participate must, generally speaking, contribute little owing to the smallness of the anharmonicity parameter (4). The only exception is processes of the type $2 \rightleftarrows 3$; in Refs. 14 and 15 it is indicated that they can play a part in establishing equilibrium in the numbers of phonons.[4)]

The possibility of using perturbation theory greatly simplifies the calculation of the collision operator involving a large number of phonons, and also of the damping of phonons. Namely, out of the whole set of diagrams we need include only those that contain renormalized vertices, i.e., vertices of the type of trees, not containing any internal loops. This statement also relates in particular to the problem of calculating the damping in the immediate neighborhood of the point $k = k^{(\infty)}$, where it is necessary to consider decay processes involving an arbitrarily large number of phonons.[9,7]

In summary, we wish to emphasize the main conclusion of the paper: There is a quantum lower limit on the size of the total dispersion Δ_k, which is derived independently of the magnitude and sign of the bare dispersion. The difficulties that arise in describing a system of phonons with an almost linear dispersion law are due to the fact that the spectrum of the phonons is calculated by means of perturbation theory, in which there is a certain freedom in the choice of the bare spectrum. If we use the renormalized, that is, the observed, spectrum, then perturbation theory is always suitable for the calculation of phonon-phonon collisions.

The proof of the assertions we have made and the construction of the quantitative theory will begin with the calculation with perturbation theory of higher order corrections to the three-photon vertex (Fig. 2). For this it is convenient to use the technique of Keldysh,[17,18,19] since this enables us to avoid analytic continuation with respect to frequencies. We write the bare three-phonon vertex in the form

$$g(q_1, q_2, q_3) = \frac{1}{2}\left(\frac{i\hbar}{2\rho}\right)^{1/4} b(q_1, q_2, q_3). \tag{5}$$

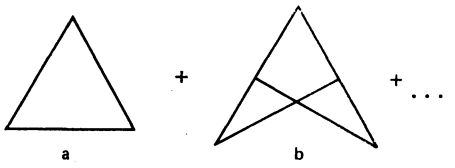

FIG. 2.

Here q denotes the set of variables ω, $\mathbf{k}$, and $b(q_1, q_2, q_3)$ is given, with dispersion neglected and for almost parallel wave vectors, by

$$b(q_1, q_2, q_3) = 2(u+1)c^{1/2}\sqrt{k_1 k_2 k_3}. \tag{6}$$

Here $u = (\rho/c)\partial c/\partial\rho$, which is equal to 2.84 at normal pressure. The expression (6) remains correct as to order of magnitude for arbitrary angles between the wave vectors.[5)] To evaluate the diagrams (Fig. 2) we use order-of-magnitude estimates of the bare vertices and Green's functions, taking only one pole term into account:

$$D \sim (\omega - \Omega_{\mathbf{k}} \sim i\Gamma)^{-1}. \tag{7}$$

Here $\Omega_{\mathbf{k}}$ is the renormalized frequency and Γ is the damping of the phonon spectrum. For graph a, after integrating over the internal frequency we get an expression which is, in order of magnitude,

$$G^{(a)} \sim \int \frac{g^2\, d^3k}{(\omega_1 + \Omega_{\mathbf{k}} - \Omega_{\mathbf{k}+\mathbf{k}_1} + i\Gamma)(\omega_2 + \Omega_{\mathbf{k}-\mathbf{k}_2} - \Omega_{\mathbf{k}} + i\Gamma)}, \tag{8}$$

where the variables q_1 and q_2 relate to the external photons. We remember that we are interested only in the case of small dispersion and damping of the phonons:

$$\Delta/\Omega \ll 1, \quad \Gamma/\Omega \ll 1. \tag{9}$$

Under conditions for which ordinary perturbation theory is valid

$$\Delta_k = c_0 k \xi(k). \tag{10}$$

Accordingly, the inequalities (9) mean that the bare dispersion and phonon damping are small.

To calculate real processes involving phonons we are interested in the properties of the vertex part near the mass shell, i.e., for

$$|\delta\omega| \ll ck, \tag{11}$$

where $\delta\omega \equiv \omega - \Omega_{\mathbf{k}}$. In virtue of the inequalities (9) and (11) both denominators in the integrand in Eq. (8) are small when $k < k_2$ and the vectors $\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2$ are parallel. The complete vertex $G^{(a)}$ is symmetrical in $\mathbf{k}_1$ and $\mathbf{k}_2$. The symmetry becomes obvious if besides (7) we include terms containing frequency denominators of the same type but with different labels and signs on the frequencies. For some of them the region where the denominators are small is bounded not by the inequality $k < k_2$, but by $k < k_1$. As for the integrals over the ranges $k > k_1$ and $k > k_2$, they have either only one such denominator or none at all.

