ρ^N at the nucleus and its derivative ρ'_N, and with them also the matrix elements V_P and V_{CP}, increase by $(Z_{nuc}/Z_N)^4$ times, respectively. So large an increase is due to the fact that in molecules with inversion doubling of the levels the effect is determined by all the electrons, including the internal ones. From this point of view it is more convenient to use the molecules PH_3, and particularly AsH_3 or $BiCl_3$. It is necessary, however, that the inversion splitting be not too small, so that the compensating field can be monitored in the experiment. Its value for PH_3 is 1.5×10^{-4} cm^{-1} (corresponding to a field $F\sim1-10$ V/cm). The increase of Z_{nuc} yields in this case for the PH_3 and AsH_3 an amplification by 10 and 400 times, respectively.

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Vacuum polarization in a strong field and the renormalization group

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Using the renormalization group, the behavior of the polarization operator for a photon in a static external electromagnetic field is found for the following two asymptotic cases: a) in the presence of a strong crossed field, and b) at low energies in the presence of a strong magnetic field. For soft photons, the polarization operator is expressed in terms of the effective Lagrangian and is calculated in the two-loop approximation for the asymptotic cases of weak and strong magnetic fields. The structure of the perturbation series in the radiation field is determined in both asymptotic regions for the three constituents of the amplitude: the one containing the vacuum part, and the two purely field constituents. It is found that the behavior of the vacuum and field constituents is similar to those of the polarization operator and of the invariant charge in quantum electrodynamics with no external field, respectively, at large squared momenta. One of the field amplitudes is exceptional: for it, the "massless hypothesis" underlying the renormalization-group analysis is not satisfied in region b).

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1. INTRODUCTION

A number of papers have recently appeared, whose results once more confirm the importance of investigating field theories in the presence of static classical fields in order to solve fundamental theoretical problems.

In the well-known paper of Coleman and Weinberg,[1] and in a series of papers that followed it, a dynamical symmetry-breaking mechanism was proposed in which scalar fields acquire nonvanishing vacuum expectation values. One of the methods of investigating this problem involves calculating the radiative corrections to the potential, i.e. calculating the effective potential for different variants of the theory.

Interesting results in another direction were obtained in Refs. 2 and 3 by Ritus, who found the next radiative correction after the Heisenberg-Euler correction to the Maxwell Lagrangian for a static field. Using the effective Lagrangian $L(F)$ as an example, he investigated the structure of the perturbation-theory (PT) series in the radiation field of quantum electrodynamics (QED) for an intense field whose field strengths E and H greatly exceed the critical value $F_{cr}\sim m^2/e$ (m and e

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are the electron mass and charge) as well as the problem of the closure of the theory, and traced the relation between the QED of an intense field and QED at small distances.

The use of renormalization-group (RG) methods in these papers was based on the realization that the RG technique; which is usually employed to analyze the asymptotic behavior of various processes at large and small momenta, can also be used to investigate the asymptotic behavior of processes with respect to any dimensional parameter for which the renormalization law is known, and in particular, with respect to the strength of an external field.[1]

The application of these considerations to QED is facilitated by the fact that the renormalization of charge and of the electromagnetic (EM) field are effected by the same constant, so that the anomalous dimension of the photon field coincides with the Callen-Symanzik (C-S) β function.

As was shown in Ref. 4, the simplest situation occurs for the n-photon vertices $\Gamma^{(n)}$ in QED with an external field, and in particular, for the effective Lagrangian $L=\Gamma^{(0)}$, for which the C-S equations and their solutions have a certain universality, being equally suited for analyzing asymptotic behavior with respect both to momenta and to an external field.

The present paper contains an application of these general RG considerations[4] to the study of the asymptotic properties of the exact polarization operator (PO) for a photon in an external EM field. This quantity describes the radiative corrections to the motion of a photon in an external field, and for $k^2=0$ it determines the amplitude for elastic scattering of a photon in the field with polarization change. Vacuum polarization may turn out to have an important effect on the characteristics of the radiation from certain astrophysical objects that have strong magnetic fields (see Ref. 5 and, for example, Ref. 6). As compared with the three-photon vertex treated in Ref. 4, the PO has a number of features—in particular, features associated with the presence of a vacuum contribution that does not vanish at $F=0$.

The polarization operator and the Green's function for a photon in static external fields were obtained in the lowest one-loop approximation of the Furry picture in Ref. 7 for a general EM field, in Refs. 8 and 9 for a crossed field with $E\perp H$ and $E=H$, and in Refs. 5, 10, and 11 for a pure magnetic field. The solutions of the dispersion equations for an EM wave in a magnetic field were investigated in Refs. 12. The calculation of the radiative corrections to the one-loop approximation, i.e., the calculation of multiloop contributions, is beset with great difficulties, so at present only the mass radiative correction has been obtained in the α approximation for the case of a crossed field at $k^2=0$.[13]

From a general point of view, on the other hand, the RG provides answers to many questions, such as how the radiative corrections depend on the field strength under various asymptotic conditions, what is an efficient parameter for an asymptotic expansion, and the like.

The paper is organized as follows. The RG technique as applied to the PO is introduced in Sec. 2. As compared with L, the PO has a complicated structure, being expressed in terms of three invariant amplitudes P_i. External fields of a special class (which includes crossed fields and pure magnetic fields), namely those in which the P_i depend on the invariant variables

$$\chi=e[(F_{\mu\nu}k_\nu)^2]^{1/2}/m^3,\quad x=e(F_{\mu\nu}^2)^{1/2}/2^{1/2}m^2,\ \nu=k^2/m^2,$$

are considered and asymptotic formulas of two types are examined.

