Nondemolition measurements in gravitational-wave experiments

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In connection with the problem of raising the sensitivity of gravitational-wave experiments, a study is made of the quantum limitations that can arise when a classical force is measured by the response of a quantum oscillator. Following up work done by the groups at Caltech and Moscow, and also by Unruh, attention is drawn to a class of nondemolition measurements that are free of quantum limitations on the accuracy with which a force can be measured. It is shown that such measurements can be realized in the case of observation of operators that are quantum integrals of the motion of the investigated system. The physical reasons for the presence or absence of a quantum sensitivity limit for an arbitrary choice of an observable are elucidated; they reside in the degree of uncertainty of the initial state of the system. In the case of integrals of the motion, this uncertainty can be reduced to zero by an initial precise measurement and subsequently remains zero (quasinondemolition measurement). It is shown further that one can make a choice of observables that do not depend on the initial state of the quantum system at all but retain information about an external influence. In this case, a precise continuous measurement can be realized without special preparation of the state of the system (strictly nondemolition measurement). The general rule for the construction of such an optimal variable is identical to the recommendations of the quantum theory of filtration.

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§1. INTRODUCTION

Questions relating to the limiting sensitivity in macroscopic experiments have attracted considerable interest in recent years in connection with the significant progress made in experimental techniques and also the need to make a number of measurements of fundamental and applied nature (for a detailed discussion of the state of the art see Refs. 1 and 2). A new stimulus in this connection has been provided by gravitational-wave experiments and relates to the elucidation of the existence (or absence) of quantum limitations on the detection of weak gravitational signals from space by macroscopic terrestrial antennas.[2-6] According to the estimates of Thorne *et al.*,[3] the deformation of a gravitational detector of the Weber type under the influence of a burst of radiation from the Virgo Cluster must be less than the characteristic quantum dispersion of the coordinate of the detector, regarded as a quantum oscillator. *A priori*, the very possibility of measurements at this "quantum" level was not obvious. However, in the papers of Braginskiĭ *et al.*[4] it was shown that such a sensitivity can be achieved by means of a definite measuring procedure, which has become known as quantum nondemolition or nondisturbative measurement. Later, Unruh[5] and Thorne *et al.*[3] studied the mathematical features of nondemolition measurements, and also gave examples of variables—the real and imaginary parts of the complexamplitude of an oscillator—suitable for continuous precise measurement and, therefore, for the detection of an arbitrarily weak external force applied to the quantum oscillator. Recently, the same problem was investigated from the point of view of the theory of optimal filtration of a signal in quantum noise.[7,8]

The second generation of gravitational antennas is characterized by a temperature $T\sim 1\,^\circ$K at frequency $\omega_\mu \lesssim 10^4$, which corresponds to a "classical" many-quantum state with mean level $\langle n\rangle \sim kT/\hbar\omega_\mu \sim 10^7$. However, the pursuit of accuracy calls for a further lowering of the temperature to $10^{-3}\,^\circ$K, so that a situation for which $\langle n\rangle \sim 1$ is conceivable. In addition, Braginskiĭ has pointed out[1,9] that if the detector has a high Q quantum properties can be manifested at short observation times $\tau \ll Q_\mu/\omega_\mu$ provided

$$nkT\omega_\mu\tau/Q_\mu<\hbar\omega_\mu. \qquad (1)$$

The fulfillment of this condition can be achieved by artificial cooling of the oscillator, i. e., by the preparation of a state with $n\sim 1$ (Refs. 2 and 9). The majority of the papers quoted above contain, in various modifications, an analysis of the key problem of detecting a weak classical force from the response of a quantum system. Such analysis is based on the principle of "fluctuation back reaction" of the measuring device on the measured system proposed by Braginskiĭ[10] (for application to a gravitational antenna, see Ref. 11). The limitations to sensitivity associated with this principle are important if the experimentalist is, for example, interested in the continuous measurement of the trajectory of some basic canonical variables such as the coordinate, momentum, or energy. A possible way of avoiding these limitations involves the introduction of the concept of a nondemolition measurement.

The aim of the present paper is to discuss the physical reasons that lead to a nondemolition measuring procedure. We shall show that if the continuous establishment of the trajectory of a canonical variable such as $\hat{q}(t), \hat{p}(t)$, or $\hat{E}(t)$ is not an aim in itself but is required only to measure an external force, it is possible to surmount the limitations due to the fluctuation back reaction by extending the class of allowed measuring operations. We draw attention to the fact that a quantum nondemolition measurement reduces to the finding of a variable that is "convenient" for measurement, such being, for example, an integral of the motion of the measured system. We shall see that the extension to the class that we consider corresponds to the rules of optimal quantum filtration of a classical system on a background of

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quantum fluctuations as set forth in Ref. 12. Note that in all that follows the concept of a "measurement" is considered in the framework of the fundamental postulates of quantum mechanics, which it is appropriate to recall word for word in this connection (see, for example, Ref. 13).

1. *Postulate of the state.* A physical system is described by a vector in a Hilbert space H, this vector satisfying the evolution equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle=\hat{H}|\psi\rangle;\quad |\psi(t)\rangle=e^{-i\hat{H}t/\hbar}|\psi_0\rangle. \tag{2a}$$

2. *Postulate of observables.* An observable is an Hermitian operator $\hat{A}$; the result of measurement of $\hat{A}$ is a real eigenvalue α_r of this operator, $A|\varphi_r\rangle=\alpha_r|\varphi_r\rangle$; the (*a priori*) probability of obtaining α_r as a result of the measurement is

$$P(A=\alpha_r/t)=\sum_j|\langle\varphi_{rj}|\psi(t)\rangle|^2, \tag{3a}$$

where j is a degeneracy index.