In either case the integrand increases with increasing k like some power of k. Accordingly, the main contribution to these integrals comes from values of k of the order of atomic values, and cutting off of the integrals occurs because of the change of character of the spectrum of damping and interaction on this scale. In the integrals containing one small denominator the integration over the angles can lead to the appearance of a large logarithm, but the small denominator itself exists only in a phonon region whose size is small in comparison with the phase volume that makes the main contribution to the integral. For values of k in the atomic range we can neglect the quantities k_1 and k_2 in comparison with k, and the result is simply a renormalization of the bare vertex. We shall assume hereafter that the appropriate subtraction has already been carried out in the integral (8).

Then the integral (8) has a maximum when the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel, and the main contribution here comes from a region of k values close to the direction of $\mathbf{k}_1$ and $\mathbf{k}_2$. The maximum value of the integrand is reached at angles between the vectors of the order of ϑ_c, which is given by the relation

$$\vartheta_c^2 \approx \frac{\Gamma}{\Omega} + \frac{|\delta\omega + \Delta|}{\Omega} \ll 1. \tag{12}$$

Let us estimate the value of the integral (8) for a very small angle between $\mathbf{k}_1$ and $\mathbf{k}_2$. We resolve all vectors into components parallel and perpendicular to $\mathbf{k}_1 + \mathbf{k}_2$:

$$\mathbf{k}_1 = \mathbf{k}_{1\parallel} + \mathbf{k}_{1\perp}, \quad \mathbf{k}_2 = \mathbf{k}_{2\parallel} - \mathbf{k}_{1\perp}, \quad \mathbf{k} = \mathbf{k}_\parallel + \mathbf{k}_\perp. \tag{13}$$

As we shall verify, the main contribution to the integral (8) comes from the region of values $k_\perp \ll k_\parallel$. In this region

$$|\omega_1 - \Omega_{\mathbf{k}} - \Omega_{\mathbf{k}-\mathbf{k}_1} + i\Gamma| \sim (\vartheta^2 + \vartheta_1^2 - 2\vartheta\vartheta_1 \cos\varphi + \vartheta_c^2)\, ck,$$
$$|\omega_2 + \Omega_{\mathbf{k}} - \Omega_{\mathbf{k}+\mathbf{k}_2} + i\Gamma| \sim (\vartheta^2 + \vartheta_2^2 - 2\vartheta\vartheta_2 \cos\varphi + \vartheta_c^2)\, ck,$$

where

$$\vartheta = k_\perp/k, \quad \vartheta_1 = k_{1\perp}/k_1, \quad \vartheta_2 = k_{2\perp}/k_2, \tag{14}$$

and φ is the angle between $\mathbf{k}_{1\perp}$ and $\mathbf{k}_\perp$. Then

$$G^{(a)} \sim g\frac{\hbar k^4}{\rho c}\int_0^{2\pi}\int_0^1 \frac{\vartheta\, d\vartheta\, d\varphi}{(\vartheta^2 + \vartheta_1^2 + \vartheta_c^2 - 2\vartheta_1\vartheta\cos\varphi)(\vartheta^2 + \vartheta_2^2 + \vartheta_c^2 - 2\vartheta_2\vartheta\cos\varphi)}. \tag{15}$$

In the denominator of the integrand the values of the angles denoted by ϑ_c in each parentheses are in general different, so that after integrating over φ we get

$$G^{(a)} \sim g\frac{\hbar k^4}{\rho c}\int_0^1 \frac{2\pi\, d\vartheta}{(\vartheta_2 - \vartheta_1)(\vartheta^2 - \vartheta_1\vartheta_2) + (\vartheta_1 + \vartheta_2)\vartheta_c^2}$$
$$\times\left[\frac{\vartheta^2 + \vartheta_1^2}{[(\vartheta^2 - \vartheta_1^2)^2 - 2\vartheta_1^2\vartheta_c^2 + \vartheta_c^4]^{1/2}} - \frac{\vartheta^2 + \vartheta_2^2}{[(\vartheta^2 - \vartheta_2^2)^2 - 2\vartheta_2^2\vartheta_c^2 + \vartheta_c^4]^{1/2}}\right]. \tag{16}$$