The situation in which $\chi\gg1$ and the invariant x and ν can be neglected is investigated in Sec. 3; this situation obtains when $k\gg m$. Asymptotic expressions $P_{i,as}^{(1)}$ are calculated for the one-loop terms. In this section there is also established the structure of the asymptotic (in $\ln\chi^{2/3}$) loop series for the amplitude $P_{1,as}$ and the ratios $\rho_{2,3,as}=P_{2,3,as}/P_{2,3,as}^{(1)}$ of the exact asymptotic amplitudes $P_{2,as}$ and $P_{3,as}$ to their one-loop approximations. It is shown that, in the asymptotic region $k^2\gg m^2$, the expansion for $P_{1,as}$ has the same structure as the PT series for the ordinary vacuum PO $k^2\pi(k^2/m^2,\alpha)$ and that the expansions for the ratios $\rho_{2,3,as}$ have the same structure as the PT series for the invariant charge $\alpha d(k^2/m^2,\alpha)$ associated with the photon propagator $D=k^{-2}d$ in QED with no external field.

In Sec. 4 we consider the limiting case $k\ll m$ for strong magnetic fields $H\gg H_{cr}$ in which the variable x is important. Starting from the effective-action formalism, we obtain the relation between the n-photon vertices $\Gamma^{(n)}$ in the low-energy limit $k_i\to0$ and the quantity L, and with the aid of the resulting formulas we calculate the amplitudes in the one- and two-loop approximations for the weak- and strong-field regions. The results of the calculations show that the "massless hypothesis" that lies at the basis of the RG analysis of asymptotic behavior is not valid for the amplitude $P_{2,as}$ in this limiting case, although all the RG results obtained in Sec. 3 are applicable to the quantities $P_{1,as}$ and $P_{3,as}$.

2. GENERAL PART

It is convenient to express the renormalized PO in the field $F_{\mu\nu}$ in the form

$$\Pi_{\mu\nu}(k,k',F)=(2\pi)^4\delta^4(k-k')\Pi_{\mu\nu}(k,eF,\alpha,m),\tag{1}$$

which takes account of the fact that $\Pi_{\mu\nu}$ depends on the external field through the product eF; the PO is related to the photon Green's function

$$D_{\mu\nu}^{-1}(k,eF,\alpha,m)=k^2\delta_{\mu\nu}-k_\mu k_\nu-\alpha\Pi_{\mu\nu}(k,eF,\alpha,m)\tag{2}$$

and is a manifestly gauge invariant function of $F_{\mu\nu}\equiv F$ and $k^2=\mathbf{k}^2-k_0^2$. The variable $\alpha=e^2/4\pi$ determines the contribution from radiative corrections in the radiation field, and in diagram language it is the expansion parameter for the loop expansion:

$$\Pi_{\mu\nu}(k,eF,\alpha,m)=\sum_{n=0}^{\infty}\alpha^n\Pi_{\mu\nu}^{n}(k,eF,m).\tag{3}$$

The quantities $\Pi_{\mu\nu}^{(n)}$ are represented by n-loop diagrams

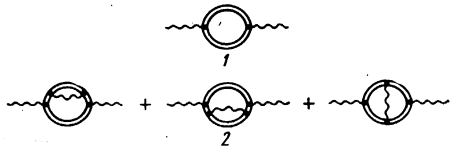

FIG. 1.

with Green's functions for an electron in the external field. For example, the lowest contributions to $\alpha\Pi_{\mu\nu}$, i.e. the quantities $\alpha\Pi^{(1)}_{\mu\nu}$ and $\alpha^2\Pi^{(2)}_{\mu\nu}$, correspond to diagram 1 and 2, respectively, in Fig. 1, where the double line represents an electron in the external field.

The convenience of the present choice of arguments for the PO is associated with the renormalization invariance of the product $eF = e_0F_0$ under multiplicative renormalization transformations:

$$D^{-1}_{\mu\nu}(k, eF, \alpha, m) = ZD^{-1}_{(0)\mu\nu}(k, e_0F_0, \alpha_0, m, \Lambda), \qquad e = Z^{1/2}e_0, \quad F = Z^{-1/2}F_0, \quad Z = Z(e_0, \Lambda/m), \tag{4}$$

as a result of which the argument eF enters on equal footing with the momentum. Here we indicate the unrenormalized values of the various quantities by a subscript zero, and mass renormalization has already been carried through on the right-hand side of (4), Λ being the cutoff parameter.

We shall consider only the physically interesting case of an external field with $G = 0$:

$$\mathcal{F} = {}^1/_4 F_{\mu\nu}^2 = {}^1/_2 (H^2 - E^2), \quad G = {}^1/_4 F_{\mu\nu}F_{\mu\nu}^* = \mathbf{EH},$$

where $F^*_{\mu\nu} = (1/2) ie_{\mu\nu\rho\sigma}$; this case includes the special cases of a crossed field ($\mathcal{F} = 0$), a pure magnetic field ($\mathcal{F} > 0$), and a pure electric field ($\mathcal{F} < 0$). In this case the following expression[9] for the PO, which follows from the general relativistic, gauge, and charge invariance properties of the theory, is valid:

$$\Pi_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) P_1 + \frac{\varphi_\mu^*\varphi_\nu^*}{\varphi^2} P_2 + \frac{\varphi_\mu\varphi_\nu}{\varphi^2} P_3, \tag{5}$$

where $\varphi_\mu = F_{\mu\nu}k_\nu$, $\varphi^*_\mu = F^*_{\mu\nu}k_\nu$, $\varphi = \varphi^* = [(F_{\mu\nu}k_\nu)^2]^{1/2}$, and the P_i $(i = 1, 2, 3,)$ are scalar functions of three invariants[1)]:

$$P_i = P_i(k^2, e\mathcal{F}^{1/2}, e\varphi, \alpha, m).$$

In the limit $F \to 0$, expression (5) reduces to the PO for a photon in vacuo:

$$\Pi_{\mu\nu}|_{F\to 0} = (k^2\delta_{\mu\nu} - k_\mu k_\nu)\pi(k^2/m^2, \alpha).$$

The amplitudes $P_{2,3}$ vanish at $F = 0$, while $P_1(F = 0) = k^2\pi(k^2/m^2, \alpha)$; moreover, in view of Eq. (5) the quantities P_1/k^2 and $P_{2,3}/\varphi^2$ are finite at $k^2 = 0$ and $\varphi^2 = 0$, respectively.