3. *Reduction postulate.* After the measurement, the state vector goes over into an eigenvector of the given operator; thus, when $\hat{A}$ is measured and α_r is obtained, we have

$$\psi\to\psi',\ \text{where}\ |\psi'\rangle=|\varphi_r\rangle. \tag{4a}$$

In the case of mixed states, the relations (2a)–(4a) are expressed in terms of the density matrix $\hat{\rho}(t)$ and takes the form

$$1.\ i\hbar\frac{\partial\hat{\rho}}{\partial t}=[\hat{H},\hat{\rho}],\quad \rho(t)=e^{-i\hat{H}t/\hbar}\hat{\rho}_0e^{i\hat{H}t/\hbar};$$

$$\hat{\rho}_0=|\psi_0\rangle\langle\psi_0|. \tag{2b}$$

$$2.\ P(A=\alpha_r|t)=\mathrm{Sp}[\hat{p}_r,\hat{\rho}(t)],\quad \hat{p}_r=\sum_j|\varphi_{rj}\rangle\langle\varphi_{rj}|. \tag{3b}$$

3. After the measurement $\hat{A}\to\alpha_r$, we have

$$\hat{\rho}(t)\to\rho^{?}=|\varphi_r\rangle\langle\varphi_r|, \tag{4b}$$

if the observer knows the result, and

$$\hat{\rho}(t)\to\hat{\rho}'=\sum_s p_s|\psi_s\rangle\langle\psi_s|,\quad \sum p_s=1,$$

if a measurement is made but the observer does not know the result.

The final (4b) reveals the greater generality of the formalism of mixed states, namely, reduction is possible either to a pure state or to a state with diagonal density matrix depending on the conditions of the experiment.

It is important to emphasize the following points. First, the postulates 1, 2, and 3 refer only to a single measuring (observing) act. The process of repeated or continuous measurements requires independent investigation. Second, the posulates assert the possibility of an absolutely precise single measurement of observables described by Hermitian operators (it is clear that one presupposes a state that is an eigenstate for the given operator). Third, the postulates presuppose an ideal observer (von Neumann's classical instrument[14]), which does not introduce during the measuring process any distortions (noise) apart from the collapse of the wave function. Real instruments do, of course, have inherent noise. This however does not have any relation to the quantum-mechanical features of the measuring process discussed in the present paper. Although the analysis which follows below is valid for an arbitrary quantum system, we shall above all have in mind the key problem for a gravitational-wave experiment, viz, the detection of a classical force from the response of a quantum oscillator. The paper is arranged as follows: In Sec. 2, we formulate the criteria of nondemolition measurement; in Sec. 3, there follows an analysis of the physical causes of the quantum limitations to sensitivity; in Sec. 4, we establish the connection between a nondemolition measurement and the recommendations of the quantum theory of optimal filtration; finally, in the Conclusions, we formulate the main results of the paper.

§2. NONDEMOLITION MEASUREMENT OF A QUANTUM OBSERVABLE

The expression "quantum nondemolition (or nondisturbative) measurement"[1)] was introduced by Braginskiĭ and Vorontsov in Ref. 2 to emphasize the need to find a quantum variable or measurement procedure for which the measurement at the time t_1 does not have an uncontrollable random influence on the same observable at the subsequent time t_2.[2)] It is in fact required that the quantum system at both times, t_1 and t_2, be in an eigenstate of the operator of the measured variable. (Note that in the book of Landau and Lifshitz[15] the expression "predictable measurement" is used in this case.) It was shown in Ref. 4 for the example of a quantum oscillator that the coordinate operator does not satisfy this condition, since for arbitrary t_1 and t_2 the measurement of $\hat{x}(t_1)$ disturbs $\hat{p}(t_1)$ which introduces an error in the result of the second measurement of $\hat{x}(t_2)$. However, in a state that is an eigenstate for the operator $\hat{x}(t)$ there exist times t_k, which are separated by half-periods, which admit nondemolition measurement. Thorne *et al.*[3] have given examples of variables that admit continuous precise quantum nondemolition measurement, and they have proposed a mathematical condition that such variables must satisfy. In this section, we attempt to give a logical and invariant mathematical definition of a nondemolition variable for quantum systems in a pure state. The main difference between measurements in quantum and classical mechanics is the presence of "measurement noise". This last is in no way related to the actual properties of the instrument, which may be an ideal (absolutely exact) classical instrument (a von Neumann instrument). The "measurement noise" is a consequence of the reduction postulate (see 3 in the Introduction).

Let us explain this more fully. In the classical view, fluctuations are a property of the measured object, and therefore the potential accuracy of a measurement can be estimated by calculating the evolution of the object without an instrument. In the quantum case, this, in general, is impossible, since the measurement introduces its "noise" by disturbing the free evolution of the object through the reduction effect. Whatever inherent dispersion the object may have, the measurement destroys it (results in its reduction), introducing an indeterminate perturbation in the subsequent states of the object. The upshot is that when one is estimating the result (accuracy) of a continuous or repeated measurement in a quantum system one must always take into account the "perturbation on the part of the instrument".

One can however find conditions under which this difference between quantum and classical measurement disappears. Indeed, if the accuracy of a measurement is small—greater than the inherent dispersion of the measured parameter—reduction does not occur.[3)] It is obvious that this situation is preserved when the accuracy of the instrument is increased until it is equal to the inherent dispersion of the object. One can say that the reduction exists but is trivial, since it carries the object into itself. This will then be a "nondemolition measurement".[4)] If during the process of evolution the inherent dispersion of the object changes, so must the accuracy of the instrument (the observation) in order to remain "nondemolition".

Of course, in this paper we shall be interested, not in nondemolition measurement in general, but only in precise nondemolition measurement in which the condition of nondemolition is preserved and the measurement is without error. It is clear that for this the only suitable observables are those that have zero inherent dispersion, i.e., are in an eigenstate for the measured operator. Such a state can be prepared in accordance with the reduction postulate by, for example, an initial measurement. Such observables are the operators corresponding to integrals of the motion of the quantum system. In accordance with Ref. 16, any system with N degrees of freedom possessing an Hermitian Hamiltonian $\hat{H}$ has $2N$ independent integrals of the motion $\hat{I}_i(t)$ whose expectation values do not depend on the time:

$$\frac{d}{dt}\langle\psi(t)|\hat{I}_i(t)|\psi(t)\rangle=0; \qquad (5)$$

$\psi(t)$ is an arbitrary state of the system satisfying the Schrödinger equation. If $\hat{u}(t)$ is a unitary evolution operator, the integrals of the motion can be readily calculated in accordance with