For $\vartheta_1 \sim \vartheta_2 \gg \vartheta_c$ the integrals of the terms in the brackets contain the large logarithms $\ln(\vartheta_1/\vartheta_c)$ and $\ln(\vartheta_2/\vartheta_c)$, which are gathered from the neighborhoods of the values $\vartheta = \vartheta_1$ and $\vartheta = \vartheta_2$, i.e., in the regions $k_1\vartheta_1 \ll |\mathbf{k}_{1\perp} - \mathbf{k}_\perp| \ll k_{1\perp}$ and $k_2\vartheta_2 \ll |\mathbf{k}_{2\perp} - \mathbf{k}_\perp| \ll k_{2\perp}$. However, the coefficients of these two logarithms are identical, so that the subtraction gives a quantity of the order of unity. Accordingly, the remaining integral can be estimated from dimensions, and the result is

$$G^{(a)} \sim g\frac{\hbar k^4}{\rho c}\frac{1}{\vartheta_1^2 + \vartheta_c^2}. \tag{17}$$

In estimating the higher-order graphs, we begin with dimensional arguments. Any graph for three external lines contains an odd number of internal vertices. Let

us denote by G_n such a graph with $2n+1$ internal vertices. The value of G_n contains $3n$ propagators and n independent internal four-dimensional integrations. After the integration over frequencies there remain $2n$ propagators and n three-dimensional integrations. We shall be interested only in the value of a three-terminal graph with small angles between the momenta of the entering phonons. Just as for the simplest graph (Fig. 2), we must separate out the contribution from large internal momenta (of the order of atomic values), which corresponds to the bare vertex. In complicated graphs there also exist contributions corresponding to large values of some of the internal momenta and small values of the rest. Suppose, for example, in the graph of Fig. 3a we have $p_1 \gg p_2$. In each of the lines that include the momentum p_1 we can neglect the other momenta compared with this one and shrink the contour of these lines to a point. In this way the graph of Fig. 3a reduces to that of Fig. 3b. By this procedure graphs appear that contain internal many-terminal vertices. As we have shown in an earlier paper,[3] the contribution of these graphs is small, proportional to the anharmonicity parameter (4).

After subtracting that part of a graph which comes from values of the internal momenta of the order of atomic values, we break the integration up into integrations over transverse and longitudinal components. In the integral over the transverse components the main contribution comes from the lower limit, since in each such two-dimensional integration there are at least two denominators, each of which after the expansion in transverse momentum components contains squares of these components. It follows that if the angles between incoming momenta are of the order of ϑ_c, then

$$G_n \sim g\left(\frac{\hbar k^4}{\rho c}\frac{1}{\vartheta_c^2}\right)^n. \tag{18}$$

In this case G_n cannot contain logarithms, since logarithms appear owing to large ranges of integration, and here all the ranges of integration over solid angles are of the order of ϑ_c^2. If an angle ϑ_1 between incoming momenta becomes larger than ϑ_c, then ϑ_1 appears in at least some denominators in the integrand. The result of this is that some, if not all, of the ranges of integration over solid angles are of the order of ϑ_1^2. It is important to note that each internal momentum runs through at least two propagators (of course after the integration over frequencies). If ϑ_1 is involved in all of the propagators, the characteristic values of the variable of integration are $\vartheta \sim \vartheta_1$. If ϑ_1 appears in all but one of the propagators, the characteristic range is $\vartheta_c < \vartheta < \vartheta_1$, and the integral contains a factor $\vartheta_1^{-2m}\ln(\vartheta_1/\vartheta_c)$, where $m \geq 1$. If there are two or more propagators without ϑ_1 the characteristic range of integration is $\vartheta \sim \vartheta_c$. There is no necessity for obtaining an accurate estimate for G_n; we merely note that since ϑ_1 is involved in not fewer than two propagators (after integration over the frequencies), in the range $\vartheta_1 \gg \vartheta_c$ G_n falls off with increasing ϑ_1 not more slowly than $1/\vartheta_1^2$. We recall that we are considering a vertex from which the contribution from large internal momenta has already been subtracted.

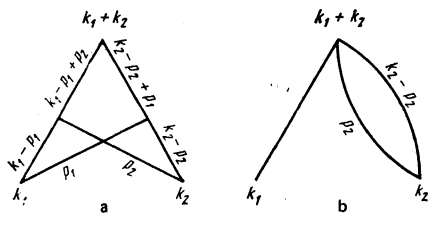

FIG. 3.