As follows from Eqs. (2) and (4), the amplitudes $P_{2,3}$ are renormalized by multiplication and are RG invariants:

$$P_{2,3}(k^2, e\mathcal{F}^{1/2}, e\varphi, \alpha, m) = P_{(0)2,3}(k^2, e\mathcal{F}^{1/2}, e\varphi, \alpha_0, m, \Lambda), \tag{6}$$

whereas the amplitude P_1, like the vacuum part of π, is renormalized by subtraction:

$$\rho_1(k^2, e\mathcal{F}^{1/2}, e\varphi, \alpha, m) = \lim_{\Lambda\to\infty} [\rho_{(0)1}(k^2, e\mathcal{F}^{1/2}, e\varphi, \alpha_0, m, \Lambda) - \pi_{(0)}(0, \alpha_0, m/\Lambda)]. \tag{7}$$

Here $\rho_1 = P_1/k^2$ is a dimensionless quantity, while $\pi_{(0)}(0, \alpha_0, m/\Lambda)$ is the unrenormalized value of the PO $\pi(k^2/m^2, \alpha)$ at $k^2 = 0$.

In this paper we use RG equations of the C-S type, which are obtained by a standard procedure[14-16] and, because of the condition $eF = e_0F_0$ noted above (which entails $e\mathcal{F}^{1/2} = e_0\mathcal{F}_0^{1/2}$ and $e\varphi = e_0\varphi_0$), contain no derivatives with respect to the argument F:

$$\left[m\frac{\partial}{\partial m} + \beta(\alpha)\alpha\frac{\partial}{\partial\alpha}\right]\left(\frac{k^2}{\alpha} - P_1\right) = -m\frac{\partial}{\partial m}P_{(0)1}, \tag{8}$$

$$\left[m\frac{\partial}{\partial m} + \beta(\alpha)\alpha\frac{\partial}{\partial\alpha}\right]P_{2,3} = m\frac{\partial}{\partial m}P_{(0)2,3} \tag{9}$$

(with fixed values of the parameters k, $eF = e_0F_0$, α_0, and Λ on the right-hand side) where $\beta(\alpha) = m\partial(\ln Z)/\partial m$ for α_0 and Λ constant. Equations (8) and (9) have the same form as in the absence of a field and are well known from the RG analysis of asymptotic behavior at large momenta.

The "massless hypothesis," which plays a central part in the RG analysis of any asymptotic behavior, is used in investigating the asymptotic forms $P_{i,as}$ of the functions P_i at $k^2 \gg m^2$, $e\mathcal{F}^{1/2} \gg m^2$, and $e\varphi \gg m^3$. According to this hypothesis the unrenormalized functions $P_{(0)i}$ do not depend on m in the asymptotic region, so that $\partial P_{(0)i,as}/\partial m = 0$ and Eqs. (8) and (9) reduce to equations in the asymptotic region with zeros on the right. The massless hypothesis can be justified only in asymptotically free theories; in other theories this hypothesis must be verified each time to each order of PT. In QED with an external field it is satisfied for L in the two-loop approximation.[2]

Some of the consequences of the asymptotic equations, which we can immediately verify by direct calculation, concern the one- and two-loop contributions to the asymptotic expansion

$$P_{i,as} = \sum_{n=0}^{\infty} \alpha^n P^{(n+1)}_{i,as}(k^2, e\mathcal{F}^{1/2}, e\varphi, m). \tag{3a}$$

Making use of the expansion

$$\beta(\alpha) = 2\sum_{n=1}^{\infty}\left(\frac{\alpha}{\pi}\right)^n \beta_n, \tag{10}$$

(of which only the first three coefficients are known at present[17]: $\beta_1 = 1/2$, $\beta_2 = 1/4$, and $\beta_3 = 121/288$) we obtain

$$\frac{\partial}{\partial m}P^{(n)}_{2,3,as} = 0, \quad n = 1, 2, \tag{11}$$

for the first two terms in the expansions of the renormalized functions (3a), provided the massless hypothesis holds. Thus, one of the interesting consequences is that the quantities $P^{(1,2)}_{2,3,as}$ are independent of the mass m, i.e. they have the "scale invariance" property (in other words, (11) indicates that the quantities $P^{(1,2)}_{2,3}$ approach finite limits as $m \to 0$). According to another consequence, the asymptotic amplitude $P^{(1)}_{1,as}$ has the form $P^{(1)}_{1,as} \approx \pi^{-1}k^2\beta_1\ln\lambda_{max}$ to logarithmic accuracy, where λ_{max} is the "important variable" in the asymptotic region concerned (see Sec. 3 below).

In subsequent sections the above mentioned properties will be verified in two asymptotic regions of the variables k^2, $e\mathcal{F}^{1/2}$, and $e\varphi$. Getting ahead of ourselves, we may say that these properties hold for all the amplitudes P_i in the limiting case a) (see below), and this is to a certain extent a confirmation of the massless hypothesis. This hypothesis is not satisfied for the amplitude P_2 in the other asymptotic region b).

In order to use dimensional considerations, it is desirable to transform to the dimensionless functions

$$\rho_1(\nu, x, \chi, \alpha) = P_1/k^2, \quad \rho_{2,3}(\nu, x, \chi, \alpha) = P_{2,3}/P_{2,3,as}^{(1)} \tag{12}$$

of the dimensionless ratios $\nu = k^2/m^2$, $x = 2^{1/2}e\mathscr{F}^{1/2}/m^2$, and $\chi = e\varphi/m^3$.