$$I_i(t)=\hat{u}(t)\hat{I}_i(0)\hat{u}^{-1}(t), \qquad (6)$$

where $\hat{I}_i(0)$ is a time-independent operator of any physical observable corresponding to the considered system. Formula (6) gives the operator of the integral of the motion in the Schrödinger picture. Going over to the Heisenberg picture, we find

$$u^{-1}(t)\hat{I}_i(t)\hat{u}(t)=\hat{I}_i(0)=\text{const},$$

and the property (5) is trivial, since here the wave function is also independent of the time. We note that the process of measurement under real laboratory conditions can be more perspicuously described in the Schrodinger representation.[17] Using the definition (5), we can readily show that an integral of the motion is a variable for which all moments do not depend on the time.[18] Then, choosing a state of the system that is an eigenstate for the operators $\hat{I}_i$, we can make all the higher centered moments—variance, excess, and so forth—vanish. In such a case, the operators $\hat{I}_i$ constitute an observable that admits a precise nondemolition measurement. It is noteworthy that the definition (5) is invariant with respect to the picture in which the quantum system is described (the other definitions could be formulated in only the Heisenberg or only the Schrödinger representation; see below, and also Refs. 3 and 5).

The function ψ in (5) is defined up to the initial conditions, the choice of which corresponds precisely to the transition to an eigenstate of the operator $\hat{I}_i$. We find explicitly the operators $\hat{I}_i$ for an harmonic oscillator in the absence of a thermal bath:

$$\hat{H}=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}+\frac{m\omega^2q^2}{2}, \quad \hat{u}=\exp\left(-\frac{i}{\hbar}\hat{H}t\right). \qquad (7)$$

As initial operator $\hat{I}_i(0)$ one can choose either the coordinate operator $\hat{q}=\hat{I}_1(0)$ [so that $\hat{q}\psi(q)=q\psi(q)$], or the momentum operator $\hat{p}=\hat{I}_2(0)$ [so that $\hat{p}\psi(q)=-i\hbar\partial\psi/\partial q$]. Then, using (6) and the well-known expansion of an operator exponential, we obtain

$$\hat{I}_1(t)=\hat{q}\cos(\omega t)-\frac{\hat{p}}{m\omega}\sin(\omega t),$$
$$\hat{I}_2(t)=\left[\hat{q}\sin(\omega t)+\frac{\hat{p}}{m\omega}\cos(\omega t)\right]m\omega. \qquad (8)$$

The Schrödinger operators $\hat{I}_1(t)$ and $\hat{I}_2(t)$ (they are denoted by $\hat{x}_1$ and $\hat{x}_2$ in Refs. 3 and 5) have a simple physical meaning, namely, the values that are obtained by their measurement are the initial mean coordinates of the oscillator in its phase space. [In this connection, the operator $\hat{I}_1(t)$ can be called the operator of the initial coordinate, and $\hat{I}_2(t)$ the operator of the initial momentum, and they can be denoted by $\hat{q}_0=\hat{I}_1(t)$ and $\hat{p}_0=\hat{I}_2(t)$].

We now consider what must be the commutation properties of an operator that represents a nondemolition variable which admits precise measurement of its "trajectory" during the process of evolution. In quantum mechanics, a necessary and sufficient condition for one to be able to make a simultaneous exact measurement of two physical variables described by Hermitian operators $\hat{A}$ and $\hat{B}$ is that they should commute: $[\hat{A},\hat{B}]=0$ (Ref. 15). We generalize this theorem to the case of joint measurability of observables referring to different times. By joint measurability, we understand the following.

Suppose that at t_1 a state $\psi(x,t_1)=\psi_1$ which is an eigenstate for the operator $A(t_1)=A_1$ is prepared. Measurement of $A(t_1)$ gives the result a_1. Thereafter, the undisturbed wave function ψ_1 evolves in accordance with the Schrödinger equation, so that the state which arises at the time t_2 is constructed in accordance with the rule $\psi_2=\psi(x,t_2)=\hat{u}(t_2,t_1)\psi_1$, where $\hat{u}(t_2,t_1)=\hat{u}$ is the evolution operator. This new state is an eigenstate for the operator $\hat{A}(t_2)=\hat{A}_2$, so that a measurement gives the eigenvalue a_2 without disturbing the wave function and so forth. Mathematically, if at t_1 we had

$$\hat{A}_1\psi_1=a_1\psi_1, \qquad (9a)$$

then at t_2

$$\hat{A}_2\psi_2=a_2\psi_2. \qquad (9b)$$

The fulfillment of Eqs. (9) is the mathematical condition that the "physical variables $\hat{A}(t_1)$ and $\hat{A}(t_2)$ are jointly measurable," and the observable $\hat{A}(t)$ admits a continuous precise measurement of its trajectory.

We apply to Eq. (9a) the operator $\hat{A}_2\hat{u}$ from the left; then, inserting the identify operator between $\hat{A}_1$ and ψ_1, we obtain

$$\hat{A}_2(\hat{u}\hat{A}_1\hat{u}^{-1})\psi_2=a_1\hat{A}_2\psi_2, \qquad (10)$$

The operator obtained in the brackets, $\hat{B}_2=\hat{u}\hat{A}_1\hat{u}^{-1}$, is taken at t_2, $\hat{B}_2=\hat{B}(t_2)$, and for it ψ_2 is an eigenfunction with eigenvalue a_1. All the quantities in (10) are referred to the single time t_2, and therefore the condition under which ψ_2 will also be an eigenfunction of the operator $\hat{A}_2$ reduces to the theorem on the simultaneous measurability of $\hat{A}_2$ and $\hat{B}_2$, i.e., to the requirement $[\hat{A}_2, \hat{B}_2]=0$. This requirement can be expressed more fully in the two equivalent forms

$$[\hat{A}(t_2),\ \hat{u}(t_2,\ t_1)\hat{A}_1\hat{u}^{-1}(t_2,\ t_1)]=0, \qquad (11a)$$

$$[\hat{u}^{-1}(t_2,\ t_1)\hat{A}(t_2)\hat{u}(t_2,\ t_1),\ \hat{A}(t_1)]=0. \qquad (11b)$$

Thus, formulas (11) give the necessary and sufficient condition for joint measurability of $\hat{A}(t_1)$ and $\hat{A}(t_2)$; the arbitrariness of t_1 and t_2 guarantees the possibility of continuous precise measurement of the "trajectory" of $\hat{A}(t)$. It is easy to see that for Schrödinger operators it is only in the special case of the equality

$$\hat{A}(t_1)=\hat{u}(t_2,\ t_1)\hat{A}_1\hat{u}^{-1}(t_1,\ t_2)$$

that joint measurability amounts to commutativity of the operator $\hat{A}$ at different times:

$$[\hat{A}(t_2),\ \hat{A}(t_1)]=0. \qquad (12)$$

In contrast, for Heisenberg operators, the condition (12) always corresponds to the requirement of joint measurability, as can be seen from Eq. (11b). Formulas (11) and (12) can serve as criteria of a nondemolition observable, but, unlike (5), they are not invariant under a change from the Heisenberg to the Schrödinger picture.