Let us proceed to the calculation of the renormalized dispersion law of the phonons, $\Lambda(k, ck) = \delta ck + \lambda_k$, where

$$\Lambda(q) = \mathrm{Re}\,\Pi^r(q) = \mathrm{Re}\,[\Pi^c(q) + \Pi^+(q)], \tag{19}$$

and Π^r, Π^c, and Π^+ are matrix elements of the polarization operator (see Ref. 3). The lowest order of perturbation corresponds to the diagram of Fig. 4a. Substituting the expressions for the phonon Green's function in terms of the phonon distribution $N_{\mathbf{k}}$ (see Ref. 3), we get (after integration over the frequency ω)

$$\Lambda^{(1)}(q) = \frac{\hbar}{16\rho}\int\frac{d^3k'}{(2\pi)^3}\Big\{b^2(\mathbf{k}, ck; \mathbf{k}', ck'; \mathbf{k}-\mathbf{k}', c|\mathbf{k}-\mathbf{k}'|) \times \frac{N_{\mathbf{k}'}+N_{\mathbf{k}-\mathbf{k}'}+1}{\omega-\Omega_{\mathbf{k}'}-\Omega_{\mathbf{k}-\mathbf{k}'}} + [b^2(\mathbf{k}, ck; \mathbf{k}+\mathbf{k}', c|\mathbf{k}+\mathbf{k}'|; \mathbf{k}', c|\mathbf{k}+\mathbf{k}'|-ck) + b^2(\mathbf{k}, ck; \mathbf{k}+\mathbf{k}', ck+ck'; \mathbf{k}', ck')]\frac{N_{\mathbf{k}'}-N_{\mathbf{k}+\mathbf{k}'}}{\omega-\Omega_{\mathbf{k}+\mathbf{k}'}+\Omega_{\mathbf{k}'}} - b^2(\mathbf{k}, ck; \mathbf{k}', ck'; \mathbf{k}+\mathbf{k}', ck+ck')\frac{N_{\mathbf{k}'}+N_{-\mathbf{k}-\mathbf{k}'}+1}{\omega+\Omega_{\mathbf{k}'}+\Omega_{\mathbf{k}+\mathbf{k}'}}\Big\}. \tag{20}$$

The expression for the matrix elements $b(\mathbf{k}, \omega; \mathbf{k}', \omega'; \mathbf{k}'', \omega'')$ are given in the Appendix. In the matrix elements we have neglected the difference between the frequencies ω and Ω_k, and also the dispersion.[6]

The integrals of the terms in (20) that contain N are cut off at values of k' of the order of a few times k_T, if N is the Planck equilibrium distribution. Let us examine in more detail the two remaining integrals [of the first and last terms in Eq. (20)], which do not contain N. They diverge at large k' as $\int k'd^2k'$, and the main contribution to them is from values of k' of the order of atomic values. Expansion of the integrand in powers of k'/k leads to the following well known[19] expression for the "divergent" part of the integral:

$$\Delta\Lambda = Ak + Bk^3 + Ck^5\ln(k_0/k), \tag{21}$$

where k_0 is a momentum of atomic order and $\Delta\Lambda$ is the renormalization of the bare dispersion law.[7)] If the vertex is given by Eq. (5), then $C = \hbar(1+u)^2/30\pi^2\rho$. But in the integral that leads to the expression (21) there are contributions from large values of k, for which there is no justification for using the expression (5). Analogous divergences arise in higher orders of perturbation theory, so that strictly speaking there is no reason to suppose that when all such terms were summed the term proportional to $k^5\ln(k_0/k)$ would be preserved in the bare spectrum.

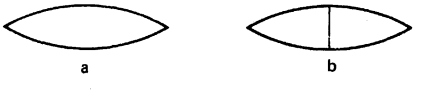

FIG. 4.

The integral that remains after subtracting the quantity (21) from Eq. (20) converges on k' of the order of k or k_T. The values of k' close to the direction of k make a logarithmically large contribution to the integral, of order $\sim\ln(1/\vartheta_c^2)$. Let us calculate this contribution. Precisely as in the calculation of the vertices, we break up the integration over $\mathbf{k}'$ into integrations in the direction parallel to $\mathbf{k}$ and in the transverse plane. The logarithmically large contribution comes from terms with small frequency denominaotrs, which are obtained with $k_\perp' \ll k$. Small values of $k_\perp'$ in the matrix elements can be neglected, and then

$$\Lambda^{(1)}(q)\approx\frac{\hbar c(u+1)^2}{4\rho(2\pi)^3}\left\{\int_0^k kk'(k-k')(2N_{k'}+1)dk'\fint\frac{d^2k_\perp'}{\omega-\Omega_{k'}-\Omega_{k-k'}}\right.$$
$$\left.+2\int_0^\infty(N_{k'}-N_{k+k'})kk'(k+k')dk'\fint\frac{d^2k_\perp'}{\omega-\Omega_{k+k'}+\Omega_{k'}}\right\}. \tag{22}$$

