Spectral distribution of the radiation in planar and axial channeling of ultrarelativistic electrons

V. A. Bazylev, V. I. Glebov, and N. K. Zhevago

I. V. Kurchatov Institute of Atomic Energy
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The spectral, angular, and polarization characteristics of radiation from channeled ultrarelativistic electrons are calculated. Using realistic models of the continuous potential of the planes and axes of the crystal, analytic expressions are obtained for the spectral-angular power density of the radiation. A critical analysis is made also of some existing results of the theory of radiation produced in channeling.

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INTRODUCTION

The channeled-particle energy connected with the transverse motion takes on discrete values, since this motion is finite. Thompson[1] has pointed out that one of the effects that lead to the change of the level population may be the spontaneous transitions between the transverse energy levels of the protons. In this case the radiation, according to Thompson, should lie in the infrared region. He did not perform, however, the corresponding calculation of the spectrum. A more detailed analysis shows that nonrelativistic channeled protons should actually emit in the infrared region in transitions between tranverse-energy levels. Unfortunately, the emission intensity turns out to be too small for this phenomenon to be observable in practice.

In contrast, as first show theoretically by Kumakhov,[2] ultrarelativistic electrons and positrons emit mainly in the x-ray and gamma bands. The emission intensity is so high that it can exceed considerably the intensity of the known types of radiation, such as synchrotron and transition radiation or bremsstrahlung. This strong difference between the emission of protons and positrons having identical total energies is due to the Doppler effect. It was therefore natural for A. Vorob'ev, Kaplin, and S. Vorob'ev, who did not take this effect into account, to arrive at incorrect conclusions concerning the intensity and spectral distribution of the emission from channeled electrons (for details see Ref. 4).

Following Kumakhov's pioneering work,[2] in view of the great practical importance of channeled particles as a high-power source of x rays and gamma rays, the theory of this phenomenon was diligently developed. Thus, for example, Kumakhov and his co-workers[4-7] made a detailed analysis of the angular, spectral, and polarization characteristics of the dipole radiation of channeled particles, and reviewed also the effect of stimulated emission. Bazylev and Zhevago[8-10] considered the influence of the frequency and spatial dispersion of the electromagnetic field in the crystal on the emission process, and analyzed the characteristics of the emission as functions of the energy of the channeled particles and of the form of the effective potential of the crystal. In another paper, one of us[10] considered the effect of the parametric coupling of the transverse and longitudinal motions, as well as the influence of the quantum recoil and the interaction of the particle spin with the effective radiation field, which become appreciable for emission from particles of sufficiently high energy. It was shown in the same paper that there exists an optimal channeled-particle energy at which the spectral density of the radiation power reaches a maximum.

It should be noted that the character of the particle motion in the case of channeling in crystals has much in common with motion in special periodic electromagnetic fields, which are produced in devices called undulators. A sufficiently complete review of the results of the theory of undulator radiation can be found, for example, in Refs. 11 and 12. In cases when the motion of the channeled particles can be considered within the framework of classical mechanics, some results of the theory of undulator radiation can be used also to calculate the intensity of dipole radiation in planar channeling.

At the same time, as will be shown below, the emission of channeled particles exhibits a number of qualitative differences from undulator radiation.

The most quantitatively reliable results were obtained so far for the case of planar channeling of positrons.[4-6,8-10] The harmonic-oscillator potential, which is usually used in this case to calculate the emission spectra, is close enough to the actual potential of the channel. This cannot be said in the case of electron channeling. Therefore the results of the theory of emission by channeled electrons, obtained to date,[4-9,13] are only qualitative in character. Yet greatest practical interest attaches precisely to emission by electrons.

In this paper we develop the theory of emission from high-energy electrons in planar and axial channeling. We present the results of the calculation of spectra of radiation having different polarizations, for several models of effective potentials of the crystal planes and axes, which are closest to the actual form of the potential. Our results will be compared in detail with the corresponding results by others, and it will be shown in particular that the theory of emission from planar channeled particles, developed by Baryshevskii and others,[15,16] is on the whole in error.

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1. GENERAL EXPRESSIONS FOR THE EMISSION PROBABILITY

The spectral-angular probability density of the emission of a photon of energy ω and polarization e in a unit time by an electron (positron) can be represented in the following general form (see, e.g., Ref. 17, Sec. 44):

$$\frac{d^2w}{d\omega d\Omega}=\frac{e^2k}{2\pi}\sum_F (j_Ie^{i\mathbf{kr}})_{IF}(j_ne^{-i\mathbf{kr}})_{FI}e_ie_n\delta(\omega-E_I-E_F). \quad (1)$$

Here k is the momentum of the emitted photon; $d\Omega$ is the differential of the solid angle; F is the aggregate of the quantum numbers of the final state of the electron; j is the electron current-density operator; $i,n=1,2,3$ are indices that number the spatial components of the vectors ($\hbar=m=c=1$). The quantity in the parentheses denotes a matrix element of the operator of the interaction of the particles with the radiation field, describing the states $|I\rangle$ and $|F\rangle$ with energies E_I and E_F.

We assume the electron to be ultrarelativistic ($E\gg 1$), so that the interaction of the electron spin with the electric field of the planes or axes of the crystal can be neglected. This neglect is made possible by the smallness of the quantity $(U_0/E)^2$, where U_0 is the characteristic value of the interaction of the energy with the planes or axes of the crystal (for details see Refs. 6 and 9). As will be made clear below (see also Ref. 10), emission by channeled particles takes place mainly at photon energies much lower than the particle energies ($\omega\ll E$). It is then possible to neglect the recoil due to the photon emission, and also the interaction of the electron spin with the effective field of the radiation.[1)]

As a result, the electron or positron can be regarded in this case as a spinless particle. The matrix elements of the current take the form

$$(je^{i\mathbf{kr}})_{IF}=i\int[\nabla\Psi_I^*(\mathbf{r})\Psi_F(\mathbf{r})-\Psi_I^*(\mathbf{r})\nabla\Psi_F(\mathbf{r})]e^{i\mathbf{kr}}d^3r, \quad (2)$$

while the wave functions Ψ_I and Ψ_F satisfy the equation

$$[-\Delta+1-2EU(\mathbf{r})]\Psi(\mathbf{r})=0. \quad (3)$$

This equation is obtained from the exact second-order Dirac equation after neglecting the terms of order $(U_0/E)^2$.

2. EMISSION IN PLANAR CHANNELING

In planar channeling, the effective crystal potential $U(\mathbf{r})$ depends only on the distance x to the crystallographic plane. Therefore the wave function of the channeled particle is of the form

$$\Psi_I(\mathbf{r})=(2E_i^{\parallel})^{-1/2}\exp(i\mathbf{p}_i^{\parallel}\boldsymbol{\rho})\psi_i(x). \quad (4)$$

Here $I=\{\mathbf{p}_i^{\parallel},i\}$, $\mathbf{p}_i^{\parallel}$ denotes the projection of the initial particle momentum on the channeling plane, ρ is the radius vector and lies in the channeling plane. The initial wave function $\psi_i(x)$ of the transverse motion satisfies an equation of the Schrödinger type

$$\left[-\frac{1}{2E_i^{\parallel}}\frac{d^2}{dx^2}+U(x)\right]\psi_i(x;E_i^{\parallel})=\varepsilon_i(E_i^{\parallel})\psi_i(x;E_i^{\parallel}), \quad (5)$$

where $E_i^{\parallel}=(1+p_i^{\parallel 2})^{1/2}$ is the initial energy of the longitudinal motion and $\varepsilon_i(E_i^{\parallel})$ is the quantized energy of the transverse motion and depends parametrically on the longitudinal energy $E_i^{\parallel}$.