We recall that the quantities $P_{2,3,as}^{(1)}$ are independent of m and α [Eqs. (11) and (3a)]; hence the C-S equations for the ratios (12) have the following form in the asymptotic region:

$$\left[2\nu\frac{\partial}{\partial\nu} + 2x\frac{\partial}{\partial x} + 3\chi\frac{\partial}{\partial\chi} - \beta(\alpha)\alpha\frac{\partial}{\partial\alpha}\right]\rho_{2,3,as}(\nu, x, \chi, \alpha) = 0, \tag{8a}$$

$$\left[2\nu\frac{\partial}{\partial\nu} + 2x\frac{\partial}{\partial x} + 3\chi\frac{\partial}{\partial\chi} - \beta(\alpha)\alpha\frac{\partial}{\partial\alpha}\right]\left(\frac{1}{\alpha} - \rho_{1,as}(\nu, x, \chi, \alpha)\right) = 0. \tag{9a}$$

On introducing the dilation parameter λ for the momenta k and the field F (to the first and second powers, respectively, according to their dimensions) the general solutions to these equations can be conveniently written in the following form (also see Ref. 4):

$$\rho_{1,as}(\nu, x, \chi, \alpha) = \alpha^{-1} - \alpha_{\text{eff}}^{-1}(\ln\lambda, \alpha) + \rho_{1,as}\left(\frac{\nu}{\lambda^2}, \frac{x}{\lambda^2}, \frac{\chi}{\lambda^3}, \alpha_{\text{eff}}(\ln\lambda, \alpha)\right), \tag{13}$$

$$\rho_{2,3,as}(\nu, x, \chi, \alpha) = \rho_{2,3,as}\left(\frac{\nu}{\lambda^2}, \frac{x}{\lambda^2}, \frac{\chi}{\lambda^3}, \alpha_{\text{eff}}(\ln\lambda, \alpha)\right), \tag{14}$$

where the effective coupling constant α_{eff} is a solution of the equation

$$\partial\alpha_{\text{eff}}(t, \alpha)/\partial t = \alpha_{\text{eff}}(t, \alpha)\beta(\alpha_{\text{eff}}(t, \alpha)), \quad \alpha_{\text{eff}}(0, \alpha) = \alpha, \quad t = \ln\lambda. \tag{15}$$

Assuming that the variable k^2 is bounded, we shall later consider two different asymptotic cases in which the important variable is effectively one or the other of the field variables χ and x. These cases correspond to solutions of Eqs. (13)–(15) in which it is convenient to choose $\lambda = \lambda_{max} = \max(x^{1/2}, \chi^{1/3})$.

Case a) corresponds to the asymptotic region $\chi \gg 1$, $\nu/\chi^{2/3} \ll 1$, and $x/\chi^{2/3} \ll 1$ with the choice $\lambda = \chi^{1/3}$. This situation is of course realized for a crossed field when $x = 0$ and $\chi = m^{-3}e|\mathbf{k}|E(1-\cos\gamma) \gg 1$ in the high-energy or strong-field limit under the condition $(k^2)^{3/2} \ll e|\mathbf{k}|E(1-\cos\gamma)$, where γ is the angle between the vectors $\mathbf{k}$ and $\mathbf{E}\times\mathbf{H}$, and it is also realized for a pure magnetic field, when $\chi = m^{-3}e|\mathbf{k}|H\sin\theta \gg 1$ under the conditions $e|\mathbf{k}|H\sin\theta \gg (k^2)^{3/2}$ and $|\mathbf{k}|\sin\theta \gg (eH)^{1/2}$, where θ is the angle between $\mathbf{k}$ and $\mathbf{H}$.

For the reasons given in Ref. 18, when these conditions are satisfied (and they are satisfied for $F \sim F_{cr}$ when, roughly speaking, $k \gg m$) one may neglect the invariants x and ν as compared with χ in calculating the probabilities of processes. In applications to the solutions (13)–(15) one can make an analogous assumption,[4] which is a generalization of the massless hypothesis, that the limits $\nu/\chi^{2/3} \to 0$ and $x/\chi^{2/3} \to 0$ exist provided, as is usual in QED, that α_{eff} increases as $\ln\chi^{1/3}$:

$$\rho_{1,as}(\nu, x, \chi, \alpha) = \alpha^{-1} - \alpha_{\text{eff}}^{-1}(\ln\chi^{1/3}, \alpha) + \rho_{1,as}(0, 0, 1, \alpha_{\text{eff}}(\ln\chi^{1/3}, \alpha)), \tag{16a}$$

$$\rho_{2,3,as}(\nu, x, \chi, \alpha) = \rho_{2,3,as}(0, 0, 1, \alpha_{\text{eff}}(\ln\chi^{1/3}, \alpha)), \tag{17a}$$

i.e. the asymptotic behavior is the same as that for a crossed field at $k^2 = 0$.

Case b) corresponds to the asymptotic region $x \gg 1$, $\nu/x \ll 1$, and $\chi/x^{3/2} \ll 1$, with $\lambda = x^{1/2}$. It is appropriate for a strong magnetic field $eH \gg m^2$ provided the conditions $k^2 \ll eH$ and $|\mathbf{k}|\sin\theta \ll (eH)^{1/2}$ are satisfied, and they are indeed satisfied for soft photons with $k \ll m$ provided $H \sim H_{cr}$.