In the first integral in Eq. (22) the integral over k' from k to ∞ has been dropped, since in this range the frequency denominator does not have small values. Expanding the frequency denominators in $k_\perp'$, we get the following expression for the renormalization of the velocity:

$$\delta c^{(1)}=\frac{\hbar(u+1)^2}{2\pi^2\rho}\int_0^\infty N_{k'}\ln\frac{1}{\vartheta_c^2}k'^3dk' \tag{23}$$

and for that of the dispersion we get

$$\lambda_k^{(1)}=-\frac{\hbar(u+1)^2}{16\pi^2\rho}\int_0^k k'^2(k-k')^2\ln\frac{1}{\vartheta_c^2}dk'. \tag{24}$$

The logarithm in these two equations has appeared because the characteristic range of integration over the angle ϑ between $\mathbf{k}$ and $\mathbf{k}'$ turned out to be large:

$$\int_{\vartheta_c}^1 d\vartheta/\vartheta=\ln(1/\vartheta_c).$$

The calculation of higher order terms in $\Lambda(q)$ reduces to making one of the vertices in Fig. 4a more complicated. However, we have seen that corrections to vertices fall off not more slowly than $1/\vartheta^2$ as the angle ϑ between the external momenta increases. Therefore the corresponding integrals over the angles in $\Lambda(q)$ are concentrated at $\vartheta\sim\vartheta_c$, i.e., no logarithms such as $\ln(1/\vartheta_c)$ appear in the higher orders. The result is

$$\Lambda(q)=\Lambda^{(1)}(q)+\frac{\hbar k^5}{\rho}\left[C_1\left(\frac{\hbar k^4}{\rho c}\frac{1}{\vartheta_c^2}\right)+C_2\left(\frac{\hbar k^4}{\rho c}\frac{1}{\vartheta_c^2}\right)^2+\ldots\right]. \tag{25}$$

In estimating the remaining terms we make no distinction between k and k_T; C_1 and C_2 are constants of the order of unity.

It is hard to expect the two terms in the expression (2) for Δ_k to cancel each other over the whole range of values of k. In particular the bare dispersion contains a term proportional to k^3, owing to short-range forces, and a term proportional to k^4, owing to the van der Waals interaction between the atoms of helium.[22,23] The expression for $\lambda_k^{(1)}$ contains only a term proportional to $k^5\ln(1/\vartheta_c^2)$, associated with the interaction between phonons. It follows from this that although at particular points Δ_k may be equal to zero, its characteristic value satisfies the condition

$$|\Delta|\geqslant|\lambda_k^{(1)}|\sim\frac{\hbar k^5}{\rho}\ln\frac{1}{\vartheta_c^2}. \tag{26}$$

Being interested in the relaxation of the phonons (i.e., real processes), we shall calculate the phonon Green's function in a small neighborhood of the mass shell

$$|\delta\omega|\leqslant\Gamma. \tag{27}$$

By means of precisely the same sort of estimates as led to Eq. (26), we can estimate the damping of the phonons. The result is

$$\Gamma\sim\frac{\hbar k^5}{\rho}\left[1+O\left(\frac{\hbar k^4}{\rho c}\frac{1}{\vartheta_c^2}\right)+\ldots\right]. \tag{28}$$

This expression differs from the analogous expression for the renormalization of the dispersion, Eq. (28), by the absence of the large logarithm in the first term, and this makes possible the solution of the problem. In fact, it follows from Eqs. (26)–(28) that

$$\vartheta_c^2\sim\frac{|\Delta|}{\Omega}\geqslant\frac{\hbar k^4}{\rho c}\ln\frac{1}{\vartheta_c^2}\gg\frac{\hbar k^4}{\rho c}. \tag{29}$$

It is reasonable to use only the first term $\Lambda^{(1)}(q)$ of the expression for $\Lambda(q)$. Accordingly, neglecting terms of the order of unity in comparison with the large logarithm, we get the following expression for the renormalization of the velocity (for the case of thermal equilibrium):

$$\delta c=\frac{\hbar(u+1)^2}{2\pi^2\rho}\int_0^\infty N_{k'}\ln\frac{ck}{|\Delta_k+\Delta_{k'}-\Delta_{k+k'}|}k'^3dk'=\frac{\pi^2(u+1)^2T^4}{30\rho\hbar^3c^4}\ln\frac{ck_T}{\Delta(k_T)} \tag{30}$$

and also the integral equation for Δ_k:

$$\Delta_k=ck\xi(k)-\frac{\hbar(u+1)^2}{16\pi^2\rho}\int_0^k k'^2(k-k')^2\ln\frac{ck}{|\Delta_k-\Delta_{k'}-\Delta_{k-k'}|}dk'. \tag{31}$$

In the first term of Eq. (30) we do not distinguish between c and c_0 because $\delta c\ll c$.