On going from Eq. (3) to Eq. (5), we have represented the total energy in the form of the sum $E=E^{\parallel}+\varepsilon$, and have neglected, as we should, the quantity $(\varepsilon/E^{\parallel})^2$.

The wave function $\Psi_F(\mathbf{r})$ of the final state of the particle is similar in form to (4). We note, however, that the wave functions $\psi_n(x;E_i^{\parallel})$ and the energy levels $\varepsilon_n(E_i^{\parallel})$ differ generally speaking from the corresponding functions $\psi_n(x;E_f^{\parallel})$ and levels $\varepsilon_n(E_f^{\parallel})$, because of the parametric coupling of the transverse motion with the longitudinal motion.

Since the photon energy is assumed to be low enough ($\omega\ll E$), we can use the relations

$$\begin{aligned}&E_i^{\parallel}-E_f^{\parallel}\approx\mathbf{k}_{\parallel}\mathbf{v}_{\parallel},\\ &\varepsilon_f(E_f^{\parallel})\approx\varepsilon_f(E_i^{\parallel})-\mathbf{k}_{\parallel}\mathbf{v}_{\parallel}\partial\varepsilon_f(E_i^{\parallel})/\partial E_i^{\parallel},\end{aligned} \quad (6)$$

where $\mathbf{v}_{\parallel}=\partial E_i^{\parallel}/\partial\mathbf{p}_i^{\parallel}$ is the longitudinal particle velocity.

After calculating with the aid of wave functions of the type (4) the matrix elements (2) of the current we arrive at the following results for the spectral-angular probability density of the emission per unit time:

$$\frac{d^2w}{d\omega d\Omega}=\frac{e^2\omega}{2\pi}\sum_f[l_\sigma|j_{if}^{(x)}(k_x)|^2\sin^2\varphi+l_\pi|j_{if}^{(x)}(k_x)\theta-j_{if}^{(z)}(k_x)\cos\varphi|^2]\delta\left[\omega\left(\frac{\theta^2+E^{-2}-\chi'(\omega)}{2}-\frac{\partial\varepsilon_f(E_i^{\parallel})}{\partial E_i^{\parallel}}\right)-\tilde{\omega}_{if}\right]. \quad (7)$$

Here $\tilde{\omega}_{if}=\varepsilon_i(E_i^{\parallel})-\varepsilon_f(E_i^{\parallel})$, $E_f^{\parallel}\approx E_i^{\parallel}-\mathbf{k}_{\parallel}\mathbf{v}_{\parallel}$ [see (6)]

$$\begin{aligned}j_{if}^{(x)}(k_x)&=\frac{i}{E_i^{\parallel}}\int\frac{d\psi_i^*(x;E_i^{\parallel})}{dx}\psi_f(x;E_f^{\parallel})\exp(ik_xx)\,dx,\\ j_{if}^{(z)}(k_x)&=\int\psi_i^*(x;E_i^{\parallel})\psi_f(x,E_f^{\parallel})\exp(ik_xx)\,dx.\end{aligned} \quad (8)$$

For photons that are linearly polarized in the emission plane[2)] we have $l_\sigma=0$ and $l_\pi=1$, while for photons polarized perpendicular to this plane we have $l_\sigma=1$, $l_\pi=0$, $k_x=k\theta\cos\varphi$ is the projection of the photon momentum on the direction normal to the channeling plane; $\theta\ll 1$ and φ are the polar and azimuthal emission angles, the $0z$ axis is chosen along the longitudinal velocity $\mathbf{v}_{\parallel}$, $\chi'(\omega)=\omega_p^2/\omega^2\ll 1$ is the average dielectric susceptibility of the crystal, and ω_p is the plasma frequency of the crystal electrons.

The derivative $\partial\varepsilon/\partial E$ depends on the concrete form of the potential and is of the order of the ratio ε/E. Thus, under the condition[3)] $\varepsilon E\gtrsim 1$ the parametric dependence of the levels of the transverse energy on the energy of the longitudinal motion becomes quite appreciable. The same can be stated also concerning the parametric dependence of the wave functions of the transverse motion.

The transverse-motion wave functions can pertain to either the discrete or to the continuous spectrum of the transverse energy. If the initial state of the particle is in the continuous spectrum and the final in the discrete spectrum, then expression (7) describes radiation accompanied by capture of the particles into the channeling regime. If both states lie in the continuous spectrum, then expression (7) corresponds to bremsstrahlung in successive collisions of particles with the crystal planes, and in the latter case an important role may be assumed by the influence of the periodicity of the

potential $U(x)$, which leads to coherence of the emission from different planes. In the remaining cases, the periodicity of the potential can as a rule be neglected.[6,10]

3. CONNECTION WITH OTHER RESULTS

After summation over the photon polarizations ($l_\sigma = l_\pi = 1$), expression (7) becomes a particular case of a more general result obtained by Zhevago[10] [see Eqs. (10) and (11) in Ref. 10].

If $\varepsilon E < 1$, the matrix elements of the current (a) can be expressed in terms of the dipole moment of the transition

$$ej_{if}^{(x)} = i\tilde{\omega}_{if} d_{if}, \quad ej_{if}^{(z)} = ik_x d_{if}.$$

If we disregard also the frequency dispersion of the electromagnetic field [$\chi'(\omega) = 0$], then we get the result (24) and (25) of the paper of Beloshitskiĭ and Kumakhov.[7]

In the quasiclassical limit, when the quantum numbers of the states of the discrete spectrum are large ($i, f \gg 1$), and their difference is relatively small ($|i-f| \ll i$), the quantum quantities in (7) and (8) can be replaced by the corresponding classical quantities, and the summation over the final states can be replaced by summation over the harmonics, in accordance with the scheme:

$$\tilde{\omega}_{if} \to n\omega_0, \quad \partial\varepsilon_f(E_i^{\parallel})/\partial E_i^{\parallel} \to -\varepsilon_{kin}/E,$$

$$j_{if}^{(x)} \to j_n^{(x)}(k_x) = \frac{1}{T}\int_0^T \exp[-in\omega_0 t + ik_x x(t) + ik_z \Delta z(t)]\, v_x(t)\, dt, \qquad (9)$$

$$j_{if}^{(z)} \to j_n^{(z)}(k_x) = \frac{1}{T}\int_0^T \exp[-in\omega_0 t + ik_x x(t) + ik_z \Delta z(t)]\, dt.$$

Here n is the number of the harmonic of the radiation, ω_0 is the frequency of the classical periodic transverse motion, $T = 2\pi/\omega_0$, $\overline{\varepsilon_{kin}}$ is the kinetic energy of the transverse motion averaged over the period,

$$\Delta z(t) = \int_0^t \frac{\overline{v_z^2} - v_z^2(t)}{2}\, dt, \quad \overline{v_z^2} = \frac{1}{T}\int_0^T v_z^2(t)\, dt,$$

and $v_x(t) = dx(t)/dt$. The equation for the transverse component of the particle trajectory $x(t)$ is

$$t = \left(\frac{E_\parallel}{2}\right)^{1/2} \int_0^x [\varepsilon - U(x)]^{-1/2} dx.$$

In this case expression (7) agrees with the analogous result of the theory of undulator radiation, obtained by Alferov *et al.* [see expressions (2) and (14) in Ref. 12].

Estimates show that at relatively low particle energies ($\lesssim$100 MeV) there are only several transverse-energy levels, so that the classical description of the motion of the channeled particles becomes utterly inapplicable. At sufficiently high energies, when $U_0 E \gtrsim 1$, the character of the motion of the channeled particles becomes classical, but the condition of applicability of the dipole approximation for the calculation of the radiation no longer holds. This is one of the characteristic features of radiation produced by channeling when compared with radiation in undulators, where the particle motion is practically always classical.

