Plasma polarization in a magnetic field

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The possibility of spin polarization of a plasma in a magnetic field is considered. The quasiclassical approximation for the motion of an electron in an inhomogeneous magnetic field is compared with the drift approximation. The Stern-Gerlach experimental setup is considered for the measurement of the magnetic moment of a free electron. The feasibility of polarizing a thermal plasma in a nonstationary magnetic field is demonstrated.

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The possibility of measuring the spin and the magnetic moment of a free electron has been widely discussed in the literature. The point of view initially advanced was that neither the spin nor the magnetic moment can be determined by experiments in which the electron has a classical behavior, i.e., when the concept of a trajectory can be introduced (see Refs. 1–3). An experimental setup aimed at determining the spin of an electron moving classically was subsequently indicated.[4-6] As to the magnetic moment, a Stern-Gerlach experimental setup was considered, in which the electron passes once through a region with homogeneous field, i.e., the Larmor radius R is larger than the field-inhomogeneity scale l, $R > l$. It is easy to verify that at $R > l$ the uncertainty principle does not permit the electron-beam splitting needed for the Stern-Gerlach experiment. The opposite limiting case $R \ll l$ was also considered[7] and it was shown that in this case the Stern-Gerlach experiment can be performed in principle; the electron motion is classical and the beam can be split into two polarized ones.

The present article considers the possibility of polarizing large particle bunches with dimension d known to be larger than the Larmor radius (cf. Ref. 7, where the beam diameter $D \ll R$). It is known that when conditions $d \gg R$ and $l \gg R$ are satisfied a system of charged particles exhibits hydrodynamic properties (see, e.g., Ref. 8). It is therefore convenient to call this system a plasma. On the other hand, the plasma will be regarded as a system of noninteracting particles drifting in a magnetic field (since the condition $l \gg R$ agrees with the condition for the applicability of the drift approximation), i.e., the collective properties of the plasma will be disregarded. In other words the particles behave like free ones, and by the same token we are dealing with the magnetic moment of a free electron.

In Sec. 2 below is discussed a concrete Stern-Gerlach experimental setup for electrons drifting in the magnetic field of electrons. In Sec. 3 we consider a plasma with a thermal spread. The possibility of polarizing such a plasma is demonstrated, and this circumstance may be of interest for astrophysical applications.

1. FUNDAMENTAL CLASSICAL-APPROXIMATION EQUATIONS

The greater part of the calculations in Rubin's paper[7] is valid not only for a narrow beam $D \ll R$, but also for a cluster with $d \gg R$. We report therefore briefly the results in a form convenient for our purposes. In the nonrelativistic limit, the Hamiltonian of the particle is written as

$$H=\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2-\mu\boldsymbol{\sigma}\mathbf{B} \tag{1}$$

(see Ref. 9). The solution of the Schrodinger equation is set in the form

$$\psi=e^{iS/\hbar}\xi, \tag{2}$$

where $\xi(\mathbf{x}, t)$ is a spinor; the equation itself is then rewritten in the form

$$-\frac{\partial S}{\partial t}\xi+i\hbar\frac{\partial}{\partial t}\xi=\left[\frac{1}{2m}\left(\nabla S-\frac{e}{c}\mathbf{A}\right)^2-\mu\boldsymbol{\sigma}\mathbf{B}\right]\xi$$
$$-i\hbar\left[\left(\nabla S-\frac{e}{c}\mathbf{A}\right)\nabla\xi+\frac{1}{2m}\Delta S\xi\right]-\frac{\hbar^2}{2m}\Delta\xi. \tag{3}$$

Equation (3) is expanded in powers of $\hbar$ (assuming formally that the term $\mu\boldsymbol{\sigma}\cdot\mathbf{B}$ does not contain $\hbar$). In the zeroth approximation in the local coordinate system k' with the Z' axis parallel everywhere to the vector $\mathbf{B}$, we have two Hamilton-Jacobi equations:

$$\frac{\partial S_1}{\partial t}+\frac{1}{2m}\left(\nabla S_1-\frac{e}{c}\mathbf{A}\right)^2-\mu B=0, \tag{4}$$

$$\frac{\partial S_2}{\partial t}+\frac{1}{2m}\left(\nabla S_2-\frac{e}{c}\mathbf{A}\right)^2+\mu B=0. \tag{5}$$

Accordingly we have two Hamiltonians and two kinetic equations for the distribution functions

$$H_1=\frac{1}{2m}\left(\mathbf{p}-\mathbf{A}\frac{e}{c}\right)^2-\mu B,\quad H_2=\frac{1}{2m}\left(\mathbf{p}-\mathbf{A}\frac{e}{c}\right)^2+\mu B,$$
$$\partial f_1/\partial t+\{H_1, f_1\}=0,\quad \partial f_2/\partial t+\{H_2, f_2\}=0; \tag{6}$$

here $\mathbf{p}$ is no longer an operator as in (1), but a generalized momentum, and $\{\ldots,\ldots\}$ are Poisson brackets (see Ref. 10).

We can formally assume that there are two types of plasma with distribution functions f_1 and f_2. The type to which a given electron belongs depends on the direction of its spin, parallel or antiparallel to the field $\mathbf{B}$. In the adiabatic approximation (4), (5) these two types do not interact with each other. We shall henceforth refer to plasma particles of type 1 or type 2 (and correspondingly to a plasma of type 1 or 2).