The expression (30) has been derived by Andreev and Khalatnikov,[24] and also by Pethick and ter Haar.[25] In Eq. (31) we can, with our chosen degree of accuracy, take the logarithm out from under the integral sign; then

$$\Delta_k=ck\xi(k)-\frac{(u+1)^2\hbar k^5}{480\pi^2\rho}\ln\frac{ck}{\bar\Delta}, \tag{32}$$

where $\bar\Delta$ is a characteristic value of Δ_k for the range $[0,k]$. Havlin and Luban[26] were the first to call attention to the existence of a correction to the law of dispersion proportional to k^5. We note that the renormalization of the dispersion is due to the zero-point vibrations, i.e., is an essentially quantum effect.

If in the important range of values the first term of Eq. (32) is larger than the second, the small parameter is of the form (1), corresponding to ordinary perturbation theory. If, on the other hand, the first term is smaller than the second, the improved perturbation theory applies, and the small parameter is a quantity inversely proportional to $\ln(\rho c/\hbar k^4)$.

We note that because of Eq. (27) we have $\delta\omega\ll\Delta$. This condition shows the essential difference between our situation and that analyzed by Iordanskiĭ and Pitaevskiĭ,[7] who considered the behavior of the polarization operator far from the mass shell, i.e., for $0<\omega-ck\ll\Delta$.

In conclusion we point out that it is obviously interesting to trace the behavior of the phonon-phonon collision operator in the neighborhood of the point $k=k^{(2)}$. In fact, for $k>k^{(2)}$ the spectrum is nondecaying, and we can use the four-phonon collision operator given by Landau and Khalatnikov.[14,15] For $k<k^{(2)}$ this collision operator diverges, and the sum of an infinite series of diverging terms of that type reduces to a renormalization of the spectrum in a three-phonon collision operator, while the remaining part of the four-phonon operator converges. Accordingly, the four-phonon operator takes different forms for $k>k^{(2)}$ and $k<k^{(2)}$. It is interesting to see how one expression goes over into the other. Since, as we have shown, in calculating vertices it is sufficient to use the lowest order of perturbation theory, in the entire range of values of k the sum of the three-phonon and four-phonon collision operators is the sum of the diagrams in Fig. 4, a and b, where each line corresponds to the complete Green's function. Separation of the three-phonon and four-phonon operators from this sum leads to extremely cumbersome calculations. This problem arises only in the region of temperatures and pressure where the two operators are in competition, and we shall not deal with it in the present paper.

We thank V. S. L'vov for a discussion of this work and helpful comments.

APPENDIX

The Hamiltonian of the three-phonon interaction contains parts associated with the kinetic and potential energies[8)]:

$$H_3=\frac{1}{2}\int\left[\mathbf{v}(\mathbf{r})\rho'(\mathbf{r})\mathbf{v}(\mathbf{r})+\frac{c^2}{\rho^2}(2u-1)\rho'^3(\mathbf{r})\right]d^3r, \tag{A.1}$$

where

$$\mathbf{v}(\mathbf{r})=\sum_{\mathbf{k}}\left(\frac{\hbar\omega_k}{2\rho Vk^2}\right)^{1/2}\mathbf{k}[\hat{b}_{\mathbf{k}}\exp(-i\omega_k t)-\hat{b}_{-\mathbf{k}}^{+}\exp(i\omega_k t)]e^{i\mathbf{kr}} \tag{A.2}$$

is the velocity operator, V is the volume of the system, and

$$\rho'=\sum_{\mathbf{k}}\left(\frac{\hbar\rho k^2}{2V\omega_k}\right)^{1/2}[\hat{b}_{\mathbf{k}}\exp(-i\omega_k t)+\hat{b}_{-\mathbf{k}}^{+}\exp(i\omega_k t)]e^{i\mathbf{kr}} \tag{A.3}$$

is the operator of density fluctuations.