We compare now our result (7) with the second term of expression (15) of the paper of Baryshevskii and Dubovskaia,[15] which, in accordance with their own statement, describes the radiation due to transitions between discrete states of transverse motion. The presence of the factor $|C_\gamma|^2$ and the summation over γ in the indicated expression of Ref. 15 correspond to averaging over the initial states of the transverse motion (see below). The factor

$$\left|\frac{1-\exp(-iq_{z\gamma\eta}L)}{q_{z\gamma\eta}}\right|^2$$

for a thick crystal ($L \to \infty$) and for weak absorption ($\mathrm{Im}\, n(\omega) \to 0$) goes over into a product of the δ function from expression (7) of the present paper and the length of the crystal L. In all other respects the two compared results are substantially different.

According to Ref. 15, the radiation is always linearly polarized in the plane containing the normal n_x to the channeling planes and the momentum of the photon. According to our results (7), such a polarization can be observed in the general (non-dipole) case only in directions parallel to the channeling planes.

The dependence of the emission probability (7) on the angles θ and φ likewise does not agree with the corresponding relation obtained in Ref. 15. In addition, the matrix elements of the transverse component of the current, which are designated $I_{\gamma\eta}$ in Ref. 15, do not contain the transverse part of the wave function of the emitted photon $\exp(ik_x x)$ in the integrand, in contrast to $j_{if}^{(x)}$ of our Eq. (8).

Thus, the results of Sec. 3 of Ref. 15, as well as the analogous results of the paper by Baryshevskiĭ and others[16] must be regarded as in error. In our opinion, the general formula obtained by Bethe and Maximon[18] for the calculation of the bremsstrahlung was used by the authors of Ref. 15 and 16 incorrectly. The point is that the wave functions of the channeled particles do not have the asymptotic form of a superposition of plane and spherical waves, as was assumed by Baryshevskiĭ *et al.*,[15,16] but are localized in the region of the channel. The transverse component of the particle momentum in the crystal does not have a definite value, inasmuch as the transverse motion in the channeling is finite. Accordingly, the general quantum formulas for the calculation of the radiation in channeling [see, e.g., Eqs. (4)–(7) of Ref. 10] differ substantially from the analogous equations for the calculation of the bremsstrahlung.

The problem of the radiation accompanied by the capture of a relativistic electron in the planar channeling regime was considered by Koptelov and Kalashnikov,[19] and also by Ol'chak and Kalashnikov.[20] Unfortunately, their calculation must be admitted to be in error. A comparison of the general expressions (28) in Ref. 19 and of 2.31 in Ref. 20 with our result (7) shows that the cited authors do not take into account the longitudinal component of the current $j_{if}^{(z)}$, although the order of magnitude of its contribution is the same as that of the transverse component $j_{if}^{(x)}$. Therefore the angular dependence of the radiation obtained in Refs. 19 and 20 is incorrect.[4)] The subsequent calculation of the emission

spectrum in these papers contains also several gross errors. First, it was assumed that the initial wave function of the electron entering into the crystal parallel to the crystal planes maintains during the instant of emission its initial plane-wave form. On the other hand, it is well known (see, e.g., Ref. 21), that under the corresponding conditions a very strong realignment of the wave function of the particle in the crystal takes place. The field of the planes can be regarded as a perturbation only at entry angles θ_0 much larger than the Lindhard critical angle θ_L. At these angles, however, radiation of the type considered becomes insignificant. Second, the wave function of the final state was chosen in Refs. 19 and 20 to be a superposition of transverse-motion wave functions. In fact, the emission process cannot lead to establishment of any coherence of the wave functions whatever, and the final state should be one of the "pure" quantum states. These errors made it impossible to estimate correctly the role of this radiation. For example, in the opinions expressed in Refs. 19 (page 14) and 20 (page 25) this radiation mechanism is claimed not to operate on positively charged particles, although there are no grounds whatever for this claim. It is also stated (Ref. 19, p. 6; Ref. 20, p. 28) that this radiation mechanism is the principal one for electrons entering the crystal at angles much smaller than the Lindhard angle ($\theta_0 \ll \theta_L$). That this statement is wrong is clear even from the fact that under the condition $\theta_0 \ll \theta_L$ the population of the continuous-spectrum states is negligibly small compared with the population of the bound states (see Sec. 5). However, even at entry angles $\theta_0 \approx \theta_L$, when the above-barrier states are predominantly populated, the principal mechanism (in the sense of the value of the spectral density) of the radiation is that occurring in transitions between above-barrier states. The corresponding estimates were made by Akhiezer and co-workers.[22]

An earlier paper by Kalashnikov, Koptelov, and Ryazanov,[23] in which the spectrum of the radiation produced by capture of nonrelativistic electrons in the channeling regime, also seems wrong to us. First, the very formulation of the problem in the case of nonrelativistic electrons becomes meaningless, since no channeling effects exist for such electrons. If the electron energy becomes less than several MeV, then the channeling effects give way rapidly to diffraction effects (see, e.g., Ref. 24). Second, even if no account is taken of this important circumstance, and heavier nonrelativistic particles are considered in place of electrons, the emission spectrum should, according to (6) and (7), differ substantially from the corresponding spectrum calculated in Ref. 23. The Doppler effect for nonrelativistic particles is insignificant ($v_\parallel \ll 1$), and the frequency of the emitted photon coincides with the transverse-motion energy difference $\omega \approx \tilde{\omega}_{if}$. For the potential model used in Ref. 23 and at a fixed angle of electron entry into the crystal, the emission spectrum should therefore be monochromatic, contrary to the results of Kalashnikov *et al.*[23] The energy of the emitted photon is in this case comparable with the depth of the potential well, i.e., it ranges from several electron volts to several dozen electron volts (depending on the material and on the channel), and is not equal to several keV, as stated in Ref. 23.

Thus, the observed orientational maxima in the x-ray part of the radiation produced by nonrelativistic electrons was explained[23] on the basis of erroneous theoretical results. At the same time, the role played by channeling as a whole in the radiation process was incorrectly represented in Ref. 23. For example, it was stated that the action of the averaged potential of the planes cannot lead by itself to emission in the case of transitions between above-barrier states. This contradicts completely our conclusion (see Sec. 2) as well as the results of Akhiezer *et al.*[22]

4. POPULATION OF TRANSVERSE-MOTION STATES

The expressions obtained for the emission probability should be averaged over the initial states $|i\rangle$ of the transverse motion. The distribution over the levels is formed when the particles enter the crystal. If all the beam particles enter at a definite angle $\theta_0 = p_x/E$ to the plane, then the relative probability P_i of the capture on a level is determined by the corresponding coefficient of the expansion of the plane wave in the transverse-motion wave functions:

$$P_i(p_x) = |\bar{\psi}_i(p_x)|^2, \quad \bar{\psi}_i(p_x) = \int_{-\infty}^{\infty} \exp(ip_x x)\psi_i(x; E_i^{\parallel})\,dx. \tag{10}$$

The correlation of the phases of the various coefficients of $\bar{\psi}_m$ and $\bar{\psi}_n$, which are the wave functions in the momentum representation, is substantial over distances $l_{nm} \sim 2\pi/\omega_{nm}$ from the crystal boundary. Thus, the state $|i\rangle$ of the transverse motion with discrete energy is completely formed over distances $l \gtrsim 2\pi/\omega_{i,i+1}$.[21]

As the particles move into the interior of the crystal, the initial distribution over the levels is generally speaking changed. This change is both the result of radiative processes and the result of particle scattering by quasifree electrons of the crystal and by the vibrating lattice sites. Formally, to determine the distribution over the levels, it is necessary to solve the kinetic equation for the density matrix of the transverse-motion states. This, however, is beyond the scope of the present paper.