The kinetic equations (6) are partial differential equations and their characteristics are called the classical trajectories of the particle. We note that Rubin[7] considered particle beams in the geometrical-optics ap-

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proximation, so that the classical trajectory was identified with a beam of small diameter D. Since our subject is not beams, we shall regard the classical trajectories as just the characteristics for the solution of Eqs. (6). The characteristic dimension over which the wave function (2) varies is determined by the distribution function $\psi^*\psi = \int f d\mathbf{v}$. More specifically,

$$\psi = \exp(iS_1/\hbar)\xi_1 + \exp(iS_2/\hbar)\xi_2, \quad \xi_{1,2} = \hat{D}(g^{-1})\chi_{1,2},$$
$$n_1 = \chi_1\chi_1^* = \int f_1\, d\mathbf{v}_1, \quad n_2 = \chi_2\chi_2^* = \int f_2\, d\mathbf{v}_2,$$
$$\mathbf{v}_{1,2} = \frac{1}{m}\left(\nabla S_{1,2} - \mathbf{A}\frac{e}{c}\right), \tag{7}$$
$$\chi_1 = \begin{pmatrix} \nu_1^{(0)} \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ \chi_{-1}^{(0)} \end{pmatrix},$$

g is the rotation from the system k' to the laboratory system k, and $\hat{D}$ is the finite-rotation matrix for the spin $\frac{1}{2}$.[9]

Repeating the calculations of Ref. 7, we obtain the conditions for the applicability of the adiabatic approximation (4), (5)

$$\lambda \ll R \ll l \ll d^2/\lambda, \tag{8}$$
$$L \ll d^2/\lambda, \; L \ll l^2/R; \tag{9}$$

$L = vt$, t is the time of the process in question, and $\lambda = \hbar/mv$ is the de Broglie wavelength.

The estimates (8) can be obtained by using directly the exact equation (3). In fact R, which is the smallest "classical" dimension in the problem, should be larger than λ—this is the condition under which the concept of a trajectory is applicable. We now divide all the terms into three groups proportional to $\hbar^0$, $\hbar^1$, and $\hbar^2$. The adiabatic approximation corresponds to terms $\sim\hbar^0$. The smallest of these terms

$$\mu\sigma\mathbf{B}\xi \approx \hbar\omega_e\xi/2 \quad (\omega_e = e\mathbf{B}/mc)$$

is in fact due to the intrinsic angular momentum of the electron. It should be larger than all the terms $\sim\hbar^1$. The latter can be estimated at $\hbar v\xi/l$ or $\hbar v\xi/d$. This leads to the second condition of (8), $l \gg R$ (we recall that $R = v/\omega_e$); the other ensuing condition, $d \gg R$, was assumed by us from the very beginning. Finally, the term $\sim\hbar^2$ is estimated at $\hbar^2\xi/2md^2$, and it must be less than the terms $\sim\hbar^1$. This leads to the third condition of (8): $d^2/\lambda \gg l$, since we assume henceforth that $d \lesssim l$.

We proceed now to a discussion of the inequalities (9). The first inequality is the well known condition that the quantum corrections have not yet managed to accumulate enough to distort the distribution function calculated from the classical trajectories with characteristic cluster dimension of the order of d. As to the second inequality (9), it can be obtained exactly from the same considerations that are used in drift theory (see Ref. 11, §3) using the small parameter R/l [see (8)]. Therefore exactly the same limitation exists in drift theory on the particle path length L.

We now make a remark concerning the validity of the use of the drift approximation. The drift velocity and the drift invariance are calculated accurate to the parameter R/l, and the quantities of order $(R/l)^2$ are discarded. We examine now the accuracy, with respect to the parameter R/l, to which the classical approximation (4), (5) corresponds. As shown in Ref. 7, in first-order approximation we obtain a correction for the wave function (7):

$$\chi_1 = \chi_1^{(0)} + \hbar\chi_1^{(1)}, \quad \chi_2 = \chi_2^{(0)} + \hbar\chi_2^{(1)},$$

and the estimates of the upper row of the spinor $\hbar\chi_1^{(1)}$, i.e., $\hbar\nu_1^{(1)}$, and the lower row of the spinor $\hbar\chi_2^{(1)}$, i.e., $\hbar\chi_{-1}^{(1)}$, lead exactly to the inequalities (9) (see (19) in Ref. 7). As to $\hbar\nu_{-1}^{(1)}$ and $\hbar\chi_1^{(1)}$, they are estimated at

$$\hbar\nu_{-1}^{(1)} \approx \nu_1^{(0)} R/l, \quad \hbar\chi_1^{(1)} \approx \chi_{-1}^{(0)} R/l$$

[see (15) of Ref. 7].