We show that the conditions (11) are a common property of any integral of the motion. For this, we write an arbitrary Hermitian integral of the motion $\hat{I}(t)$ in the form (6). The problem of joint measurability of $\hat{I}(0)$ and $\hat{I}(t)$ ($t_1=0,\ t_2=t$) is solved on the basis of the constructions

$$\begin{gathered}\hat{A}(t_1)=\hat{I}(0),\quad \hat{A}(t_2)=\hat{I}(t);\\ \hat{B}(t_2)=\hat{u}(t)\hat{I}(0)\hat{u}^{-1}(t)=\hat{I}(t)=\hat{A}(t_2);\\ [\hat{A}(t_2),\ \hat{B}(t_2)]=0,\end{gathered} \qquad (13)$$

i.e., any integral of the motion satisfies the condition (11) of a nondemolition measurement.

Conversely, if, for example, we consider the coordinate operator $\hat{A}=\hat{q}$ for a harmonic oscillator, then, although $[\hat{q}(t_1), \hat{q}(t_2)]=0$ always holds because $\hat{q}$ does not depend on the time, the requirement (11) is not satisfied in the general case. To verify this, we construct the operator

$$\hat{B}(t_2)=\hat{B}(t)=u(t)\hat{x}\hat{u}^{-1}(t)=\hat{q}\cos\omega t-\frac{\hat{p}}{m\omega}\sin\omega t=\hat{I}_1(t)=\hat{q}_0. \qquad (14)$$

It can be seen that (11) is satisfied only for fixed times t_k such that $\omega t=\pi k$, $k=1, 2, \ldots$. Hence, a continuous precise measurement of the coordinate operator (and similarly of the momentum) is impossible; only a stroboscopic measuring process at the times t_k will represent an exact nondemolition measurement, as was noted in Ref. 4.

It is also important to note that the condition (11) encompasses not only quantities such as integrals of the motion. Thus, if

$$F(\hat{A}(t_2))=\hat{B}(t_2), \qquad (15)$$

where F is an arbitrary function of the operator $\hat{A}(t_2)$, then (11) is also satisfied. The operator $\hat{A}(t)$ need not be an integral of the motion but nevertheless the times t_1 and t_2 are chosen such that (11) is satisfied. At these times, $\hat{A}(t_1)$ and $\hat{A}(t_2)$ are jointly measurable. One can have not only periodic points (as in stroboscopic measurement of the coordinate of an oscillator), but also a finite set of arbitrarily arranged points. In what follows, we shall given an example of operators that are measurable continuously and precisely but which are not integrals of the motion of the investigated system. To conclude this section, we note that a nondemolition observable could be defined in a more general manner from, for example, the point of view of the criterion of maximal information extracted from a measurement, as is discussed in Ref. 19.

§3. PHYSICAL REASONS FOR THE QUANTUM SENSITIVITY LIMITATIONS

We now analyze the problem of detecting a classical force from the response of a quantum oscillator. This example will show how recourse to the measurement of a nondemolition variable enables one to obtain an arbitrary high sensitivity. In contrast to the previous section, we extend the scope of the investigation and consider mixed states. Our analysis will reveal the physical criteria that select a nondemolition observable from arbitrarily chosen variables.

We shall give our treatment in the framework of a definite model; although this is not unique, it is in accord with the principles of quantum mechanics and the intuitive idea of the behavior of a damped quantum oscillator. We use a model equation of Fokker–Planck type, although other models are possible (see the review Ref. 20 and also Ref. 21). Introducing in place of the density matrix the Wigner function[22]

$$W(q,p)=\int(\rho(q+\xi/2,\ q-\xi/2))\,e^{-ip\xi/2\hbar}\,d\xi, \qquad (16)$$

we can write down for it the following kinetic equation (for the details, see Ref. 23):

$$\frac{\partial W}{\partial t}=-\frac{p}{m}\frac{\partial W}{\partial q}+\frac{\partial}{\partial p}\{[m\omega^2q+2\gamma p-f(t)]W\}+D\frac{\partial^2 W}{\partial p^2}, \qquad (17)$$

where $\gamma=\omega/2Q$ is the damping coefficient, $f(t)$ is the classical force applied to the oscillator, and, finally the coefficient $D=\gamma m\hbar\omega\coth(\hbar\omega/2kT)$ is chosen to make the equilibrium Wigner function[24] satisfy Eq. (17). As is shown in Ref. 25, Eq. (17) satisfactorily describes the relaxation of an oscillator under the condition that the relaxation time of the thermal bath is much shorter than γ^{-1} and the random forces acting on the oscillator due to the thermal fluctuations of the thermal bath are Gaussian and delta correlated.