In the classical approach, the initial amplitude of the transverse oscillations of the particles in the channel is given by

$$E\theta_0^2/2 + U(x_0) = \varepsilon.$$

The averaging of the emission probabilities over the initial states of the transverse motion is replaced in this case by averaging over the initial transverse coordinate x_0. The quantum condition that the correlation of the phase shifts of the wave functions become weaker in the classical limit corresponds to the requirement that the particle executes a sufficient number of oscillations in the crystal to be able to regard the channeling process as assuming a steady state. When the particle leaves the crystal the localized wave functions of the transverse motion $\psi_n(x)$ are restructured back into a superposition of plane waves with different directions of the transverse momentum. This restructuring takes

place over the same lengths l as at the entry.

It follows from the foregoing that the condition for the applicability of the developed theory is, on the one hand, a sufficiently large crystal thickness $L \gg l$, and on the other, a sufficiently weak absorption of the photons in the crystal $l_c(\omega) \gg l$ (l_c is the photon mean free path), so that the main contribution to the radiation is made by the internal region of the crystal. Estimates show that the last condition is always satisfied for photon energies $\omega \sim \omega_{max}^{(1)}$, corresponding to the maximum spectral distribution of the radiation. The influence of the crystal boundary on the spectrum of the considered radiation can therefore be neglected in the essential frequency region.

5. PLANAR CHANNELING OF POSITRONS

In this case the potential $U(x)$ takes the form of a parabola practically everywhere within the limits of the channel

$$U(x) = 4U_0 x^2/d^2,$$

where the coordinate x is reckoned from the median plane. For the quantities that enter in the expression for the spectral-angular distribution (7) we get

$$\bar{\omega}_{if} = \omega_0(i-f), \quad \omega_0 = \frac{1}{d}\left(\frac{8U_0}{E}\right)^{1/2}, \quad \frac{\partial \varepsilon_f}{\partial E_i^{\parallel}} = -\frac{1}{2}\frac{\varepsilon_f}{E_i^{\parallel}},$$
$$j_{if}^{(x)} = \sum_{f'} C_{ff'} J_{if'}^{(1)}(k_x), \quad j_{if}^{(z)} = \sum_{f'} \frac{\bar{\omega}_{if'}}{k_z} C_{ff'} J_{if'}^{(1)}(k_x), \tag{11}$$

where $C_{ff'}$ and $J_{if'}^{(1)}$ are defined by ($J_{if'} = J_{f'i}$):

$$J_{if'}^{(1)} = \left(\frac{f'!}{i!}\right)^{1/2} (-2\xi)^{(i-f')/2} e^{-\xi} L_i^{i-f'}(2\xi) \,(\operatorname{sign} k_x)^{i-f'}, \quad i > f',$$
$$C_{ff'} = \left[\frac{f_<!}{f_>!}(1-\rho)^{1/2}\right]^{1/2} P_{(f+f')/2}^{|f-f'|/2}((1-\rho)^{1/2}), \quad \rho \approx \left(\frac{\omega}{4E}\right)^2 \ll 1.$$

The remaining symbols are defined in Ref. 10 [Eqs. (21) and (22)].

In the classical limit we have

$$j_n^{(x)} = \sum_{m=-\infty}^{\infty} J_m(\beta) J_{n+2m}(\alpha), \quad j_n^{(z)} = \sum_{m=-\infty}^{\infty} \frac{\omega_0}{k_z}(n+2m) J_m(\beta) J_{n+2m}(\alpha),$$

where J_ν is a Bessel function, $\alpha = (k_x d/2)(\varepsilon/U_0)^{1/2}$, $m = |f-f'|/2$, and $\beta = \omega\varepsilon/4\omega_0 E$. The maximum value of the spectral density of the energy radiated at the first harmonic is reached at the frequency

$$\omega_{max}^{(1)}(\varepsilon) = 2E^2\omega_0/(1+\varepsilon E),$$

corresponding to $\theta = 0$. For positrons entering the crystal at an angle $\theta_0 \ll \theta_L$, the maximum contribution to the radiation at the frequency $\omega_{max}^{(1)}$ is made by positrons with transverse energy $\varepsilon \approx U_0$, which have the maximum oscillation amplitude. For example, for the (110) planes of diamond $d = 1.26$ Å, $U_0 = 23$ eV. For this case the frequencies $\omega_{max}^{(1)}(U_0)$ corresponding to the maximum of the radiation at the first harmonic are listed in the table for different positron energies. The table gives also the results of the measurements of Miroshnichenko *et al.*[25] As shown by a detailed calculation based on the developed theory, there is also good agreement between the form of the emission spectrum and the experimental results. A more detailed comparison of the results of the theory and experiment[25] will be published by us in a separate paper.

TABLE I.

E, GeV	$\hbar\omega_{max}^{(1)exp}$, MeV	$\hbar\omega_{max}^{(1)theor}$, MeV
4	23	31
6	42	51
10	90	92
14	120	126

The emission by above-barrier positrons ($\varepsilon > U_0$) is determined in the classical limit by the equations

$$\frac{d^2W}{d\omega d\Omega} = \frac{e^2\omega^2}{2\pi}\sum_{n=1}^{\infty} |e j_n|^2 \delta\left(\frac{\omega(\theta^2 + E^{-2} + \overline{v_x^2})}{2} - \omega_0\left(n + \frac{k_x d}{2\pi}\right)\right),$$
$$j_n^{(x)} = \frac{2}{dk}\left(\frac{2U_0}{E}\right)^{1/2} \sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty} J(g) J(h) m \frac{\sin D}{D},$$
$$j_n^{(z)} = \sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty} J(g) J(h) \frac{\sin D}{D}, \quad \omega_0 = \frac{2\pi}{d}\left(\frac{2U_0}{E}\right)^{1/2} \frac{1}{\arcsin(U_0/\varepsilon)^{1/2}},$$
$$g = k_x d(\varepsilon/4U_0)^{1/2}, \; h = k_z d\varepsilon/8(2EU_0)^{1/2},$$
$$D = \pi n + k_x d/2 + k_z d/4[2E/(\varepsilon - U_0)]^{1/2} + (m+2m')\arcsin(U_0/\varepsilon)^{1/2}, \tag{11'}$$
$$\overline{v_x^2} = \frac{\varepsilon}{E} + \left[\frac{U_0(\varepsilon - U_0)}{E^2}\right]^{1/2} \frac{1}{\arcsin(U_0/\varepsilon)^{1/2}}.$$

6. PLANAR CHANNELING OF ELECTRONS

The parabolic-well approximation can be used also to describe planar channeling of electrons with transverse energies close to their minimum, when the distance from the electron to the channeling plane does not exceed the radius of the crystal atoms. However, it is precisely in this case that a relatively strong scattering of the electrons by the individual atoms of the plane and by the electrons of the crystals should take place, and with it also relatively rapid dechanneling.

In a larger range of distances x from the plane, as noted in Ref. 10, a potential closer to the real one is of the form

$$U(x) = -U_0/\operatorname{ch}^2(x/b). \tag{12}$$

The well depth U_0 and its width b should be chosen such as to make (12) as close as possible to the exact form of the potential, for example to the potential obtained by Appleton *et al.*[26] in the Moliere approximation for the potential of an atom of the plane, with account taken of the isotropic thermal vibrations of the lattice.