3. ASYMPTOTIC PROPERTIES OF THE POLARIZATION OPERATOR FOR $\chi \gg 1$

We begin our study of the asymptotic properties of the PO for $\chi \gg 1$ by calculating asymptotic expressions for the one-loop contributions for the case of a crossed field. Employing the equations

$$P_1^{(1)}(\chi, k^2) = -\frac{4k^2}{\pi}\int_4^\infty \frac{dv}{v^2[v(v-4)]^{1/2}}\left[f_1(z) - \ln\left(1 + \frac{k^2}{m^2 v}\right)\right],$$

and

$$P_{2,3}^{(1)}(\chi, k^2) = -\frac{2m^2}{3\pi}\int_4^\infty dv \frac{2v+1\pm3}{v[v(v-4)]^{1/2}}\left(\frac{\chi}{v}\right)^{2/3} f'(z)$$

(which were derived in Ref. 9) for $\chi \gg 1$, which corresponds to small values of the variable

$$z = \left(\frac{v}{\chi}\right)^{2/3}\left(1 + \frac{k^2}{m^2 v}\right) \ll 1,$$

and making use of the asymptotic expressions for the special functions $f_1(z)$ and $f'(z)$,

$$f_1(z)|_{z\to0} = \ln z + \frac{2}{3}c + \frac{1}{3}\ln 3 + \frac{i\pi}{3},$$

and

$$f'(z)|_{z\to0} = \frac{3^{2/3}}{6}\Gamma\left(\frac{2}{3}\right)(1 - i3^{1/2}),$$

(which were also derived in Ref. 9), we obtain[2)]

$$P_{1\,as}^{(1)} = k^2\pi^{-1}({}^1/_3\ln\chi^{2/3} + b_{00}), \tag{18}$$

where

$$b_{00} = -{}^2/_9 c - {}^1/_9\ln 3 - {}^{10}/_{27} + {}^1/_3 i\pi,$$

and

$$P_{2,3,as}^{(1)} = -\frac{m^2}{\pi^2}(3\chi)^{2/3}3^{1/2}\Gamma^4\left(\frac{2}{3}\right)(1 - i3^{1/2})\frac{5\pm1}{28}. \tag{19}$$

It is difficult to see that the results (18) and (19) actually confirm the massless hypothesis in the sense of the consequences noted above. Moreover, it also follows from these results that the parameter k^2 drops out of the asymptotic expressions $\rho_{i,as}^{(1)}$ in accordance with the "generalized massless hypothesis." On the other hand, the field dependence in Eq. (19), as well as that in the two-loop contribution, follows directly from Eq. (11) via dimensional considerations:

$$P_{2,3,as}^{(1)} \sim P_{2,3,as}^{(2)} \sim (e\varphi)^{2/3} = m^2\chi^{2/3}.$$

Turning now to the study of the general solutions (16a) and (17a), we note that the effective parameter for asymptotic expansions is $\alpha_{eff}(\ln\chi^{1/3}, \alpha)$. For these solutions one can also write expansions in double series in α and $\ln\chi^{2/3}$ that are valid when $(\alpha/3\pi)\ln\chi^{2/3} \ll 1$. The expansion

$$\rho_{1,as} = \frac{1}{\pi}(b_{00} + b_{10}\ln\chi^{2/3}) + \frac{1}{\pi}\sum_{n=2}^{\infty}\left(\frac{\alpha}{\pi}\right)^{n-1}\sum_{k=0}^{n-1} b_{n,k}\ln^k\chi^{2/3} \tag{20}$$

has the same structure as the analogous PT series in α for the vacuum PO in the asymptotic region $k^2 \gg m^2$ and for the ratio $(L_M - L)/\alpha L_M$ in the asymptotic region $F \gg F_{cr}$, where $L_M = -\mathcal{F}$ is the Maxwell Lagrangian.[2,3] The coefficients in the principal logarithmic terms are also the same in these expansions and can be expressed in terms of the known coefficients β_1 and β_2:

$$b_{10}=\beta_1, \quad b_{n,n-1}=\beta_1^{n-2}\beta_2/(n-1), \quad n\geq 2. \tag{21}$$

The other coefficients are related to the β_k as follows:

$$kb_{n,k}-\delta_{1k}\beta_n=\sum_{q=k}^{n-1}(q-1)\beta_{n-q}b_{q,k-1}, \quad n\geq 2, \quad k\geq 1. \tag{22}$$

The expansion

$$\rho_{2,3,as}=1+\sum_{n=1}^{\infty}\left(\frac{\alpha}{\pi}\right)^n\sum_{k=0}^{n-1}c_{n,k}\ln^k\chi^{2/3} \tag{23}$$

without its unity term suppressed is similar to the asymptotic series in the region $k^2 \gg m^2$ for the invariant charge $\alpha d(k^2/m^2, \alpha)$ associated with the exact vacuum photon propagator $D = k^{-2}d$. The coefficients satisfy the following recursion relations:

$$kc_{n,k}=\sum_{p=k}^{n-1}pc_{p,k-1}\beta_{n-p}, \quad n\geq 2, \quad k\geq 1, \tag{24}$$

and in particular, we have

$$c_{n,n-1}=c_{1,0}\beta_1^{n-1} \tag{25}$$

for the coefficients in the principal logarithmic terms.

It should be emphasized that the one-loop radiative corrections associated with vacuum polarization increase with increasing field strength as $\chi^{2/3}$, in contrast to the logarithmic increase of the vacuum radiation corrections with increasing momenta. For the two-loop contribution we have

$$P^{(2)}_{2,3,as}=c_{1,0}P^{(1)}_{2,3,as}, \quad P^{(2)}_{1,as}\approx\frac{3}{4\pi}P^{(1)}_{1,as} \tag{26}$$

(the second of these expressions is valid to logarithmic accuracy) but, beginning with the term in α^2, the ratio of successive corrections becomes of the order of $\alpha \ln\chi^{2/3}$. A similar situation also obtains (and independently of the field strength) in the limiting case $k \ll m$ (see below), and also for the effective Lagrangian we have $L^{(2)} \approx \alpha L^{(1)}$ (Refs. 2 and 3). The amplitudes $P_{2,3,as}$ are scale invariant only in the zeroth and first approximations in α, the invariance being broken by the terms in $\ln\chi^{2/3}$.