From Eq. (17), we can readily obtain the following system of equations for the variances $\sigma_q=\langle q^2\rangle-\langle q\rangle^2$, $\sigma_p=\langle p^2\rangle-\langle p\rangle^2$, $\sigma_{pq}=\langle pq\rangle-\langle p\rangle\langle q\rangle$:

$$\dot\sigma_p=-2m\omega^2\sigma_{pq}-4\gamma\sigma_p+2D,$$

$$\dot\sigma_q=\frac{2}{m}\sigma_{pq},\quad \dot\sigma_{pq}=\frac{1}{m}\sigma_p-m\omega^2\sigma_q-2\gamma\sigma_{pq}. \tag{18}$$

The solutions of the system (18) have the form

$$\sigma_q(t)=\sigma_q^{(eq)}+e^{-2\gamma t}\left\{\Delta\sigma_q\left(\cos\Omega t+\frac{\gamma}{\Omega}\sin\Omega t\right)^2+\frac{\Delta\sigma_p}{m^2\Omega^2}\sin^2\Omega t+\frac{2\sigma_{pq}^{(i)}}{m\Omega}\sin\Omega t\left(\cos\Omega t+\frac{\gamma}{\Omega}\sin\Omega t\right)\right\}, \tag{19a}$$

$$\sigma_p(t)=\sigma_p^{(eq)}+e^{-2\gamma t}\left\{\Delta\sigma_p\left(\cos\Omega t-\frac{\gamma}{\Omega}\sin\Omega t\right)^2+\frac{m^2\omega^4}{\Omega^2}\Delta\sigma_q\sin^2\Omega t-\frac{2m\omega^2}{\Omega}\sigma_{pq}^{(i)}\sin\Omega t\left(\cos\Omega t-\frac{\gamma}{\Omega}\sin\Omega t\right)\right\}, \tag{19b}$$

$$\sigma_{pq}(t)=-e^{-2\gamma t}\left\{\frac{\sigma_{pq}^{(i)}}{\Omega^2}(\gamma^2-\omega^2\cos 2\Omega t)+\Delta\sigma_q\frac{m\omega^2}{\Omega}\sin\Omega t\times\left(\cos\Omega t+\frac{\gamma}{\Omega}\sin\Omega t\right)-\frac{\Delta\sigma_p}{m\omega}\sin\Omega t\left(\cos\Omega t-\frac{\gamma}{\Omega}\sin\Omega t\right)\right\}; \tag{19c}$$

here, $\sigma_\alpha^{(eq)}$ denotes the equilibrium values of the variances:

$$\sigma_p^{(eq)}=m^2\omega^2\sigma_q^{(eq)}=\frac{m\omega\hbar}{2}\operatorname{cth}\frac{\hbar\omega}{2kT}.$$

The index (i) indicates the initial conditions: $\Delta\sigma_\alpha=\sigma_\alpha^{(i)}-\sigma_\alpha^{(eq)}$, $\Omega=(\omega^2-\gamma^2)^{1/2}$.

Corollary 1. We show that the variance of a canonical variable cannot be made arbitrarily small at all moments of time. Suppose $T=0$ and $\sigma_{pq}^{(i)}=0$, and we ignore γ/ω. Then, for example, it follows from (19a) that

$$\sigma_q(t)=\sigma_q^{(eq)}(1-e^{-2\gamma t})+\frac{e^{-2\gamma t}}{2}\left[\sigma_q^{(i)}+\frac{1}{m^2\omega^2}\sigma_p^{(i)}\right]+\lambda\cos 2\omega t. \tag{19d}$$

Considering short times $\gamma t\ll 1$ (without $\omega t>\gamma t$) and using the uncertainty relation $(\sigma_q^{(i)}\sigma_p^{(i)})^{1/2}\geq\hbar/2$, we obtain on the average over a period ω^{-1} for the minimal estimate of $\sigma_q(t)$

$$\sigma_q(t)\approx\sigma_q^{(eq)}2\gamma t+\hbar/2m\omega,\quad \sigma_q^{(eq)}=\hbar/2m\omega. \tag{19e}$$

The first term in (19e) describes the fluctuations due to the interaction with the thermal bath, and its value decreases as $\gamma t\to 0$; physically, the second term is due to the quantum uncertainty of the initial conditions and cannot be reduced to zero by the reduction of the initial state.

By analogy with the case of pure systems, it is natural to assume that the best accuracy can be achieved when integrals of the motion are measured (for a discussion of integrals of the motion for open systems, see Ref. 26). If there is no external force, the integral corresponding to the coordinate $\hat q$ is the operator[26]

$$\hat I_q=e^{\gamma t}\left[\left(\cos\Omega t-\frac{\gamma}{\Omega}\sin\Omega t\right)\hat q-\frac{\sin\Omega t}{m\Omega}\hat p\right]. \tag{20}$$

Using formulas (19), we can obtain the following expression for σ_I [for simplicity, we set $\sigma_{pq}^{(i)}=0, \sigma_I(0)=\sigma_q^{(i)}$]:

$$\sigma_I(t)\approx\sigma_q^{(i)}+\frac{\hbar}{2m\omega}\operatorname{cth}\frac{\hbar\omega}{2kT}\left\{e^{2\gamma t}\left[1-\frac{2\gamma}{\Omega}\sin\Omega t\left(\cos\Omega t-\frac{\gamma}{\Omega}\sin\Omega t\right)\right]-1\right\}. \tag{21a}$$

Corollary 2. We show that the variance of the integral of the motion $\hat I_q(t)$ can be arbitrarily small for sufficiently small damping. Suppose $T=0$; then, assuming $\gamma t\ll 1$ and $\gamma/\omega\ll 1$, we obtain from (21a) the estimate

$$\sigma_I(t)=\sigma_q^{(i)}+\frac{\hbar}{m\omega}\gamma t. \tag{21b}$$

Choosing a state with known initial coordinate $\sigma_q^{(i)}=0$, we see that $\sigma_I(t)\to 0$ as $\gamma t\to 0$.

When an external force $f(t)$ acts, the expectation value $\langle\hat I_q\rangle$ no longer remains equal to $q^{(i)}$, but is shifted in time by an amount [which is the same for all states $W(q,p,t)$]

$$\delta(t)=\frac{1}{m\Omega}\int_0^t e^{\gamma\tau}[\sin\Omega\tau]f(\tau)d\tau,\quad \langle\hat I_q\rangle=q^{(i)}-\delta(t). \tag{22}$$

Measuring the value of $\hat I_q(t)$ continuously, we can in principle recover the form of the external force:

$$f(t)=-\frac{m\Omega}{\sin\Omega t}e^{-\gamma t}\frac{d}{dt}\langle\hat I_q(t)\rangle. \tag{23}$$

Before we estimate the accuracy of the measurement of $f(t)$, let us compare the quantum properties of the coordinate $\hat q(t)$ with the properties of the integral $\hat I_q(t)$. This can be readily done on the basis of Corollaries 1 and 2. A cardinal difference is the fact that the variance of $\hat q$ depends on both the initial variances of the canonical pair $\sigma_q^{(i)}$ and $\sigma_p^{(i)}$, whereas the variance of $\hat I_q(t)$ depends only on $\sigma_q^{(i)}$. This leads to the existence of a minimal variance $\sigma_q(t)=\hbar/2m\omega$ and to the possibility of the vanishing ($\gamma t\to 0$) of the variance σ_I by the reduction effect, i.e., by the choice of an initial state with known coordinate $q^{(i)}$ (or preparation of it in the process of the measurement itself). These properties are preserved for $f(t)\neq 0$ and do not depend on the form of the initial distribution $W(q,p,0)$.