The solid curve of Fig. 1 shows the potential of the (100) plane of carbon at normal temperature, with account taken of the influence of the neighboring planes, as calculated in accordance with the results of Ref. 26. The dotted curve is the model relation (12) with $U_0 = 12.8$ eV and $b = 0.174$ Å. The figure shows also the transverse-energy levels of an electron with total energy 1 GeV.

The solutions of the Schrödinger equation (5) with the potential (15) are, as is well known,[27]

$$\psi_n(x, E) = \frac{\Gamma(2s-2n+1)}{\Gamma(s-n+1)}\left[\frac{(s-n)n!}{b\Gamma(2s-n+1)}\right]^{1/2} (1-\xi^2)^{(s-n)/2} C_n^{s-n+1/2}(\xi). \tag{13}$$

The functions ψ_n in (13) are normalized by the condition $\langle\psi_n|\psi_m\rangle = \delta_{mn}$. Corresponding to these functions are the

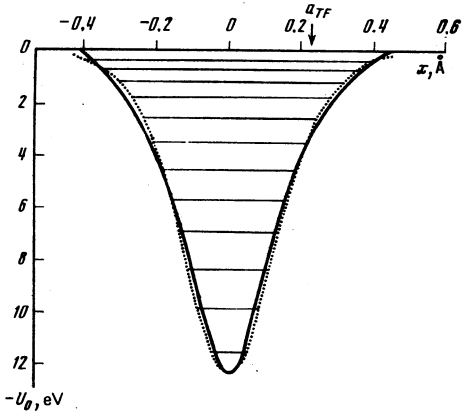

FIG. 1. Potential of the (100) plane of carbon as a function of the distance to the plane. The solid curve is the result of the calculation of Appleton *et al.*[19] The dotted curve is the model function $U(x)=U_0\cosh^{-2}\alpha x$ with parameters $U_0=12.8$ eV and $\alpha^{-1}=0.174$ Å. The horizontal lines show the discrete level of the transverse energy of an electron of energy $E=1$ GeV channeled by the (100) plane in carbon.

transverse-energy eigenvalues

$$\varepsilon_n(E)=-(2Eb^2)^{-1}(s-n)^2. \tag{14}$$

In (13) and (14) we have $s=s(E)$, where

$$s(E)=-{}^1/_2+({}^1/_4+2b^2U_0E)^{1/2},\quad \xi=\mathrm{th}\,(x/b),\quad n=0,1,2,\ldots[s],$$

$C_n^\lambda(\xi)$ is a Gengenbauer polynomial in ξ, and $[s]$ is the integer part of s. Thus, for the quantities that determine the emission frequency [see (7)] we obtain the following expressions:

$$\bar\omega_{if}=\frac{2s-i-f}{2Eb^2}(i-f),\quad \frac{\partial\varepsilon_f(E_i^{\parallel})}{\partial E_i^{\parallel}}=-\frac{\varepsilon_f(E_i^{\parallel})}{E_i^{\parallel}}-\frac{2(s-f)U_0}{(2s+1)E_i^{\parallel}}. \tag{15}$$

To calculate the matrix elements of the current (8), we represent each of the polynomials in the corresponding wave functions as finite sums in the powers of the quantity $(1-\xi^2)$.[28] It then becomes possible to integrate in (8) with respect to the variable ξ. As a result we get

$$j_{if}^{(x)}(k_x,s,s')=\left[\frac{(s-i)(s'-f)\Gamma(s+b_i)\Gamma(s'+b_f)}{i!\,f!}\right]^{1/2}\sum_{l=0}^{[i/2]}\sum_{m=0}^{[f/2]}(-1)^{l+m}2^{2(l+m)-1}$$
$$\times C^l_{[i/2]}C^m_{[f/2]}\frac{(a_i)_l(a_f)_m}{\Gamma(b_i+l)\Gamma(b_f+m)}\frac{|\Gamma(r-q+l+m)|^2}{\Gamma(2r+2l+2m)}g_{if}^{lm},$$
$$j_{if}^{(z)}(k_x,s,s')=\frac{\alpha}{E}\frac{(-1)^{1/2}}{2s+1}[(s+1)[i(2s-i)]^{1/2}j_{i-1,f}^{(z)}(k_x,s-1,s')$$
$$-s[(i+1)(2s+1-i)]^{1/2}j_{i+1,f}^{(z)}(k_x,s+1,s')]. \tag{16}$$

We have introduced here the notation

$$b_i=s-i+1,\ a_i=s-[i/2]+{}^1/_2,\ r={}^1/_2[s+s'-(i+f)],$$
$$b_f=s'-f+1,\ a_f=s'-[f/2]+{}^1/_2,\ q=(-1)^{1/2}k_xb/2,$$

$$g_{if}^{lm}=\begin{cases}1 & i,f-\text{even},\\ \dfrac{r+l+m+2q^2}{2(r+l+m)(r+l+m+{}^1/_2)} & i,f-\text{odd},\\ \dfrac{q}{r+l+m} & i,f-\text{of different parity},\end{cases}$$
$$s'\approx s-\omega b^2U_0/({}^1/_2+s),$$

C_k^l are the binomial coefficients, and $(a)_k=a(a+1)\ldots(a+k-1)$ is the Pochhammer symbol.

The dipole approximation corresponds to relatively small parameters $q\ll1$. From the orthonormality condition of the wave function follows the equality[5)]

$$j_{if}^{(x)}|_{q=0}^{s=s'}=\delta_{if}.$$

Calculations of the capture probability amplitudes $\bar\psi_n(p_x)$ (10) lead to the following result:

$$\bar\psi_n(p_x)=\frac{1}{2}\left|\frac{\Gamma((s-n)/2-ip_xb/2)}{\Gamma(s-n)}\right|^2\left|\frac{\Gamma(2s-n+1)b}{n!(s-n)}\right|^{1/2}$$
$$\times\sum_{k=0}^{n}\frac{(-n)_k(2s-n+1)_k((s-n)/2-ip_xb/2)_k}{(s-n+1)_k(s-n)_k\,k!}. \tag{17}$$

The obtained expressions (16) and (17) are particularly suitable for practical calculation of the electron emission spectrum when the number of transverse-energy levels is not too large: $i\lesssim10$. This corresponds to electron energies $E\lesssim1$ GeV.

Numerical calculation of (16) shows that the probability of the radiative transitions in the potential (12) decreases more slowly than in a parabolic potential when the difference between the quantum numbers $(i-f)$ increases. Therefore in planar channeling the electron radiation should have a broader spectrum than the positron radiation. Contributing to this is also (when averaging over the initial states) the non-equidistant character of the transverse-energy spectrum (14). Comparison of (15) and (11) shows that the characteristic frequencies of the emission by the electrons is several times higher than that by positrons, owing to the smaller width of the potential well $(b<d/2)$.

The emission spectrum of the above-barrier electrons is determined by Eqs. (11′), in which the substitution $U_0\to-U_0$ must be made.

7. EMISSION IN AXIAL CHANNELING

In axial channeling of an electron it can be assumed in first approximation that the electron moves in the field of one atomic chain. The states of the electron are characterized in this case by a definite projection of the momentum on the crystal axis. In addition, owing to the axial symmetry of the potential, the projection of the angular momentum on this axis is conserved in the course of the motion. Positrons move in the case of axial channeling in the field of several crystal axes that are close to one another. In this case, generally speaking, the potential has no axial symmetry.