Next, in all the quasiclassical expressions (density, velocity) the wave function enters quadratically in the form of expressions of the type

$$\psi^*\psi, \quad (\psi\nabla\psi^* - \psi^*\nabla\psi)\,i\hbar/2m - \psi\psi^*\mathbf{A}e/mc.$$

In addition, the drift approximation is obtained by averaging the distribution function over the short time $2\pi/\omega_c$. This means that when ψ from (7) is substituted in these expressions (which are quadratic in ψ), no cross terms $\xi_1^*\xi_2$ and $\xi_1\xi_2^*$ are produced. In fact, the averaged expressions

$$\langle \exp[i(S_1 - S_2)\hbar] \rangle$$

(the angle brackets mean averaging over the small time $2\pi/\omega_e$) vanish; $S_1/\hbar$ and $S_2/\hbar$ differ by at least the term

$$\int_0^t \omega_e(\mathbf{x} = \mathbf{x}(t_1))\,dt_1,$$

where $\mathbf{x}(t_1)$ is the trajectory and

$$\left\langle \exp i\int_0^t \omega_e(\mathbf{x} = \mathbf{x}(t_1))\,dt_1 \right\rangle = 0.$$

We are left only with the expressions for $\xi_1^*\xi_1$ and $\xi_2^*\xi_2$; these contain the above-mentioned corrections $\hbar\nu_{-1}^{(1)}$ and $\hbar\chi_1^{(1)}$, as can be easily seen, raised to the second power (since $\xi_1^{*(0)}\xi_1^{(1)} = 0$, $\xi_2^{*(0)}\xi_2^{(1)} = 0$). The corrections themselves are $\sim R/l$, and their squares $\sim(R/l)^2$.

Thus, the terms discarded in the quasiclassical approximation are of the order of $(R/l)^2$ [in addition to other small parameters in accordance with inequalities (8)]. Consequently, the transition from the quasiclassical equations (4), (5), and (6) to the drift approximation is perfectly legitimate.

2. SPIN POLARIZATION OF A PLASMA WITH A SMALL VELOCITY SPREAD

We use the letter p for the polarization of the plasma and define it as

$$p = (n_1 - n_2)/(n_1 + n_2), \quad |p| < 1; \tag{10}$$

here $n_{1,2}$ is the density of plasma of type 1 and 2 [see (7)]. We note first that the plasma has a natural polarization. In fact, at thermodynamic equilibrium the measure of the spin polarization is given by

$$p = \frac{e^{\mu B/T} - e^{-\mu B/T}}{e^{\mu B/T} + e^{-\mu B/T}} \approx \frac{\hbar\omega_e}{2T}. \tag{11}$$

We now define a parameter that will play an important role subsequently

$$\alpha=\mu B/T=\hbar\omega_e/2T=\lambda/R. \quad (12)$$

It is assumed here that the magnetic moment of the electron, as follows from the Dirac equation, is defined as $\mu=e\hbar/2mc$. In addition, $T=mv^2/2$, where v is the total velocity of the particle; in the presence of a velocity spread, v will be taken to mean the characteristic value of the velocity (the mathematic expectation value of v). It is this velocity which is substituted in the definition of the de Broglie wavelength $\lambda=\hbar/mv$, and also of the Larmor radius $R=v/\omega_e$.

We confine ourselves to a purely classical situation, when the parameter α is quite small. In this case the natural polarization, according to (11), is negligibly small. To obtain a noticeable polarization higher than α it is necessary to consider a state far from thermodynamic equilibrium.

We place the plasma in the field of an adiabatic trap with an axially symmetrical field (see, e.g., Ref. 12). The dynamics of the particles is described by the equation of the characteristics for the kinetic equations (6):

$$\dot{\mathbf{v}}_1=[\mathbf{v}_1\times\boldsymbol{\omega}_e]+\hbar\nabla\omega_e/2m,\quad \dot{\mathbf{v}}_2=[\mathbf{v}_2\times\boldsymbol{\omega}_e]-\hbar\nabla\omega_e/2m. \quad (13)$$

We can formally assume that the particles move in an inhomogeneous magnetic field in the presence of an electric potential $\pm\hbar\omega_e/2e$, which is positive for plasma of type 1 and negative for plasma of type 2. It is known that application of an electric field produces an electric drift of the particles across the magnetic field. In our case of an adiabatic trap, this electric drift is superimposed on the azimuthal drift due to the inhomogeneity of the magnetic field. Since the electric field, as follows from (13), is given by the expression $\pm\hbar\nabla\omega_e/2e$, the electric drift velocity takes the form

$$\langle\mathbf{v}_{1,2}\rangle=\frac{c}{B^2}[\mathbf{E}\times\mathbf{B}]=\pm\frac{\hbar}{2m}\left[\frac{\nabla\omega_e}{\omega_e}\times\frac{\mathbf{B}}{B}\right]\approx\frac{\hbar}{2ml}=\frac{v\lambda}{2l}, \quad (14)$$

where the plus sign corresponds to the drift velocity $\langle\mathbf{v}_1\rangle$, and the minus sign to the velocity $\langle\mathbf{v}_2\rangle$. It follows therefore that two plasma bunches of type 1 and type 2 move apart with a velocity $\approx\hbar/ml$. As a result, the plasma vecomes polarized. However, the fastest and most effective separation of bunches of different types occurs along the force lines. We proceed not to describe this process.