We estimate the sensitivity to the external force in a measurement of $I_q(t)$, assuming fulfillment of the conditions $\gamma\ll\omega$ and $\gamma t\ll 1$, for which

$$\sigma_I(t)=\sigma_q^{(i)}+\frac{\hbar}{2m\omega}\left[\operatorname{cth}\frac{\hbar\omega}{2kT}\right]\cdot 2\gamma t. \tag{24}$$

In the case of a resonance force $f(t)=F_0\sin\Omega t$, an estimate in accordance with (22) gives

$$\delta(t)\approx F_0 t/2m\omega. \tag{25}$$

The minimal detectable amplitude $(F_0)_{min}$ is found by equating $|\delta(t)|$ and $\sigma_I^{1/2}(t)$. The result depends essentially on the choice of the initial state of the oscillator.

A. Before the measurement, the oscillator is in an equilibrium state, $\sigma_q^{(i)}=\sigma_q^{(eq)}$. In (16), the second term can be ignored; the smallest detectable force has the the limit

$$F_{min}^{(1)}=2t^{-1}\left(\frac{m\hbar\omega}{2}\operatorname{cth}\frac{\hbar\omega}{2kT}\right)^{1/2}=\begin{cases}\dfrac{2}{t}(mkT)^{1/2}, & kT\gg\hbar\omega\\ \dfrac{1}{t}(2m\hbar\omega)^{1/2}, & \hbar\omega\gg kT\end{cases}. \tag{26}$$

B. The oscillator is in a state with given coordinate $\sigma_q^{(i)}=0$. The sensitivity is increased by $(2\gamma t)^{-1/2}$ times:

$$F_{min}^{(2)}=(2\gamma t)^{1/2}F_{min}^{(1)}. \tag{27}$$

In the limit $\gamma\to 0$, an arbitrarily small force can be detected. In practice, there will evidently always be the possibility of error in the construction of $\hat{I}_q$ in the case of weak damping. However, if, as before, we form the integral of the motion (8) of an undamped oscillator instead of taking the exact operator (20), the resulting limit is fairly low, namely,

$$F_{min} \sim F_{min}^{(1)}(1/\sqrt{Q}) \to 0 \quad \text{as} \quad Q \to \infty.$$

Concluding this section, we emphasize once more the physical origin of the quantum limitations encountered in the measurement of an external force by the response of an oscillator. It resides in the quantum-mechanical uncertainty of the initial conditions $\hat{p}_0$ and $\hat{q}_0$. For a nondemolition variable such as an integral of the motion this uncertainty can be reduced by, for example, an initial accurate measurement.

§4. EXTENSION OF THE CLASS OF ADMISSIBLE VARIABLES

As was noted in the Introduction, it was already shown in Refs. 3 and 4 that there is no fundamental quantum limit to the detection of a classical force, so that, in principle, the possibilities of a gravitational antenna are unlimited. The existence of the limit (26) is only a consequence of an incorrect measuring procedure. However, a change of the procedure to an operation of "stroboscopic" type[4] or the construction of a "machine" to measure I_1 and I_2 (Ref. 3) appeared very complicated for practical use and did not have a passage to the limit of the classical methods of measurement. In this section, we show that there does exist a measuring procedure which is a quantum nondemolition procedure and links up with the prescriptions of the classical theory of optimal filtration.[27]

As we now know the physical reasons for the quantum sensitivity limit, it is natural to attempt to extend the class of variables that permit exact determination of an external force without restricting ourselves to just integrals of the motion. Namely, it is sensible to seek a variable that is free of uncertainty in the initial state of the oscillator but at the same time contains the reaction to an external force. The simplest example of such an observable is the difference between the coordinates of the oscillator at adjacent time instants:

$$\hat{y}(t, \tau) = \hat{q}(t+\tau) - \hat{q}(t), \tag{28}$$

the delay τ satisfying the condition $\Omega\tau = 2\pi l$, $l = 1, 2, \ldots$; $\gamma\tau \ll 1$.

Let us find the variance and the change of the expectation value under the influence of a force for $\hat{y}(t, \tau)$. Mathematically, it is convenient to make the calculations for Heisenberg operators. We take into account interaction with a thermal bath by introducing into the equations of motion of the operators the random forces $\hat{\varphi}(t)$ and $\hat{\theta}(t)$ (Langevin method):

$$\dot{\hat{q}} = m^{-1}\hat{p} + \hat{\varphi}(t), \quad \dot{\hat{p}} = -m\omega_0^2\hat{q} - 2\gamma\hat{p} + f(t) + \hat{\theta}(t). \tag{29}$$

The operator $\hat{\varphi}(t)$ is needed to conserve the commutator $[q(t), p(t)] = i\hbar$. In Ref. 23, it was shown that the system (29) is equivalent to the kinetic equation for the Wigner function (17) under the following conditions on the commutators and the correlation functions of $\hat{\varphi}$ and $\hat{\theta}$:

$$[\hat{\varphi}(t_1), \hat{\varphi}(t_2)] = [\hat{\theta}(t_1), \hat{\theta}(t_2)] = 0; \quad [\hat{\theta}(t_1), \hat{\varphi}(t_2)] = -2i\hbar\gamma\delta(t_1 - t_2); \tag{30a}$$

$$\langle \hat{\varphi}(t_1), \hat{\varphi}(t_2)\rangle = 0, \quad \langle \hat{\theta}(t_1), \hat{\theta}(t_2)\rangle = 2m\gamma\hbar\omega \operatorname{cth}(\hbar\omega/2kT)\delta(t_1 - t_2). \tag{30b}$$