The RG also provides for the summation of the logarithmic terms of various ranks; for example, by virtue of (25), the sum of the principal logarithmic terms in (23) has the form

$$1+\frac{\alpha}{\pi}c_{1,0}\Big/\left(1-\frac{\alpha}{3\pi}\ln\chi^{2/3}\right)$$

and indicates the appearance of the "zero charge" difficulty for the case of an external field.

4. THE REACTION $k \ll m$. THE RELATION BETWEEN THE POLARIZATION OPERATOR AND THE EFFECTIVE LAGRANGIAN

In the low-energy limit $k \ll m$, the PO may be expressed in terms of the effective Lagrangian for the electromagnetic field. We shall first obtain the relation between these two quantities for the general case of the n-photon vertices $\Gamma^{(n)}$ of QED in an arbitrary static field $F_{\mu\nu} \equiv F$:

$$\Gamma^{(n)}=\Gamma^{(n)}_{(\mu_i)}(\{k_i\},F), \quad \sum_{i=1}^{n}k_i=0,$$

starting from the basic formalism for the effective action.[1,19] Here $\{k_i\}$ is the set of momenta for the n photons, and $\{\mu_i\}$ is the corresponding set of vector indices.

Here we introduce the effective action $\Gamma(f,F)$ into QED with an external field by an additional term in the Lagrangian $\mathcal{L}^{ext} = j_\mu A_\mu$, where A_μ is the vector potential for the external field $F_{\mu\nu}$ and $f \equiv f_{\mu\nu}(p) = i(p_\nu a_\mu(p) - p_\mu a_\nu(p))$ is the "classical" field corresponding to the vector potential a_μ, and j_μ is the quantized electron current. The effective action is the generating functional for the vertex functions $\Gamma^{(n)}$:

$$(2\pi)^4\delta^4\left(\sum_{i=1}^{n}k_i\right)\Gamma^{(n)}_{(\mu_i)}(\{k_i\},F)=i^n\frac{\delta^n\Gamma(f,F)}{\delta a_{\mu_1}(k_1)\ldots\delta a_{\mu_n}(k_n)}\Bigg|_{a=0}$$
$$=2^n(k_{1\alpha_1}\ldots k_{n\alpha_n})\gamma^{(n)}_{(\alpha_i\mu_i)}(\{k_i\},F), \tag{27}$$

where the $\gamma^{(n)}$ are defined as the n-th order derivatives of Γ with respect to the $f_{\alpha_i\mu_i}$ at the point $f = 0$. The effective action has the property

$$\Gamma(f,F)=\Gamma(f+F), \tag{28}$$

and its functional expansion in the $\gamma^{(n)}$ has the form

$$\Gamma(f+F)=\sum_n\frac{1}{n!}\int\prod_{i=1}^{n}d^4k_if_{\mu_i\alpha_i}(k_i)\gamma^{(n)}_{(\mu_i\alpha_i)}(\{k_i\},F). \tag{29}$$

It is not difficult to see from (29) that the expansion $\gamma^{(n)}(\{k_i\},F) = \gamma^{(n)}(0,F) + \ldots$ of the vertices about the zero-momentum point $k_i = 0$ corresponds to the expansion of the functional $\Gamma = \Omega L(f^c + F) + \ldots$ about constant values of the functions

$$f^c_{\mu\alpha}=\int d^4k\,f_{\mu\alpha}(k)$$

and, with the aid of (28), leads to the relation

$$\gamma^{(n)}_{(\mu_i\alpha_i)}(0,F)=\frac{\Omega}{2^n}\partial^nL(F)/\partial F_{\mu_1\alpha_1}\ldots\partial F_{\mu_n\alpha_n},$$

where Ω is a four-dimensional volume. Thus, the momentum dependence of the vertices separates out in the low-energy limit $k \ll m$, and we obtain

$$\Gamma^{(n)}_{(\mu_i)}(\{k_i\},F)|_{k_i\to0}=(k_{1\alpha_1}\ldots k_{n\alpha_n})\frac{\partial^nL(F)}{\partial F_{\mu_1\alpha_1}\ldots\partial F_{\mu_n\alpha_n}}. \tag{30}$$

In particular, for the Green's function $D^{-1}_{\mu\nu} = \Gamma^{(2)}_{\mu\nu}$ this formula becomes

$$D_{\mu\nu}^{-1}(k,F)=k_\alpha k_\beta\partial^2L(F)/\partial F_{\mu\alpha}\partial F_{\nu\beta}. \tag{31}$$

Transforming to the invariants $\mathcal{F}$ and G in (31), calculating the amplitudes in the low-energy limit for the case of external field with $G = 0$ of interest to us, and denoting the amplitudes for this special case by $\tilde{P}_i$, we obtain[3)]

$$\tilde{P}_1=\frac{k^2}{\alpha}\frac{\partial}{\partial\mathcal{F}}(L-L_M)|_{G\to0}, \quad \tilde{P}_2=\frac{\varphi^2}{\alpha}\frac{\partial^2L}{\partial G^2}\Bigg|_{G\to0}, \quad \tilde{P}_3=\frac{\varphi^2}{\alpha}\frac{\partial^2L}{\partial\mathcal{F}^2}\Bigg|_{G\to0}. \tag{32}$$

It should be pointed out that in this approximation the vacuum contribution to π vanishes, i.e. $\tilde{P}_1(F=0) = 0$, so that field and charge renormalization must be effected with a Z factor of "field" type–not of "momentum" type. As was shown in Ref. 2, the part of invariant charge is played in QED with an external field without real particles by the ratio

$$\alpha l^{-1}(e\varepsilon/m^2, e\eta/m^2, \alpha)=\alpha L_M/L,$$

where the variables

$$\eta=[(\mathscr{F}^2+G^2)^{1/2}+\mathscr{F}]^{1/2}, \text{ and } \varepsilon=[(\mathscr{F}^2+G^2)^{1/2}-\mathscr{F}]^{1/2}$$

represent the magnetic and electric fields, respectively, in a coordinate system in which they are parallel; $Z^{-1}=l_0(0,0,\alpha_0, m/\Lambda)$, and this expression coincides with the usual expression $Z^{-1}=1-\alpha_0\pi_0(0,\alpha_0,m/\Lambda)$.