We write down in the required approximation an expression for the velocity component $v_\parallel$ parallel to $\mathbf{B}$ (see Ref. 13):

$$m\dot{v}_\parallel=e\mathbf{E}\frac{\mathbf{B}}{B}-\frac{m}{2B^2}v_\perp^2\mathbf{B}\nabla B=(\pm\mu-\mu_\perp)\mathbf{h}\nabla B; \quad (15)$$

here $v_\perp^2=v^2-v_\parallel^2$, $\mu_\perp=mv_\perp^2/2B$ is the transverse adiabatic invariant (classical magnetic moment), and $\mathbf{h}=\mathbf{B}/B$. The motion along the force line is described therefore in the following manner:

$$\begin{aligned}&m\dot{v}_\parallel=-\partial U_{1,2}/\partial x_\parallel,\quad U_{1,2}=-(\pm\mu-\mu_\perp)B,\\ &mv_\parallel^2/2+U_{1,2}=E,\quad mv^2/2\mp\mu B=E,\end{aligned} \quad (16)$$

where $\partial/\partial x_\parallel$ is the derivative along the direction of h. Assume that electrons with almost equal $v_\parallel$ and $v_\perp^2$ are injected into the central part of the trap (the permissible spreads of $v_\parallel$ and $v_\perp^2$ will be indicated below). In the present section we consider both trapped and untrapped particles. We first turn to the trapped particles.

The electrons of two sorts move in somewhat different potentials, U_1 and U_2, where $U_1-U_2=\hbar\omega_e$. Therefore the periods t_1 and t_2, of the oscillations of the electrons of the different sorts in the trap (i.e., the time of passage from mirror to mirror and back) are different. The exact value of t_1-t_2 is determined by the concrete model of the trap field. For estimates it suffices to use the system (16) with account taken of the fact that $\mu/\mu_\perp$ is the only small parameter that determines the difference t_1-t_2. In other words, $t_1-t_2\approx t_1\mu/\mu_2$.

The exact calculation given in Sec. 3 for a definite model confirms this model [see Eq. (28)]. We assume for the sake of argument that the initial instant we have $v_\parallel^2\approx v_\perp^2$, and then the small parameter $\mu/\mu_\perp$ is none other than α in accordance with (12), but recalculated for the center of the trap. We use the subscript zero for all the values (fields, velocities, etc.) at the center of the trap. Consequently we can write $t_1-t_2\approx t_1\alpha_0$, and therefore the two clusters of sort 1 and sort 2 move apart with velocity

$$\delta v_\parallel=l(1/t_2-1/t_1)\approx v_\parallel\alpha_0. \quad (17)$$

Comparing (17) with (14) we see that the plasma becomes polarized along the force lines faster by a factor l/R than across the lines. The measure of the polarization, defined by (10), can be estimated at

$$p=\delta v_\parallel t/d, \quad (18)$$

where the estimate (18) holds when $p\leqslant 1$; after sufficiently long times $t>d/\delta v_\parallel$ the clusters move apart completely, and $p=1$ (complete polarization).

We estimate now the minimum possible polarization, using the limitations (9) and the fact that in the given case $L\approx v_\parallel t$ (since it is assumed that $v_\parallel\approx v_\perp\approx v$):

$$p\ll d/R,\quad p\ll\alpha_0 l^2/dR. \quad (19)$$

The first inequality in (19) is not a restriction, since $d/R>1$, whereas $p\leqslant 1$ by definition. On the other hand, the second restriction admits in principle of a polarization measure $p>\alpha$ (i.e., greatly exceeding the thermodynamic-equilibrium value) and even $p=1$. The point is that $l\gg d$ and $l\gg R$, and consequently the parameter l^2/dR in (19) is much larger than unity.

By way of example we consider an adiabatic trap with $d=10R$, stipulate total polarization, $p=1$, and require, to satisfy (19), that $\alpha_0 l^2/dR^2=10$. Then $l=10R\alpha_0^{-1/2}$. For a field $B_0=10^4$ G and an energy $T=10^{-11}$ erg we have $R=5\times10^{-4}$ cm and $\alpha_0=10^{-5}$, so that l turns out to be ≈1.6 cm. The dimension $l_{cr}=1.6$ cm is the critical value of l: the inequality (19) is satisfied at $l\geqslant l_{cr}$. We recall that l is the characteristic dimension of the change of the field, i.e., in fact the dimension of the trap. It is easy to verify that in this example the inequalities (8) are satisfied with large margin (because of the small de Broglie wavelength, $\lambda=10^{-8}$ cm in this case).

It was stated above that the initial velocities of the particles should be almost equal. It is clear that the velocity spread Δv must in any case be less than $\delta v_\parallel$ in accordance with (17). Otherwise the cluster diffuses before it splits into two clusters of different types. More accurately speaking, the following conditions must be satisfied:

$$\Delta v_\parallel \ll \delta v_\parallel, \quad \Delta v_\perp \ll \delta v_\parallel. \tag{20}$$

The second condition of (20) follows from the requirement imposed on $\mu_\perp$ in (16) and from the fact that $v_\parallel \approx v_\perp$. The requirements (20) do not contradict the uncertainty relation, according to which $\Delta v \geqslant \hbar/md$. This follows in fact from relations (8) and (9). In addition, this can be verified also directly. The conditions (20) are replaced by $\Delta v \ll \delta v_\parallel \approx \alpha_0 v$, and at $d \gg R$ we have $\hbar/md \ll \alpha_0 v$, which was in fact assumed. Thus, the velocity is spread $\hbar/md$ that results from the uncertainty relation is small compared with $\delta v_\parallel$. We recall that at large $R > l$, the uncertainty relations made it completely impossible to carry out the Stern-Gerlach experiment.[2,3] The requirement $\Delta v < \delta v_\parallel$ means that $\Delta v \ll v$. This is preciesly why it can be stated that the plasma is characterized by a small velocity spread.