It is convenient to construct the diagram technique by means of the operator

$$\varphi(x)=\frac{1}{V^{1/2}}\sum_{\mathbf{k}}\varphi_{\mathbf{k}}(t)e^{i\mathbf{kr}},\quad \varphi_{\mathbf{k}}(t)=\hat{b}_{\mathbf{k}}\exp(-i\omega_k t)+\hat{b}_{-\mathbf{k}}^{+}\exp(i\omega_k t) \tag{A.4}$$

(x is the four-vector $\mathbf{r},t$), in terms of which the velocity and density operators can be expressed in the following way:

$$\mathbf{v}=\frac{1}{V^{1/2}}\sum_{\mathbf{k}}\mathbf{v}_{\mathbf{k}}e^{i\mathbf{kr}},\quad \mathbf{v}_{\mathbf{k}}=i\mathbf{k}\left[\frac{\hbar}{2\rho k^2\omega_k}\right]^{1/2}\frac{\partial\varphi_{\mathbf{k}}}{\partial t}; \tag{A.5}$$

$$\rho'=\frac{1}{V^{1/2}}\sum_{\mathbf{k}}\rho_{\mathbf{k}}e^{i\mathbf{kr}},\quad \rho_{\mathbf{k}}=\left[\frac{\hbar\rho k^2}{2\omega_k}\right]^{1/2}\varphi_{\mathbf{k}}. \tag{A.6}$$

It can be seen from these relations that one cannot construct both the velocity and the density operators in terms of φ; the velocity is expressed in terms of $\partial\varphi/\partial t$. Therefore also in the construction of the diagram technique we cannot at once introduce only one Green's function

$$D(x,x')=-i\langle T_c\varphi(x)\varphi(x')\rangle, \tag{A.7}$$

where T_c is an ordering on a contour, as described in a paper by Keldysh.[17] Three further Green's functions arise in a natural way, two vector functions

$$E_\alpha(x,x')=-i\langle T_c v_\alpha(x)\varphi(x')\rangle,$$
$$F_\alpha(x,x')=-i\langle T_c\varphi(x)v_\alpha(x')\rangle, \tag{A.8}$$

and one tensor function

$$G_{\alpha\beta}(x,x')=-i\langle T_c v_\alpha(x)v_\beta(x')\rangle. \tag{A.9}$$

For all of these functions we introduce spatial Fourier transformations

$$D(x,x')=\frac{1}{V}\sum_{\mathbf{kk'}}e^{i(\mathbf{kr}-\mathbf{k'r'})}D(\mathbf{k},\mathbf{k'})\quad\text{etc.} \tag{A.10}$$

We then have

$$D(\mathbf{k},\mathbf{k'})=-i\langle T_c\varphi_{\mathbf{k}}(t)\varphi_{\mathbf{k'}}(t')\rangle,\quad E_\alpha(\mathbf{k},\mathbf{k'})=-i\langle T_c v_{\alpha\mathbf{k}}(t)\varphi_{\mathbf{k'}}(t')\rangle,$$
$$F_\alpha(\mathbf{k},\mathbf{k'})=-i\langle T_c\varphi_{\mathbf{k}}(t)v_{\alpha\mathbf{k'}}(t')\rangle,\quad G_{\alpha\beta}(\mathbf{k},\mathbf{k'})=-i\langle T_c v_{\alpha\mathbf{k}}(t)v_{\beta\mathbf{k'}}(t')\rangle. \tag{A.11}$$

Now by means of Eq. (A.5) we can easily establish the relations

$$E_\alpha(\mathbf{k},\mathbf{k'})=ik_\alpha\left(\frac{\hbar}{2\rho k^2\omega_k}\right)^{1/2}\frac{\partial}{\partial t}D(\mathbf{k},\mathbf{k'}),$$
$$F_\alpha(\mathbf{k},\mathbf{k'})=ik_\alpha'\left(\frac{\hbar}{2\rho k'^2\omega_{k'}}\right)^{1/2}\frac{\partial}{\partial t'}D(\mathbf{k},\mathbf{k'}). \tag{A.12}$$
$$G_{\alpha\beta}(\mathbf{k},\mathbf{k'})=-k_\alpha k_\beta'\frac{\hbar}{2\rho kk'(\omega_k\omega_{k'})^{1/2}}\frac{\partial^2}{\partial t^2}D(\mathbf{k},\mathbf{k'}).$$