In the case of a pure state in the absence of coupling between the oscillator and the thermal bath, $\gamma = 0$, calculation in accordance with (29) gives for the variance of the observable $\hat{y}$ the expression (we set $\sigma_{pq}^{(i)} = 0$):

$$\sigma_y(t, \tau) = 4\sin^2\frac{\Omega\tau}{2}\left[\sigma_q^{(i)} \sin^2\Omega\left(t + \frac{\tau}{2}\right) + \frac{\sigma_p^{(i)}}{m^2\Omega^2}\cos^2\Omega\left(t + \frac{\tau}{2}\right)\right]. \tag{31}$$

Since τ is taken to be a multiple of the oscillation period, it follows from (31) that $\sigma_y(t, \tau = 2\pi l/\Omega) \equiv 0$, i.e., the variable admits a precise measurement at any time. Allowance for damping leads to a limitation on the accuracy which vanishes in the limit $\gamma \to 0$. From (29) and (30),

$$\sigma_y(t, \tau) = (\gamma\tau)^2 e^{-2\gamma t}\left[\sigma_q^{(i)} \cos^2\Omega t + \frac{\sigma_p^{(i)}}{m^2\Omega^2}\sin^2\Omega t\right] + 2\gamma\tau\sigma_q^{(eq)}, \tag{32}$$

and $\sigma_q^{(i)}$ and $\sigma_p^{(i)}/m^2\Omega^2$ do not exceed $\sigma_q^{(eq)}$ in the cases in which we are interested in the measurement of a force, and therefore the variance σ_y is effectively independent of the time and the initial state of the oscillator and is determined solely by the thermal-bath fluctuations ($\gamma \ll \omega$, $\omega \sim \Omega$):

$$\sigma_y(t, \tau) = 2\gamma\tau\frac{\hbar}{2m\omega}\operatorname{cth}\frac{\hbar\omega}{2kT}. \tag{33}$$

At the same time, the expectation value of the operator $\hat{y}$ under the influence of the external resonance force $f(t) = F_0 \sin\Omega t$ changes by the amount

$$\langle \hat{y}(t, \tau)\rangle = -\frac{F_0\tau}{2m\Omega}\cos\Omega t. \tag{34}$$

Comparison of (33) and (34) for $t = \tau$ gives the sensitivity (27), which is equal to the maximal possible sensitivity for measurement of the integral of the motion $\hat{I}_q(t)$. However, this result is now independent of the initial state of the oscillator. The invariance of the variable $\hat{y}(t)$ with respect to the initial state of the system can be interpreted as follows. The quantum fluctuations of the operators $\hat{q}(t)$ and $\hat{q}(t+\tau)$ contain a strongly correlated part due to the uncertainty of the state at the "initial" time t provided $\tau \ll \tau^* = \gamma^{-1}$. The correlation leads to the disappearance of this fraction of the fluctuations when the difference operator $\hat{y}$ is formed, which renders it indifferent to the initial state. The variance σ_y is accumulated only by the interaction with the thermal bath and is therefore proportional to γ. Formally, $\hat{y}(t)$ satisfies the definition for a quantum nondemolition measurement: Its variance remains at the zero level (when $\gamma = 0$), the commutator vanishing, i.e., $[\hat{y}(t), \hat{y}(t')] \equiv 0$; at the same time, it is not an integral of the motion. The prescription for forming $\hat{y}(t)$ is different from the rule for constructing integrals of the motion and does not require us to follow $\hat{p}(t)$ as well as observing $\hat{q}(t)$. From the point of view of the philosophy of "measuring disturbances" (Sec. 2), the construction of $\hat{y}(t)$ is a procedure that effectively uses the correlation between the disturbances introduced by the acts of measurement at different sections of the trajectory of $\hat{q}(t)$.

To get an idea of an instrument that would realize $\hat{y}(t)$, it is sufficient to recall that, as is well known in radiophysics, a difference section with delay τ through which a random process $q(t)$ is transmitted is equivalent to filtration with spectral transfer coefficient

$$|K(j\omega)| \propto \sin[(\omega-\omega_0)\tau/2], \quad |(\omega-\omega_0)| \leqslant \Delta\omega \ll \omega_0; \tag{35}$$

here, ω_0 is the oscillator eigenfrequency corresponding to the center of the filter. Thus, $\hat{y}(t)$ can be measured by continuous observation of the coordinate $\hat{q}(t)$, but through a "light filter" with the characteristic (35). This is a measurement of the spectral Fourier components of the trajectory of $\hat{q}(t)$ that (roughly) lie outside the stop band of the filter, whose width is $\Delta\omega \sim \tau^{-1}$. Through such a filter, the observer is not in a position to obtain information on the trajectory of $\hat{q}(t)$ for $f=0$, since its spectral components then lie within the stop band if $\tau \ll \tau^*$. But when a force acts at times equal to or less than τ, some of the spectral components of $\hat{q}(t)$ cross the edge of the stop band and can be detected and measured. Note that measurements of an individual spectral component of the operator $\hat{q}(t)$ leave the remaining spectral components undisturbed.

It is now important to emphasize the following circumstance: The difference-section operator $\hat{y}(t)$ is only one possible example of a variable that does not depend on the quantum uncertainty of the initial state of the measured system. A general prescription for finding such operators is given by the theory of optimal filtration of weak signals, which has been generalized to the quantum case by Stratonovich and Grishanin,[12] and also Helstrom.[28] In particular, if a deterministic function $S(t)$ is measured on the background of a quantum Gaussian variable $\hat{\xi}(t)$ with commutator $[\hat{\xi}(t_i), \hat{\xi}(t_k)] = c_{ik}$($c_{ik}$ is a numerical matrix), an optimal observable can be constructed in accordance with the prescription (Heisenberg operators)[10]

$$\hat{u}(t) = S^T K^{-1}\hat{q}/S^T K^{-1} S \propto \int_0^t \int_0^t S(t_1) K^{-1}(t_1, t_2)\hat{q}(t_2)\,dt_1\,dt_2, \tag{36}$$