We now discuss a somewhat modified Stern-Gerlach experiment. To this end we consider the particles that leave the trap. In this case the configuration of the magnetic field outside the trap proper becomes important. A section through the trap is shown in the figure. The trap proper [more accurately speaking, its half from the center (the left-hand part of the figure) to the mirror at the point A] is to the left of the point A. Outside of the trap (to the right of the point A) the field has been continued uniformly. Let now electrons with a small velocity spread $\Delta v < \delta v = \alpha_0 v$ be injected into the central part of the trap. The plasma of type 1 will then arrive at the point A with a velocity that differs somewhat from the velocity of the plasma of type 2 at the same point, namely according to (16) we have

$$[v_\parallel(A)]_{1,2}=[(v^2)_0-(v_\perp^2/B)_0 B_{max}\pm(2\mu/m)(B_{max}-B_0)]^{1/2}.$$

Here B_{max} is the field in the region of the mirror. Because of the difference between the velocities, $[v_\parallel(A)]_1$ and $[v_\parallel(A)]_2$, the plasma of type 1 will overtake the plasma of type 2. The thickness of the leading front of the plasma (i.e., of the region where the density changes from zero to the average value) depends on the method used for the injection into the central region of the trap. Let the injection of the electrons begin at $t=0$ and let the particle flux by the instant $t=\Delta t$ reach its constant (after $t>\Delta t$) value. Then the front thickness is $v_\parallel(A)\Delta t = d$ ($v_\parallel(A)$ is the velocity neglecting the intrinsic moment). Using the smallness of the intrinsic magnetic moment of the electron, we have

$$\delta v_\parallel=[v_\parallel(A)]_1-[v_\parallel(A)]_2=2\mu(B_{max}-B_0)/mv_\parallel(A). \tag{21}$$

The total polarization, i.e., the overtaking of the plasma of type 2 by the plasma of type 1 over the length d, occurs within a time $d/\delta v_\parallel$ at a distance $dv_\parallel(A)/\delta v_\parallel$ from the point A. By way of an estimate we assume that B_{max} is not much larger than B_0 and that $v_\parallel \approx v_\perp$ at the center of the trap. Then $\delta v_\parallel \approx \alpha_0 v_\parallel(A)$ and the total polarization takes place at a distance d/α_0 from the point A.

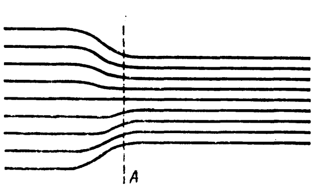

FIG. 1.

We note that the criteria (8) and (9) are certainly satisfied here if we put $d \approx l$. In practice, let the particle path L also be close to l in view of the fact that outside the trap (to the right of the point A) the field is homogeneous, we have $\nabla\omega_e=0$ in Eqs. (13) and this system becomes exact for $v_\parallel$. The criterion (9) then contains only the path of the particle inside the trap. The criteria (8) and (9) are ultimately replaced by the conditions $\lambda \ll R \ll l$.

3. SPIN POLARIZATION OF A PLASMA WITH A THERMAL VELOCITY SPREAD

It was shown above that the velocity δv at which the clusters move apart is proportional to $\hbar$; in other words, this velocity is small. It might seem therefore that in the classical case, when α is quite small in accordance with (12), the velocity δv cannot be discerned against the background of the thermal motion. This, however, is not always the case. We shall illustrate the feasibility, in principle, of polarizing a thermal plasma using what is probably the simplest example, that of an emptying adiabatic trap. The gist of the process is that the coefficients of reflection of the particles from the mirrors is different for particles of different types. The rates of emptying of the trap for the plasma of type 1 differs from that of type 2, as a result of which the residual plasma in the trap becomes polarized. We are interested here in final analysis in the number of particles that leave the trap after passing once through the trap. Therefore, just as in the Stern-Gerlach experiment (see the end of Sec. 2), the quasiclassical approximation is valid at $\lambda \ll R \ll l$.

The condition of particle containment follows directly from (16). The velocity $v_\parallel$, as seen from (16) is given by

$$(v_\parallel)_{1,2}=[(v^2)_0-(v_\perp^2/B)_0 B\pm(2\mu/m)(B-B_0)]^{1/2}. \tag{22}$$

For the particle to be reflected, the radicand of the expression must become negative inside the trap: the particle cannot pass into a region where this expression is less than zero. From this we obtain the relection condition:

$$\begin{gathered}(v^2/v_\perp^2)_0<B_{max}/B_0\mp(\mu/\mu_\perp)(B_{max}/B_0-1),\\ \sin^2\theta>[1\pm(k-1)\mu/\mu_k]/k,\\ \sin^2\theta=v_\perp^2/v^2,\quad k=B_{max}/B_0,\quad \mu_k=mv_0^2/2B_0.\end{gathered} \tag{23}$$

Naturally, neglecting the instrinsic magnetic moment [$\mu=0$ in (23)], we return to the well known reflection condition: $\sin^2\theta>1/k$. Consequently, the number of particles that leave both ends of the trap per unit time is determined by the quantity (see § 10 of Ref. 13)

$$\tfrac{1}{2}sn_v dv[1\pm(k-1)\mu/\mu_k]/k,$$

where s is the cross section of the central part of the trap, and $n_v dv$ is the density of the particles with velocities between v and $v+dv$; the velocity distribution is assumed Maxwellian.