Thus, in the general case the wave function of the particle has in the case of axial channeling the form [cf. (4)]

$$\Psi_f(\mathbf{r})=(2E_i^{\parallel})^{-1/2}\exp(ip_i^{(z)}z)\psi_i(\boldsymbol{\rho}), \tag{18}$$

where ρ is a radius vector perpendicular to the crystal axis, the latter being chosen to be the $0z$ axis. The wave function of the transverse motion satisfies the equation

$$\left[-\frac{1}{2E_i^{\parallel}}\Delta_\rho\psi_i(\boldsymbol{\rho};E_i^{\parallel})+U(\boldsymbol{\rho})\right]=\varepsilon_i(E_i^{\parallel})\psi_i(\boldsymbol{\rho};E_i^{\parallel}), \tag{19}$$

where $\Delta\rho$ is the Laplacian in two-dimensional space.

The spectral-angular density of the radiation probability can be represented in the form

$$\frac{d^2w}{d\omega\,d\Omega}=\frac{e^2\omega}{2\pi}\sum_f[l_\omega|[\mathbf{j}_{if}^{(\rho)}\mathbf{n}_\rho]|^2+l_\pi|j_{if}^{(z)}\theta-\mathbf{j}_{if}^{(\rho)}\mathbf{n}_\rho|^2]$$
$$\times\delta\left[\omega\left(\frac{\theta^2+E^{-2}-\chi'(\omega)}{2}-\frac{\partial\varepsilon_f}{\partial E_i^{\parallel}}\right)-\bar\omega_{if}\right], \tag{20}$$

where

$$j_{if}=\int \hat{v}\psi_i(\rho;E_i^{\parallel})\psi_f^*(\rho;E_f^{\parallel})\exp(ik_p\rho)d^2\rho,\quad \hat{v}=\left(\frac{i\nabla_\rho}{E},1\right), \tag{21}$$

$\mathbf{n}_p$ is a unit vector in the direction of the projection of the photon momentum on the $x0y$ plane, $\mathbf{k}_p=k\theta\mathbf{n}_p$, and the frequencies $\tilde{\omega}_{if}$ are determined by the first relation in (8).

It is known that Eq. (19) has an analytic solution only for model dependences of the potential, such as $U(\rho)\propto\rho^2$ or $U(\rho)\propto\rho^{-1}$, which are precisely close to the real potential of the crystal axis (see below). Further calculations, however, are greatly complicated because of the degeneracy of the states in such fields. We confine ourselves therefore to the classical approximation, the use of which in the case of axial channeling is justified at electron energies higher than ~100 MeV. The transition to the classical description, when the motion becomes periodic, follows a scheme analogous to (9):

$$j_{if}\to j_n(k_\rho)=\frac{1}{T}\int_0^T\exp[-in\omega_0+ik_\rho\rho(t)+ik_z\Delta z(t)]v(t)dt,$$

$$v(t)=\{\dot{\rho}(t),1\},\quad \Delta z(t)=\int_0^t\frac{\overline{v_\rho^2}-v_\rho^2(t)}{2}dt, \tag{22}$$

$$\tilde{\omega}_{if}\to n\omega_0,\quad \partial e_f/\partial E_i^{\parallel}\to-\bar{e}_{kin}/E.$$

The parameters of the transverse component of the electron trajectory in the potential U_ρ are determined by the values of the transverse energy and angular momentum M relative to the crystal axis[29]:

$$\varepsilon=U(\rho_0)+E\theta_0^2/2,\quad M=E\theta_0\rho_0\cos\Phi_0, \tag{23}$$

and also by the condition that the tangential component of the transverse momentum be continuous on the crystal boundary:

$$p_\perp^{(v)}(\rho_0,\Phi_0)=E\theta_0\cos\Phi_0. \tag{24}$$

In these equations, θ_0 is the angle of entry of the electron relative to the axis, ρ_0 is the distance from the point of entry into the crystal to the axis, $\cos\Phi_0=P/\rho_0$, and P is the impact parameter. The spectral-angular distribution of the radiation energy on moving along a definite trajectory should then be averaged over all possible trajectories, which correspond to all possible ρ_0 and Φ_0.

In axial channeling of electrons, the values of the potential near the equilibrium position of the atoms of the axis is well approximated by a parabolic well of the type

$$U(\rho)=\beta\rho^2, \tag{25}$$

where the coefficient β differs noticably from the crystal temperature.[26]

In a potential of the form (25), the electron moves along an ellipse:

$$\tilde{x}(t)=a\sin(\omega_0t+\delta),\ \tilde{y}(t)=b\cos(\omega_0t+\delta) \tag{26}$$

with frequency $\omega_0=(2\beta/E)^{1/2}$.

The Cartesian system of coordinates $\tilde{x}O\tilde{y}$ is chosen such that its axes coincide with the axes of the ellipse. The crystal axes pass through the origin. The semiaxes a and b of the ellipse, and the angle of inclination of the ellipse axes to the Ox axis of the fixed coordinate frame xOy (the Oy axis is parallel to the direction of the initial transverse momentum) are determined by conditions (23) and (24).

If we are interested only in the spectral distribution of the radiation, then the orientation of the ellipse relative to the axis of the fixed coordinate frame xOy is immaterial and can be arbitrary and convenient for integration with respect to the azimuthal angle. It must also be borne in mind that when account is taken of the anharmonic corrections to the potential (25), precession of the ellipse sets in. After averaging the inclination angle of the ellipse axis over the precession period, the spectral-angular distribution of the radiation also ceases to depend on the azimuthal angle. With the aid of (23) we obtain the values for the semiaxes of the ellipse:

$$\left.\begin{aligned}&a=\rho_0(A+B)/2,\quad b=\rho_0(A-B)/2,\\ &\text{where } A=(1+\eta^2+2\eta\cos\Phi_0)^{1/2},\ B=(1+\eta^2-2\eta\cos\Phi_0)^{1/2},\ \eta=\theta_0/\rho_0\omega_0.\end{aligned}\right\} \tag{27}$$

The condition for the applicability of expression (25) for the potential are the inequalities $a\lesssim a_{TF}$, $b\lesssim a_{TF}$, where a_{TF} is the radius of the atom. These inequalities determine the limits of the region of electron capture on orbits of the type (26) as functions of the entry angle θ_0 relative to the crystal axis.

In the case considered it is easy to obtain

$$\varepsilon_{kin}=\varepsilon/2,\quad k_z\Delta z(t)=c\sin 2\omega_0t, \tag{28}$$

where $c=\omega\omega_0(b^2-a^2)/8$. We introduce the notation

$$A_n^{(\sigma)}=|[\mathbf{j}_n^{(\rho)}\times\mathbf{n}]|,\quad A_n^{(\pi)}=|j_n^{(z)}\theta-\mathbf{j}_n^{(\rho)}\mathbf{n}_\rho|. \tag{29}$$

Calculation of these quantities leads to the result

$$A_n^{(\sigma)}=\frac{\omega_0 r}{2}\left|\sum_{m=-\infty}^{\infty}J_m(c)e^{-2im\chi}[e^{i(\psi-\chi)}J_{n-1-2m}(\omega r\theta)-e^{-i(\psi-\chi)}J_{n+1-2m}(\omega r\theta)]\right|,$$

$$A_n^{(\pi)}=\left|\sum_{m=-\infty}^{\infty}J_m(c)e^{-2im\chi}J_{n-2m}(\omega r\theta)\left(\theta-\frac{(n-2m)\omega_0}{\omega\theta}\right)\right| \tag{30}$$

$$r=[(a\cos\tilde{\varphi})^2+(b\sin\tilde{\varphi})^2]^{1/2},\quad \chi=\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg}\tilde{\varphi}\right),\quad \psi=\operatorname{arctg}\left(\frac{a}{b}\operatorname{tg}\tilde{\varphi}\right),$$

$\tilde{\varphi}$ is the angle between the major semiaxis of the ellipse and the projection of the photon momentum on the transverse plane.