where

$$\hat{q}(t) = S(t) + \hat{\xi}(t), \quad K = {}^1/_2\langle \hat{\xi}(t_1)\hat{\xi}(t_2) + \hat{\xi}(t_2)\hat{\xi}(t_1)\rangle$$

is the correlation matrix of the quantum variable $\hat{\xi}(t)$. (The matrix form of expression arises when the continuous function $\hat{q}(t)$ is replaced by the discrete sequence (column) $\hat{q}_1 = \hat{q}(t_1), \hat{q}_2 = \hat{q}(t_2)\ldots$; S^T is the transposed vector row.) Simple calculations show that the observable $\hat{u}$ satisfies the quantum nondemolition requirement, i.e., $[\hat{u}(t_1), \hat{u}(t_2)] = 0$ for arbitrary t_1 and t_2.[5)] For a single harmonic oscillator coupled weakly to a thermal bath, the rule (36) [in the case of *a priori* information only about a band $\Delta\omega \sim \tau^{-1}$ of the spectrum of $S(t)$] reduces to the difference-section operation.[27] The results of application of (36) to a gravitational antenna of given structure (coupled oscillators, passive and active variants) are presented in Ref. 8. The prescriptions of optimal filtration are constructed as the solution to the extremal problem of the best separation of one statistical set [in our case, the deterministic process $S(t)$] on the background of another [the quantum Gaussian variable $\hat{\xi}(t)$] with maximal use of *a priori* information on the nature of both variables (spectra, correlation properties, and so forth). Use is hereby made of subtle differences in the spectral coloration of $S(t)$ and $\hat{\xi}(t)$, which makes possible a considerable advance in sensitivity compared with the rough integral criterion of "fluctuation back reaction".[5,10,11] The value of spectrally selective quantum measurement has also been recently noted in Ref. 29.

§5. CONCLUSIONS

1. In our opinion, the main result of the present paper is the following. We have identified the physical reasons why integrals of the motion are suitable variables for continuous precise measurement whereas an arbitrary operator is not. They concern the uncertainty of the initial parameters of the quantum system. A selection of operators that depend on an incomplete set of canonically conjugate variables, to be precise half of them, enables one through an initial reduction to eliminate this uncertainty and ensure subsequently a precise measurement. Elucidation of this circumstance has shown that it is helpful to adopt a detection procedure that does not depend on the initial parameters and, quite generally, the initial state of the measured system. We have seen that the classical optimal methods of separation of a weak signal from noise are based on such a rule. The problem of optimal filtration has been given a quantum generalization in Refs. 12 and 28. These rules can be fully applied to the separation of a signal from the quantum fluctuations of a gravitational antenna.[8]

2. We have noted a point of principle with regard to the theory of quantum nondemolition measurement. It is as follows. The continuous measurement of a canonical variable, the coordinate or momentum, is an example of a disturbative measurement with limited accuracy, whereas the continuous measurement of integrals of the motion $\hat{I}_q$ and $\hat{I}_p$ was defined[3] as nondisturbative and, therefore, precise, but under the condition of a special initial state of the system. This state is realized by an initial precise measurement. Therefore, measurement of integrals of the motion (and the "stroboscopic" method as a special case of this) could strictly be called a quasinondisturbative measurement, since the initial act of measurement is here in the general case disturbative. We have pointed out above that there exist variables with a known prescription for construction that are immediately and always strictly nondisturbative irrespective of the state in which the quantum system is. Thus, measurement of observables constructed in accordance with the rule (36), (28) of optimal quantum filtration can be called a strictly nondisturbative (nondemolition) measurement.

3. What now follows from this for gravitational-wave experiments? It was already known from Refs. 3 and 4 that there is in principle no quantum limitation on the sensitivity of a gravitational antenna; the smallness of the response of an antenna to a burst of gravitational waves is not the criterion of a quantum limitation on the possibility of detecting such bursts. However, it followed from Refs. 3 and 4 that it would be necessary to rearrange radically the measuring procedure (as compared with the traditional methods employed with first-generation antennas) with the aim of finding procedures with a proper quantum basis. Examples of

such procedures were seen in the implementation of the stroboscopic method (connecting and disconnecting the sensor to the gravitational detector in times much shorter than a period) or the construction of a machine measuring the real and imaginary components of the complex amplitude of an oscillator[3] and so forth. It is clear from the analysis made in this paper that there is no need for this, and it suffices to follow the recommendations of the theory of optimal filtration. For a gravitational antenna of Weber type, these recommendations were explained in Ref. 8.

4. We note that in principle a problem of quantum limitations on the measurement of weak classical forces, rather than coordinates, momenta, and so forth, does not exist. The point is that, by Ehrenfests theorem, any classical force $f(t)$ can be measured by measuring the mean values of the accelerations $\langle\ddot{x}\rangle$ and mean values $\langle\partial U/\partial x\rangle$ of functions of the coordinates, $f=m\langle\ddot{x}\rangle+\langle\partial U/\partial x\rangle$, and this can be done with arbitrary accuracy in the framework of quantum mechanics. This conclusion depends neither on the temperature or on the relationship between the work of the force and the distance between the energy levels of the system. Basically, the limitations on the accuracy of the measurement of a classical force $f(t)$ in the quantum case do not differ from the limitations in the classical treatment.

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[1)]In western literature, the abbreviation QNDM is used.

[2)]Note that the concept of "nondemolition measurement" was used by Elsasser [W. Elsasser, Phys. Rev. **52**, 987 (1937)] to describe the transition from a quantum to a classical measurement [see also L. N. Brillouin, Science and Information Theory, New York (1962), Ch. 16 (Russian translation published by Fizmatgiz, 1960)].

[3)]More precisely, "almost does not take place," since in the small the state of the object is always subject to reduction somewhere in the wings of the distribution function.

[4)]We should point out that throughout this paper we consider measurement of only one observable and not a set of different observables. In addition, for the sake of rigor, we restrict ourselves to Gaussian variables so as not to complicate the discussion by an analysis of the modifications that occur in the the higher moments.

[5)]The proof is based on the circumstance that the time commutator of $\hat{u}(t)$ can be reduced by a linear transformation to the time commutator of the quantum noise $\hat{\xi}(t)$, which vanishes in the absence of dissipation, $\gamma=0$.[8]

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