Integrating over the velocities, we have

$$\frac{\partial N_{1,2}}{\partial t}=-sn_{1,2}v_T\left[1\pm\frac{3}{2}(k-1)\frac{\mu}{\bar{\mu}_k}\right]\frac{1}{k}\left(\frac{2}{3\pi}\right)^{1/2}. \qquad (24)$$

Here $v_T^2=3T/m$, T is the temperature, and $\bar{\mu}_k=mv_T^2/2B_0$, $N_{1,2}$ is the number of particles of the first and second kind in the trap. The volume of the trap is sl' ($l'\approx l$), and consequently $N_{1,2}=sl'n_{1,2}$. Substituting this $N_{1,2}$ in (24), assuming that the Maxwellian distribution is maintained in the trap by the collisions, and assuming also a constant temperature, we obtain an exponential decrease of the density. It follows from (24) that the arguments of the exponentials are somewhat different for the particles of type 1 and 2. As a result, the residual plasma becomes polarized.

According to the definition of the measure of the polarization (10), we obtain, using (24),

$$p=\text{th}\left[\frac{3}{2}\frac{v_T}{l'}\frac{1-k}{k}\frac{\mu}{\bar{\mu}_k}\left(\frac{2}{3\pi}\right)^{1/2}t\right]=\text{th}\,\nu_p t.$$

It is seen therefore that at $t\gg 1/\nu_p$ we have $|p|\to 1$, i.e., the polarization becomes complete. It must be borne in mind that the Coulomb collisions that lead to isotropization of the distribution function, depolarizes the plasma at the same time. Nonetheless, polarization can take place. The point is that the frequency ν_π of collisions in which the spin of the electron is rotated through an angle π (or the frequency of spin flip as a result of collisions) is negligibly small compared with the usual collision frequency ν. Thus, the described polarization will take place if

$$\nu_p,\ \nu>\nu_\pi.$$

We have demonstrated above the possibility, in principle, of spin polarization of a plasma that is initially in thermal equilibrium. As to the experimental realization of this possibility or its occurrence under astrophysical conditions, two difficulties arise here. First, the quantity ν_p contains a small parameter $\mu/\bar{\mu}_k=\frac{2}{3}\alpha_0$, as a result of which the time of polarization becomes quite large. Second, by the time polarization becomes substantial, the density of the plasma itself becomes exponentially small. Nonetheless, this possibility cannot be denied under conditions of interstellar clouds and over cosmological times, when large density differentials (by many orders of magnitude at long times) are usual phenomena.

We proceed now to consider plasma polarization in an alternating magnetic field, which is of greater interest from the point of view of applications. It is known that the particles become accelerated in a nonstationary field. The gist of the mechanism is that the particles of type 1 and type 2 are accelerated somewhat differently. The difference in the energy increment per unit time of the particles of the different types accumulates. After a long time interval this difference can become appreciable. We consider below the acceleration of particles in a nonstationary field of an adiabatic trap, and acceleration by MHD waves.

We consider first the nonstationary trap. It is known that a periodic perturbation of the magnetic field of the trap in the region of the mirror can accelerate the particles. We consider perturbations in the form of periodic shifts of the entire magnetic fields in the region of the mirror along the trap axis. Then the motion of the particle between the mirrors recalls the motion between walls, one of which executes periodic oscillations. The particle-energy increment is given by (see Refs. 14 and 15)

$$dv^2/dt=V^2/t', \qquad (25)$$

where v is the particle velocity (in our case this is $v_\parallel$), V is the velocity of the wall (mirror), and t' is the time of motion of the particle between the walls. The mirror oscillation frequency ω is assumed large compared with $2\pi/t'$, and in addition, $v\ll V$, for otherwise an adiabatic regime is realized, the longitudinal adiabatic variant is conserved, and there is no acceleration.

We shall see below that the values of t' differ somewhat for particles of type 1 and 2. This will cause particles of the different types to be differently accelerated. The period t' is defined according to (16) as follows:

$$t_{1,2}=(m/2)^{1/2}\int(E-U_{1,2})^{-1/2}dx_\parallel. \qquad (26)$$

The integration is carried out up to the points where the denominator vanishes, and $x_\parallel$ is the length of the path along the force line. We consider the following model:

$$B=B_0(1+x_\parallel^2/a^2). \qquad (27)$$

Here $x_\parallel$ is reckoned from the center of the trap (where $x_\parallel=0$). Then

$$t_{1,2}=\pi a[(\mu_\perp\mp\mu)B_0]^{-1/2}=t'(1\mp{}^1/_2\mu/\mu_\perp), \qquad (28)$$

where t' is the period without allowance for the intrinsic moment; the last equation in (28) is accurate to within a quantity of the order of $\mu/\mu_\perp$. Neglecting the collisions, $\mu_\perp$ is conserved and consequently $t_{1,2}$ does not depend on the energy, and $t_1\neq t_2$.