In particular, in the case of transverse motion along a circle $a=b$, $\chi=\psi=\tilde{\varphi}$, $c=0$ and the contribution to the sum over m is made only by the term with $m=0$. In this case the formulas become much simpler

$$A_n^{(\sigma)}=\omega_0 a|J_n'(\omega a\theta)|,\quad A_n^{(\pi)}=\left|\left(\theta-\frac{n\omega_0}{\omega\theta}\right)J_n(\omega a\theta)\right| \tag{31}$$

and we arrive at the result known from the bremsstrahlung theory for the distribution of the radiation in the case of motion along a helical line (see, e.g., Ref. 30, Sec. 10).

It follows from (27), however, that only a relatively small number of electrons that land in the vicinity of the points with coordinates $\rho_0=\theta_0/\omega_0$, $\Phi_0=0$ or π, is captured on orbits close to circular ($a\approx b$). The overwhelming majority of the captured electrons move along ellipses with eccentricities noticably different from zero. In this case, in contrast to motion on a circle, longitudinal oscillations of the particles $\Delta z(t)$ are excited [see (28)]. At sufficiently high energies ($\varepsilon E\gtrsim 1$), as follows from an analysis of (30), these oscillations

lead to a noticable increase of radiation at higher harmonics and, contrarywise, to some suppression of the radiation at the first harmonic.

Let now the electron move all the time at distances from the crystal axis that are larger than the radius of the thermal oscillations but smaller than half the distance to the neighboring axis $d/2$. In this region, as shown in Refs. 31 and 32, a potential close enough to the real one is of the form

$$U(\rho)=-\alpha/\rho, \tag{32}$$

where α is a constant practically independent of the crystal temperature. In this potential, the transverse motion with nonzero angular momentum M relative to the axis is along a Kepler ellipse:

$$\tilde{x}(t)=a(\cos\xi-e), \quad t=\omega_0^{-1}(\xi-e\sin\xi),$$
$$\tilde{y}(t)=b\sin\xi, \quad b=a(1-e^2)^{1/2}, \quad \omega_0=(2|\varepsilon|)^{3/2}/\alpha E^{1/2}. \tag{33}$$

The frequency ω_0 of revolution on the ellipse, in contrast to the considered case of a parabolic potential (25), (26), depends on the transverse energy ε. In addition, the crystal axis passes now through a focus of the ellipse and not through the center as in (26). The major semiaxis of the ellipse a and the eccentricity e are defined, as is well known, by the equations

$$a=\alpha/2|\varepsilon|, \quad e=(1-2|\varepsilon|M^2/E\alpha^2)^{1/2}.$$

From the conditions (23) we get the values of the major and minor semiaxes and of the eccentricity of the ellipse as functions of the initial coordinates of the electron on the input surface of the crystal:

$$a=\frac{\rho_0}{2(1-\zeta)}, \quad b=\rho_0|\cos\Phi_0|[\zeta(1-\zeta)]^{1/2},$$
$$e=[1-4\zeta(1-\zeta)\cos^2\Phi_0]^{1/2}, \quad \varepsilon=\rho_0\theta_0^2E/2\alpha. \tag{34}$$

The perihelion of the orbit $\rho_{min}=a(1-e)$ must be larger than a quantity on the order of the Thomas–Fermi radius a_{TF}, and the aphelion $\rho_{max}=a(1+e)$ must be less than half the distance $d/2$ to the neighboring axis, in order for the approximation (34) for the potential of the channel to be applicable. These conditions together with the equalities (34) determine the limits of the electron capture region on an orbit of the type (33) as a function of the entry angle θ_0. The capture region is shown in Fig. 2 for values of the parameter $R\equiv E\theta_0^2d/4\alpha=1$. The curves drawn inside this region correspond to the determined eccentricity. The circle shown on Fig. 2 has a radius $d/2$.

For the case of transverse motion in a potential (32) we obtain $\bar{\varepsilon}_{kin}=|\varepsilon|$, $k_z\Delta z=c\sin\xi$, where $c=\omega\omega_0(a^2-b^2)/2$. The calculations of the quantities $A^{(\sigma)}$ and $A_n^{(\pi)}$ [see (29)] yield

$$A_n^{(\sigma)}=\frac{\omega_0 a}{2}(1-e^2\cos^2\tilde{\varphi})^{1/2}|e^{-i\chi}J_{n-1}(\mu)-e^{i\chi}J_{n+1}(\mu)|,$$
$$A_n^{(\pi)}=\left|\theta J_n(\mu)-\frac{\omega_0 a}{2}(1-e^2\sin^2\tilde{\varphi})^{1/2}[e^{i\tilde{\chi}}J_{n-1}(\mu)+e^{-i\tilde{\chi}}J_{n+1}(\mu)]\right|. \tag{35}$$

We have introduced here the notation

$$\chi=\operatorname{arctg}\left(\frac{a}{b}\operatorname{tg}\tilde{\varphi}\right)-\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg}\tilde{\varphi}+e\frac{\omega\omega_0a^2+2n}{2\omega\theta a\cos\tilde{\varphi}}\right),$$
$$\tilde{\chi}=\operatorname{arctg}\left(\frac{a}{b}\operatorname{ctg}\tilde{\varphi}\right)-\operatorname{arctg}\left[\left(1+e\frac{\omega\omega_0a^2+2n}{2\omega\theta b\sin\tilde{\varphi}}\right)^{-1}\frac{a}{b}\operatorname{ctg}\tilde{\varphi}\right],$$
$$\mu=\left\{(\omega\theta a\cos\tilde{\varphi})^2+\left[\omega\theta b\sin\tilde{\varphi}+e\left(\frac{\omega\omega_0a^2}{2}+n\right)\right]^2\right\}^{1/2},$$

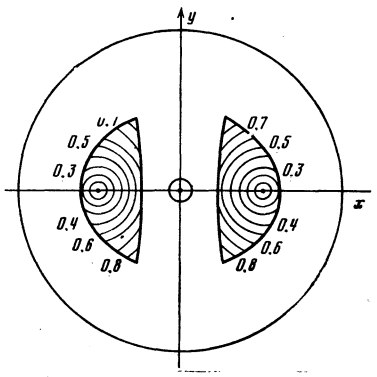

FIG. 2. Region of crystal surface near the ⟨100⟩ axis of silicon. The outer circle has a radius equal to half the distance d to the neighboring chain, while the inner circle has the radius of the thermal vibrations at normal temperature. The shaded region corresponds to capture of an electron on relatively stable elliptic orbits. The angle of entry of the electron relative to the crystal axis is equal to the critical angle $\theta_{cr}=(4\alpha/Ed)^{1/2}$. The curves inside the capture region correspond to different eccentricities of the ellipses. The Oy axis coincides with the direction of the perpendicular component of the electron velocity on entering the crystal.

$\tilde{\varphi}$ is the angle between the major semiaxis of the ellipse and the projection of the photon momentum on the transverse plane.

In the particular case of motion on a circle $e=0, a=b$, $\chi=\tilde{\chi}=0$, $\mu=a\omega\theta$ and we arrive again, as in the analysis of expression (30), at the result (31).