It follows from (32) that the amplitudes $\tilde{P}_{2,3}$, like L, are RG invariants, whereas the quantity $\rho_1=\tilde{P}_1/k^2$ is renormalized by subtraction:

$$\tilde{\rho}_1=\tilde{\rho}_{(0)1}-\alpha_0^{-1}(1-l_0(0,0,\alpha_0,m/\Lambda)).$$

Formula (32) makes it possible to calculate the $\tilde{P}_i$ in the one- and two-loop approximations (the latter being the important one) in terms of the now known Heisenberg-Euler Lagrangian[20] and the two-loop term[2] $L^{(2)}$.

It is easy to see that in the case of a crossed field we have $\tilde{P}_1=0$ and only the first two terms in the expansion in α contribute to the other amplitudes:

$$\tilde{P}_2=\tilde{P}_2^{(1)}+\alpha\tilde{P}_2^{(2)}=\frac{7}{45}m^2\chi^2\left(1+\frac{\alpha}{\pi}\frac{263\cdot45}{162\cdot14}\right),$$
$$\tilde{P}_3=\tilde{P}_3^{(1)}+\alpha\tilde{P}_3^{(2)}=\frac{4}{45}m^2\chi^2\left(1+\frac{\alpha}{\pi}\frac{360}{81}\right). \tag{33}$$

For the region of weak magnetic fields we obtain

$$\tilde{P}_1^{(1)}+\alpha\tilde{P}_1^{(2)}=\frac{2k^2}{45\pi}\left(\frac{H}{H_{cr}}\right)^2\left(1+\frac{\alpha}{\pi}\frac{360}{81}\right), \tag{34}$$

in the two-loop approximation, while the amplitudes $P_{2,3}^{(1,2)}$ are given by Eqs. (33) after the substitution

$$\chi=\frac{|\mathbf{k}|}{m}\frac{H}{H_{cr}}\sin\theta.$$

It is convenient to use the variables η and ε in studying the region $H\gg H_{cr}$. Noting that the case of a pure magnetic field corresponds to the values $\eta=H$ and $\varepsilon=0$ for the variables and making use of the following asymptotic expressions for the region in which $e\eta/m^2\gg1$ and $e\varepsilon/m^2\ll1$:

$$L_{as}^{(1)}=\frac{\alpha\eta^2}{6\pi}\left(\ln\frac{e\eta}{\gamma\pi m^2}+\frac{6}{\pi^2}\zeta'(2)\right),$$
$$L_{as}^{(2)}=\frac{\alpha^2\eta^2}{8\pi^2}\left(\ln\frac{e\eta}{\gamma\pi m^2}+4a_{2,0}\right), \tag{35}$$

where $\gamma=e^c$, $c=0.577\ldots$ is Euler's constant, $\zeta(x)$ is the zeta function, $6\pi^{-2}\zeta'(2)=-0.569\ldots$, and $a_{2,0}=0.878572\ldots$ (the constant $a_{2,0}$ was evaluated by numerical integration in Ref. 3), we obtain[4)]

$$\tilde{P}_{1,as}^{(1)}=\frac{k^2}{\pi}\left(\frac{1}{3}\ln\frac{eH}{m^2}+\tilde{b}_{00}\right),$$
$$\tilde{P}_{1,as}^{(2)}=\frac{k^2}{\pi^2}\left(\frac{1}{4}\ln\frac{eH}{m^2}+\tilde{b}_{2,0}\right), \tag{36}$$

where

$$\tilde{b}_{00}=2\pi^{-2}\zeta'(2)-{}^1/_3\ln\gamma\pi+{}^1/_6,\quad \tilde{b}_{2,0}={}^1/_8+a_{2,0}-{}^1/_4\ln\gamma\pi,$$
$$\tilde{P}_{3,as}^{(1)}={}^4/_3\pi\tilde{P}_{3,as}^{(2)}={}^1/_3\mathbf{k}^2\pi^{-1}\sin^2\theta. \tag{37}$$

It is necessary to investigate the nonasymptotic expression for $L^{(1)}$ in order to calculate the amplitude $\tilde{P}_{2,as}^{(1)}$; the calculation is carried through in the Appendix and leads to the result

$$\tilde{P}_{2,as}^{(1)}=\frac{\mathbf{k}^2}{3\pi}\sin^2\theta\left(\frac{eH}{m^2}+\frac{6}{\pi^2}\zeta'(2)-\ln2\pi+\frac{1}{2}\right). \tag{38}$$

Equations (36)–(38) enable one to draw a conclusion concerning the general solution (13)–(15) for the limiting case b) corresponding to the choice $\lambda=x^{1/2}$. We see that in this case the massless hypothesis is satisfied for the amplitudes $\tilde{P}_{1,3}$ in the two-loop approximation but is not satisfied for $\tilde{P}_{2,as}^{(1)}$, which is not scale invariant. In the same approximation, the generalized massless hypothesis, according to which the limits of formulas (13) and (14) as ν/x and $\chi/x^{2/3}$ approach zero exist, is also valid for the ratios

$$\tilde{\rho}_{1,as}(\nu,x,\chi,\alpha)=\alpha^{-1}-\alpha_{\text{eff}}^{-1}(\ln x^{1/2},\alpha)+\tilde{\rho}_{1,as}(0,1,0,\alpha_{\text{eff}}(\ln x^{1/2},\alpha)), \tag{16b}$$

and

$$\tilde{\rho}_{3,as}(\nu,x,\chi,\alpha)=\tilde{\rho}_{3,as}(0,1,0,\alpha_{\text{eff}}(\ln x^{1/2},\alpha)). \tag{17b}$$

Expansions like (20) and (23), respectively, but with $\chi^{2/3}$ replaced by x, are also valid for these ratios, as are the pertinent remarks following (20) and (23) in the text.