The solution of (25) takes the form

$$v_{1,2}^2=v_{1,2}^2(0)+tV^2/t_{1,2}.$$

We see therefore that the difference between the energies of particles of type 1 and 2 increases gradually and after a time

$$t=\frac{t'v_T^2}{V^2}\frac{\mu_\perp}{\mu}$$

their difference will be of the order of the thermal energy.

We have thus described in this example the polarization of a plasma in phase space. In addition, polarization in ordinary space takes place here simultaneously. In fact, the point of reflection in the model (27) moves away in the course of the acceleration into the region of ever increasing $x_\parallel$, $x_\parallel^{(R)}\approx v_\parallel/t'$ ($x_\parallel^{(R)}$ is the reflection point). Consequently the particles of the different types accelerate differently, and the points of reflections for the different particles move away from each other. After a time

$$t=\frac{a^2}{t'V^2}\left(\frac{\mu_\perp}{\mu}\right)^2$$

the distance between the reflection points becomes $\approx a$, and plasma of only one kind is present between the reflection points, i.e., the plasma is completely polarized.

The foregoing particle acceleration is due to the nonstationary character of the magnetic field. The field of the trap was perturbed in the region of the mirrors. One can expect a perfectly analogous effect to occur in the presence of nonstationary perturbations in the central region.

We turn now to particle acceleration in the field of MHD waves propagating against the background of a homogeneous magnetic field. The particles move mainly along the force lines, i.e., almost along the straight line of the background field, and "follow" the perturbations—the MHD waves. Larmor rotation with velocity $v_\perp$ causes the particle to wind its way around the force line. The classical magnetic moment $\mu_\perp = mv_\perp^2/2B$ is, as is well known, an adiabatic invariant. Consequently, $v_\perp^2$ fluctuates together with B, but is conserved on the average. As to the velocity $v_\parallel$ along the force lines itself, it was shown with the nonstationary trap as an example that the averaged value $\langle v_\parallel^2 \rangle$ can increase because of the nonstationary perturbations—the MHD waves. The particles of different type are differently accelerated, and this leads to spin polarization of the plasma.

We write down the equation for $v_\parallel$ (see Ref. 13):

$$m\dot{v}_\parallel = e(\mathbf{Eh}) - \mu_\perp(\mathbf{h}\nabla B) + e\frac{v_\parallel}{\omega_e}(\mathbf{E}[\mathbf{h}(\mathbf{h}\nabla)\mathbf{h}]) - \mu_\perp\frac{v_\parallel}{\omega_e}(\nabla B[\mathbf{h}(\mathbf{h}\nabla)\mathbf{h}]); \quad (29)$$

$\mathbf{h} = \mathbf{B}/B$. We represent the field in the form $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, where $\mathbf{B}_0$ is the homogeneous field and b is the fluctuation. The third and fourth terms of the right-hand side of (29) are quadratic in the fluctuation and will be disregarded. Expression (29) contains the entire information on the forces acting on the plasma. In particular, the specific force due to the presence of the magnetic moment of the electron, $\pm\hbar\nabla\omega_e/2e$ (see (13)), is included in the electric field $\mathbf{E}$.

If the intrinsic magnetic moment is neglected, Eq. (29) becomes much simpler when account is taken of the fact that for MHD perturbations, as is well known, $\mathbf{E}\cdot\mathbf{h} = 0$:

$$m\dot{v}_\parallel = -\mu_\perp(\mathbf{h}\nabla B) = -\partial U/\partial x_\parallel, \quad U = \mu_\perp B. \quad (30)$$

The acceleration is perfectly analogous to the acceleration of a particle in the field of a nonstationary potential. The physical meaning of the acceleration becomes clearer if we change to the expression for the energy:

$$\frac{d}{dt}\left(\frac{mv_\parallel^2}{2} + U\right) = \frac{\partial U}{\partial t}. \quad (31)$$

This expression reduces, naturally, to Eq. (6.13) of Ref. 13:

$$dT/dt = e(\mathbf{Ev}) - c\mu_\perp(\mathbf{h}\,\mathrm{rot}\,\mathbf{E}), \quad (32)$$

where T is the energy. We have used here the electrodynamics equation

$$\mathrm{rot}\,\mathbf{E} = -c^{-1}\partial\mathbf{B}/\partial t,$$

and in the approximation linear in the perturbation we must put $E\cdot v = 0$. It is seen directly from (32) that the acceleration is connected with the solenoidal electric field that is produced because the magnetic field is not stationary.