At sufficiently low electron energies ($|\varepsilon|E\ll 1$) we obtain the relations $c\ll n, n\gg\omega\theta(a+b)$, $\mu\approx ne$. In this limiting case, substitution of expressions (35) in the general formula (20) and integration over the angles can yield a relatively simple expression for the spectral distribution of the radiation power[6)]

$$\frac{dQ}{d\omega}=2e^2\omega\frac{|\varepsilon|}{E}\sum_{n=1}^{\infty}(1-2\Omega_n+2\Omega_n^2)\left[J_n'^2(ne)+\frac{1-e^2}{e^2}J_n^2(ne)\right]\Theta(1-\Omega_n), \tag{36}$$

where $\Omega_n=\omega/2n\omega_0E^2$, Θ is the Heaviside function and j_n' is the derivative of the Bessel function with respect to the argument.

In contrast to the parabolic potential (25), when an electron moves in a potential (32) along an ellipse ($e\neq 0$), in the dipole approximation (36) several harmonics [$\sim e^{1/2}(1-e)^{-3/2}$] with relatively high intensity are emitted, rather than one. The reason is that the transverse oscillations of the electron in the potential (32) are subject to substantial anharmonicity.

In the case of high energies, when $|\varepsilon|E\gtrsim 1$ the maximum frequency of the emission at the n-th harmonic is determined by a relation more general than (36), namely

$$\omega_{max}^{(n)}=\frac{2^{3/2}}{\alpha}\frac{(|\varepsilon|E)^{1/2}}{1+2|\varepsilon|E}n.$$

To calculate the spectral distribution of the radiation power it is necessary here to use expression (35) in its general form, i.e., to take into account the effects due to the fact that the radiation is not of the dipole type and

the parametric connection between the transverse and longitudinal motions. We note that under conditions of recent experiments on the emission of electrons on an axial channel,[33,34] the parameter $|\varepsilon|E$ was of the order of unity. Therefore these effects must be taken into account if the experimental results are to be correctly interpreted.[33,34]

A special case arises when the electrons enter the crystal strictly along the direction of the axes ($\theta_0 = 0$), i.e., with zero orbital angular momentum $M = 0$. In this case the trajectories of the electron lie in a plane passing through the entry point ρ_0 and the crystal axis. Axial channeling of this type was considered by Nip *et al.*[35] To calculate the radiation in axial channeling with zero angular momentum one can use the results obtained for planar channeling of electrons. We choose the Ox axis in the plane of the radial oscillations. Then the potential of the axis, as shown by an analysis of the results of Ref. 9, is well approximated by expression (12), where U_0 and b are suitable parameters.

The spectral-angular distribution of the radiation in axial channeling with $M = 0$ is determined by expressions (7), (15), and (16). The distribution of the radiation must then be averaged over the coordinates ρ_0 and φ_0 of the point of entry of the electron into the crystal. Averaging over φ_0 is equivalent to averaging of (7) over the azimuthal angle φ.

The electrons whose energy (reckoned from the bottom of the well) and angular momentum are relatively small move all the time in the region of the maximum density of the electrons and of the nuclei of the crystal ($\rho_{eff} \lesssim a_{TF}$). These electrons, as a result of their collisions with the electrons and nuclei of the crystals, become dechanneled more rapidly than the remaining ones or move over to orbits with $M \neq 0$ that are farther from the axis. The slowest to dechannelize are electrons with $M \neq 0$ and with sufficiently high energies, which move on orbits of the type (33). An intermediate position is occupied by electrons with $M = 0$ and sufficiently low energy, which stay only part of the time in the region $\rho \lesssim a_{TF}$. A more detailed analysis of the stability of different orbits is beyond the scope of the present paper.

CONCLUSION

1. The calculation of the emission spectra of channeling particles is based on sufficiently realistic assumptions concerning the form of the channeling potential. This calculation can be used for a detailed comparison of theory with experiment with arbitrarily high energies of the channeled particles.

2. Calculation based on the developed theory shows that the spectral radiation density of the channeled particles in spontaneous transitions between levels of the transverse motion can exceed by 2–4 orders of magnitude the corresponding value for bremsstrahlung in an amorphous target in the emission frequency region $\gtrsim 10$ MeV. The spectral density of the radiation on the dechanneling length can exceed by 3–5 orders of magnitude the spectral density of transition radiation. Thus, the more rigorous approach confirms on the whole Kumakhov's[2,4,5] initial estimates of the radiation intensity. The corresponding estimates obtained by others[3,15,16] are based on erroneous results, and have therefore predicted either a negligibly small radiation intensity in the x-ray and γ-ray bands[3] or else an intensity comparable with that of transition radiation.[15,16]

[1)] The emission spectrum for planar channeling in the region $\omega \sim E$ was calculated in Ref. 10.

[2)] This plane contains the photon momentum and the vector $\mathbf{v}_{\parallel}$.

[3)] This condition is satisfied at electron energies $\gtrsim 1$ GeV and $\gtrsim 100$ MeV for planar and axial channeling, respectively.

[4)] The same can be stated also concerning formula (3.10) of Ref. 20 for the probability of transition between discrete levels.

[5)] This equation can serve as a check on the accuracy of the computer calculations.

[6)] It must be borne in mind that at sufficiently low energies the condition for applicability of classical mechanics may not hold.

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Interaction of weak pulses with a low-frequency high-intensity wave in a dispersive medium

B. S. Azimov, Yu. N. Karamzin, A. P. Sukhorukov, and A. K. Sukhorukova

G. V. Plekhanov Institute of National Economy
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We consider three-frequency nonstationary interaction of waves in a quadratically nonlinear medium. The low-frequency high-initial-intensity pump wave is not subject to decay instability, so that the nonlinear interaction regime can be described in the given-pump-field approximation. The cases of excitation of a wave at the sum-frequency by a long pump pulse and a short signal pulse, and the converse situation, are discussed. Analytic and numerical methods are used. Effects of nonlinear disperse spreading are described, as is also the breakup of the excited pulse into subpulses.

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1. INTRODUCTION

The study of synchronous (resonant) interactions in dispersive media plays a fundamental role in various branches of physics, such as plasma physics, nonlinear optics, or hydrodynamics. In the last decade, much progress was made in the development of the theory of nonstationary interactions of modulated waves (wave packets) (see, e.g., Refs. 1–3). The most advanced is the description of the interaction of pulses in first-order approximation of dispersion theory, which takes into account the difference between the group velocities. The method of solving the inverse scattering problem yielded in this case a general analytic solution of the system of three equations for the complex amplitudes.[3] However, by virtue of the complicated form of the solution[3] at arbitrary boundary (or initial) conditions, it cannot always be used in the analysis of the concrete situations. Therefore, in addition to the approach developed by Belavin and Zakharov,[3] use is made also of other methods of solving the equations (for example, the given-field method, or asymptotic methods), and the numerical experiments are used more and more extensively.

Until recently, most attention was paid in the theory of nonlinear three-frequency interactions of pulses to the analysis of parametric processes (decay instability)[4,5] and second-harmonic generation by short pulses.[6–8] The nonstationary interaction of another type, wherein a high-intensity low-frequency wave (pump) is mixed with a weak signal of another frequency, resulting in production of a wave at the sum or difference frequency, remain practically uninvestigated. Wave generation at difference and sum frequencies plays an important role in nonlinear optics.[9] To describe the excitation of picosecond and subpicosecond pulses it is necessary to develop a nonstationary theory that takes into account the specifics of this problem. Of principal nontrivial interest in this case is the development of a theory of the nonlinear frequency-conversion regimes. In the present article we have attempted to fill this gap.

We consider nonstationary interaction of three pulses propagating in a general case with different group velocities. Since the powerful pump pulse is not subject to decay instability, the nonlinear frequency-conversion regimes are well described in the given-pump-field approximation. We discuss in the paper the physics of the interaction of a short signal pulse with a long pump

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