Accordingly, the complete (spatial and time) Fourier transformations of E_α, F_α, and $G_{\alpha\beta}$ can be expressed in terms of the complete transformation of D, but because Eq. (A.12) contains time derivatives these expressions involve frequencies. And if in constructing the technique we use these expressions and eliminate E_α, F_α, and $G_{\alpha\beta}$, leaving only the function D, which also appears in the usual versions of the diagram technique, the bare vertex is frequency dependent. It takes the following form

$$b(q_1,q_2,q_3)=c^2\frac{k_1k_2k_3}{(\omega_{k_1}\omega_{k_2}\omega_{k_3})^{1/2}}(2u-1)+\frac{\mathbf{k}_2\mathbf{k}_3}{k_2k_3}\frac{k_1(\omega_{k_2}\omega_{k_3})^{1/2}}{\omega_{k_1}^{1/2}\omega_2\omega_3}$$
$$+\frac{\mathbf{k}_1\mathbf{k}_2}{k_1k_2}\frac{k_3(\omega_{k_1}\omega_{k_2})^{1/2}}{\omega_{k_3}^{1/2}\omega_1\omega_2}+\frac{\mathbf{k}_1\mathbf{k}_3}{k_1k_3}\frac{k_2(\omega_{k_1}\omega_{k_3})^{1/2}}{\omega_{k_2}^{1/2}\omega_1\omega_3}. \tag{A.13}$$

The vertex has the property

$$b(-q_1,q_2,q_3)=b(q_1,-q_2,q_3)=b(q_1,q_2,-q_3)=b(q_1,q_2,q_3). \tag{A.14}$$

Higher-order vertices do not depend on the frequencies.

[1)] An exception is a paper by Iordanskiĭ and Pitaevskiĭ,[7] in which a quasiclassical method was used to determine the character of a singularity of the polarization operator off the mass shell near the line $\omega=c_0k$.

[2)] The quantity $k^{(2)}$ is determined from the condition $\Omega(k^{(2)})=2\Omega(k^{(2)}/2)$.

[3)] Actually because of the high powers of k that occur in the integrands the characteristic values of k are several times k_T.

[4)] For $k^{(2)}>0$ equilibrium in the phonon numbers is brought about both owing to processes $2\rightleftarrows3$ and also owing to pro-

cesses $2 \rightleftarrows 1$. These mechanisms can compete, because the former has the anharmonicity as a small factor, and the second (for $k_T > k^{(2)}$) has a small phase volume.

[5)]The exact expression for $b(q_1, q_2, q_3)$ is cumbersome, and depends not only on the wave vectors but also on the frequency arguments $\omega_1, \omega_2, \omega_3$. It is derived in the Appendix.

[6)]The last term in the expression (20) differs from the analogous term given in Refs. 19 and 20; the latter term contains a matrix element $b(\mathbf{k}, ck; \mathbf{k}', ck'; -\mathbf{k}-\mathbf{k}', ck+ck')$, which does not coincide with the term in (20) even when $\mathbf{k}$ and $\mathbf{k}'$ are parallel. The difference is due to the fact that when one constructs the diagram technique for helium II with one type of bare vertices and Green's functions the bare vertices must depend on the frequency arguments as well as on the momenta; this was not taken into account in Refs. 19 and 20.

[7)]It is interesting to note that a spectrum of precisely the same type is obtained not only in hydrodynamics but also in the direct study of a weakly nonideal Bose gas with short-range forces.[21] Kemoklidze and Pitaevskiĭ[22] and Feenberg[23] have shown that inclusion of the long-range van der Waals forces can lead to the appearance of a k^4 term.

[8)]We call attention to the fact that in this respect the situation is different from that which is usual for solids. In them, the quantization is usually carried out in a Lagrangian (comoving) coordinate system, and therefore the anharmonic terms come only from the potential energy of the lattice vibrations. On the other hand, the quantization of the phonons in helium II is done in an Eulerian (laboratory) system,[27] and therefore the anharmonic terms contain contributions from the kinetic energy also.

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Effect of a weak electric field on the dielectric losses in centrosymmetric ferroelectrics of the displacement type

A. K. Tagantsev

A. F. Ioffe Physicotechnical Institute, USSR Academy of Sciences

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Two mechanisms whereby a constant homogeneous electric field affects the losses due to lattice anharmonicity of an ideal crystal are considered. These are: 1) partial suppression of the already present processes of absorption of a measuring-field quantum with participation of two phonons from different modes, and 2) the appearance of new processes that cause absorption of this quantum, with participation of two phonons from the same mode. The influence of the finite phonon damping is discussed. A threshold is predicted for the field dependence of the first mechanism. The values of the threshold fields are estimated and their frequency dependences are obtained.

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1. INTRODUCTION

The question of dielectric losses in ferroelectrics of the displacement type was considered by a number of workers.[1-8] A detailed analysis of the losses in a centrosymmetric cubic ferroelectric in a homogeneous alternating electric field, based on the papers of Balagurov *et al.*,[4,5] is contained in Vaks's monograph.[9] The results for noncentrosymmetric cyrystals in inhomogeneous electric fields were obtained by Balagurov and Vaks.[6,7] All the authors considered the contribution made to the losses by the lattice anharmonicity

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