In concluding, the author is pleased to thank B. L. Voronov and V. I. Ritus for many discussions and much valuable advice.

APPENDIX

To calculate the amplitude $\tilde{P}_2$ in the one-loop approximation when it is written in terms of the variables η and ε,

$$\tilde{P}_2=\frac{2\varphi^2}{\alpha H^2}\left(\frac{\partial L}{\partial\eta^2}+\frac{\partial L}{\partial\varepsilon^2}\right)\Big|_{\eta=H,\,\varepsilon=0},$$

we make use of the following expression for $L^{(1)}$:

$$L^{(1)}=\frac{(e\eta)^2}{8\pi^2}\int_0^\infty\frac{dt}{t^3}e^{-izt}\left[\frac{\varepsilon}{\eta}t^2\operatorname{ctg}(t)\operatorname{cth}\left(\frac{\varepsilon}{\eta}t\right)-1+\frac{t^2}{3}\left(1-\frac{\varepsilon^2}{\eta^2}\right)\right],$$

in which $z=m^2/e\eta$. In the limit as $\varepsilon/\eta\to0$, the first two terms in the expansion of this formula take the form

$$L^{(1)}(\eta,\varepsilon)=L^{(1)}(\eta,0)+\varepsilon^2\frac{\partial L^{(1)}}{\partial\varepsilon^2}\Big|_{\varepsilon^2\to0}+\ldots,$$

$L^{(1)}(\eta,0)$ is given by Eq. (35) for $z\ll1$, while

$$\frac{\partial L^{(1)}}{\partial\varepsilon^2}\Big|_{\varepsilon^2\to0}=\frac{e^2}{24\pi^2}\int_0^\infty dt\,e^{-izt}(\operatorname{ctg}t-t^{-1})$$
$$=\frac{\alpha}{6\pi}\left[\ln\frac{m^2}{2e\eta}+\frac{e\eta}{m^2}-\psi\left(1+\frac{m^2}{2e\eta}\right)\right],$$

where $\Psi(x)$ is the logarithmic derivative of the gamma function and

$$\psi(1+z/2)|_{z\to0}\approx-c+\frac{\pi^2}{12}z+\ldots.$$

Retaining the terms that do not vanish when $\eta=H\gg H_{cr}$, we reach Eq. (38).

[1)]These amplitudes determine the poles of the photon Green's function as solutions of the equations $k^2=P_1$ and $k^2=P_1+P_{2,3}$.
[2)]Such an asymptotic formula was also calculated in Ref. 8, but using a different representation for the PO. It is easy to see

that the results of Ref. 8, when expressed in terms of the amplitudes P_i, give Eq. (19) for the $P_{2,3,as}$; however, they do not contain the logarithmic term, which is small as compared with $m^2\chi^{2/3}$, and therefore give $P_{1,as}^{(1)}=0$ instead of Eq. (18).

[3)]In deriving (32) from (31) we used the condition $(\partial^2 L/\partial\mathcal{F}\partial G)|_{G\to 0}=0$; this condition is easy to understand since the quantity concerned is a pseudoscalar expressed in terms of G and, since it is finite, it must vanish as $G\to 0$.

[4)]In the second of Refs. 12, the PO in a magnetic field was expressed in terms of scalar functions χ_i related to the P_i by the equations $\alpha\chi_1=P_1$ and $\alpha\chi_{2,3}=P_1+P_{2,3}$. From (36) and (37) we obtain the asymptotic formulas

$$\chi_{1,as}^{(1)}\approx\chi_{3,as}^{(1)}\approx\frac{\alpha k^2}{3\pi}\ln\frac{eH}{m^2},$$

which are accurate to terms that are constant in H and agree with the results obtained in the usual manner in Ref. 12.

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Nonlinear interaction between waves in strongly inhomogeneous media

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Nonlinear three-wave interaction between waves in a randomly inhomogeneous dispersive medium is considered. A kinetic equation for the probability distribution of the second-harmonic intensity is obtained in the diffusion approximation. It is shown on the basis of the diffusion equation that the random mismatching of the wave phases, violating the phase synchronism condition, results in a weakening of the nonlinear interaction, which goes over to a stationary regime corresponding to the limiting efficiency value of an optical frequency doubler (50%). The dynamics of the transition of the nonlinear interaction to the stationary conditions is analyzed. An extension to the case of nondegenerate three-frequency nonlinear interaction is presented.

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1. INTRODUCTION

Significant attention has been paid in recent years to the analysis of resonance interaction of waves in dispersive inhomogeneous media. Here the nonlinear interaction process is accompanied by regular or random mismatchings of the phases, the conditions of phase synchronism to which the resonance interaction of the waves is extremely sensitive are violated, and this leads to a weakening of the nonlinear interaction.

The greatest number of researches has been devoted to the multiplication of laser frequencies in nonlinear crystals with inhomogeneities. The interest in this problem in laser physics is due to the fact that the effect of generation of light harmonics allows one to estimate very simply, by experiment, the effect of the nonlinear inhomogeneities on the nonlinear properties of the crystal. However, because of the difficulties of theoretical analysis, the analysis of the basic question of the effectiveness of the conversion of the basic radiation into subharmonics is carried out either in the approximation of the given pump field,[1-3] or in the approximation of the given pump intensity,[4,5] which essentially reduce the considered problems to linear ones. The parametric instabilities in an inhomogeneous plasma are considered in similar fashion.[6-8]

For a systematic analytic consideration of the effect of the inhomogeneities of the medium on the process of nonlinear interaction, it is necessary to reject the approximations usually employed. An attempt to go beyond the framework of the parametric approximation has been undertaken in the recent papers of Filonenko

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