When account is taken of the intrinsic magnetic moment, a term $\pm\hbar\nabla\omega_e/2e$ is added to the electric field, E, and this changes the electric-field component parallel to the magnetic field. Now the first terms of the right-sides of (29) and (32) no longer vanish and, as can be easily verified, these first terms turn out to be of the same form as the second terms, with $\mu_\perp$ replaced by $\pm\mu$. By combining these terms we get the potential $U = (\mu_\perp \mp \mu)B$ [cf. (15)], and instead of (32) we get accordingly

$$dT/dt = c(\pm\mu - \mu_\perp)(\mathbf{h}\,\mathrm{rot}\,\mathbf{E}). \quad (33)$$

In (33) we must integrate along the trajectory

$$T = T_0 + \int_0^t f\,dt;$$

f is the right-hand side of (33). If f is a stationary (in the statistical sense) random process, then the integral of f increases, even though $\langle f \rangle = 0$. In fact,

$$\langle T^2 \rangle = T_0^2 + 2\int_0^t (t-y)\Phi(y)\,dy \approx T_0^2 + 2t\int_0^\infty \Phi(y)\,dy,$$
$$\Phi(y) = \langle f(t) f(t+y) \rangle, \quad (34)$$
$$\langle T^2 \rangle^{1/2} \approx \left(T_0^2 + 2t\int_0^\infty \Phi(y)\,dy\right)^{1/2}.$$

Thus, the energy increases like $t^{1/2}$.

The assumption that the function f is stationary along the trajectory acquires a clear physical meaning at $v_\parallel > v_A$, where v_A is the Alfven velocity. The frequency of the perturbation $\omega = v_A/l$ is much smaller here than v/l, where l is the inhomogeneity scale (the wavelength). In a time interval on the order of several correlation times of the random function f it can be assumed approximately that

$$f[x(t), t] = f[x(t), t_0] = f[v_\parallel(t-t_0) + x_0, t_0], \quad v_\parallel = \mathrm{const}.$$

In this approximation, the assumption that f is stationary is essentially the quite natural assumption that the magnetic fluctuations are homogeneous. It is convenient to change over to the variable r in the integral

$$\int_0^\infty \Phi(y)\,dy,$$

and write down (34) in differential form:

$$\frac{d\langle T^2 \rangle}{dt} = \frac{2}{v_\parallel}\int_0^\infty \Phi(r)\,dr,$$
$$\Phi(r) = \langle f(x+r) f(x) \rangle, \quad dy = dr/v_\parallel, \quad (35)$$
$$T \sim \left(\frac{5qt}{m^2}\right)^{2/5} m, \quad q = 2\int_0^\infty \Phi(r)\,dr.$$

The last expression is valid at large t.

According to (33), the expression for q in (35) is different for unlike particles. Whereas unlike particles having same initial energy $T(0)$ should, neglecting the intrinsic moment, acquire by the instant of time t and energy $T(t)$, the actual difference between the energies of the unlike particles reaches by that instant the value

$$T_- - T_+ = {}^8/_5 T(t)\,\mu/\mu_\perp.$$

We see therefore that after a time

$$t=\frac{1}{5qm^{1/4}}\left(\frac{5}{8}T_T\frac{\mu_\perp}{\mu}\right)^{4/5}$$

(T_T is the thermal energy) the energy difference reaches the value T_T.

In the situation described above, the plasma becomes spin-polarized in phase space. Polarization in ordinary space can be the consequence of this process. In fact, as stated above, the particle moves mainly along the force line, i.e., almost along a straight line. Assume that the region in which the MHD waves propagate is finite, so that the particles ultimately leave this region and land in an unperturbed homogeneous magnetic field. Because of the different velocities of particles of different type, the leading front of the plasma, on entering the homogeneous field, is polarized in analogy with the situation in the Stern-Gerlach experiment (see the end of Sec. 2). A similar situation will obtain also when accelerated particles land in the region where the plasma can no longer be regarded as collisionless (for example, the spilling of particles from the earths magnetosphere into the ionsphere). Particles of different types, having different energies, have different mean free paths, as a result of which the plasma becomes polarized over a finite length.

The described acceleration and polarization of particles in phase space can take place in an interstellar plasma, and produce in turn circular polarization of the passing electromagnetic radiation.

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Advanced magnetothermal phenomena in a laser plasma

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The possibility of spontaneous generation of sizable magnetic fields in a laser plasma within periods much shorter than hydrodynamic times is demonstrated. The field growth has a threshold that depends on the geometrical factors of the corona and on the dimensions of the heat-release region. The magnetic fields cause a qualitative restructuring of the heat front and decrease the heat transfer substantially. In particular, heat can penetrate into the plasma in the form of a magnetothermal jet.

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1. The magnetic fields produced spontaneously in a laser plasma were investigated in a number of experimental and theoretical studies.[1-21] The hertofore considered magnetic-field generation mechanisms can be broken up into two groups.

The first group includes mechanisms connected in one way or another with the process of absorption of the laser radiation. Notice should be taken of the transfer of momentum from the light wave to the electrons in resonant absorption when the radiation is obliquely incident on the critical surface,[2,3] of the electromagnetic instability due to the anisotropy of the distribution of the electron velocities in the absorption zone,[4,5] of the parametric generation of magnetic fields by beats between the high-frequency motions of the electrons,[6,7] and of the generation of magnetic fields by the advanced Langmuir motions in the plasma.[8] The foregoing field-generation methods are of considerable interest from the point of view of their influence on the light absorption, and particularly on the diffraction of the absorbed energy and its distribution over the critical surface. The

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