Static gravitational solitons

G. A. Alekseev

V. A. Steklov Mathematics Institute, USSR Academy of Sciences

V. A. Belinskiĭ

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences

(Submitted 24 September 1979)
Zh. Eksp. Teor. Fiz. **78**, 1297–1313 (April 1980)

The properties of exact soliton solutions of the gravitation equations in vacuum are considered for the case of static asymptotic plane gravitational waves. The structure of the singularities and of the horizons, as well as the behavior of the fields at infinity, are investigated. A classification of the solutions is proposed.

PACS numbers: 04.30. + x, 04.20.Jb

Zakharov[1] and one of us have described a technique of integrating the equations of gravitation in vacuum by the method of the inverse scattering problem, for stationary gravitational waves and axial symmetry. An exact $2n$-soliton solution of Einstein's equations was obtained in explicit form, and some of its general properties were considered. Here and below we designate the number of solitons in the solution by $2n$ (n is a natural number) order to emphasize the fact that this number should be even, if we require that the metric tensor have a physical signature and Euclidean properties at infinity of space (see Ref. 1 and Sec. 2 of the present paper). It was indicated in Ref. 1 that each pair of solitons in the constructed solution forms a bound state (if there are only two solitons, then this state corresponds to the Kerr solution), and the general $2n$-soliton solution describes in asymptotically flat space a stationary configuration consisting of any such two-soliton formations that interact with one another. For a sufficiently remote observer, the gravitational field of such a configuration can be regarded as an external field generated by n localized axisymmetric rotating objects, each with its own mass and its own mass center on the symmetry axis. The total mass of the field source is equal to the sum of the masses of the indicated objects, and the coordinate of their common mass center is given by the usual expression of particle mechanics. It was emphasized in Ref. 1 that this interpretation is applicable just from the point of view of a sufficiently remote observer, and generally speaking becomes meaningless in regions where horizons and singularities are located.

In the present paper we continue the investigation of Ref. 1 for the purpose of analyzing the internal (singular) regions of $2n$-soliton metrics. We point out immediately that in our desire to make our analysis lucid and to separate the principal and qualitative singularities of the solutions, we confine ourselves to the static case, i.e., to gravitational fields described by diagonal Weyl metrics. The qualitative character of many results is applicable also to a general stationary case with rotations, and this provides a good understanding of some main properties of the investigated solutions.

In the present paper we investigate the structure of singularities and horizons of static $2n$-soliton metrics, consider the causes and conditions of the equilibrium of the corresponding static configurations, and propose a scheme for classifying these solutions with the aid of clear and illustrative diagrams.

It is shown that in the general case the equilibrium conditions for a configuration of n 2-soliton particle-like formations calls for the presence, on the symmetry axis between them, of weak singularities (that do not manifest themselves in the invariants of the curvature), of the type of violation of local Euclidian behavior; this can be interpreted as the presence of "supports" between the masses. However, there exists also a class of multisoliton solutions that describe static configurations with positive total mass and do not contain such singularities. These solutions correspond to a set of 2-soliton particles, some of which have negative mass parameters (in the free state, these parameters should be interpreted as physical masses). It is clear from general considerations that under definite conditions such a configuration can have a positive total mass and be in an equilibrium state, without requiring the presence of "supports."

The difficulties with the interpretation of the internal regions of the Weyl metrics are well known. They stem from the contradiction between the equilibrium state of masses and the equations of motion, which require that these masses fall on each other. Naturally, the resolution of the contradiction calls for the presence of some singularities in the internal regions. Thus, only the external regions of the considered metrics are satisfactory from the physical point of view. They can describe external fields of static axisymmetric bodies, whose matter blocks the internal regions in which they substantially alter thereby the character of the solution.

Another result of the present paper is establishment of the fact that the Tomimatsu-Satu solution with zero angular momentum and with arbitrary integer distortion parameter is a particular case of $2n$-soliton static solutions. This is discussed in detail in Sec. 3.

We note in conclusion that this paper can be read without referring to Ref. 1 if the method of obtaining the solutions considered here is not of interest. We indicate also, to avoid misunderstandings, that the metric investigated here [formula (1)] was obtained directly from formulas (3.7), (3.8), and (5.1) of Ref. 1 as the $2n$-soli-

 0038-5646/80/040655-08$02.40

ton solution on a flat background. Formula (5.2) of Ref. 1 was not used to obtain this solution.

2. ASYMPTOTIC PLANE STATIC SOLITONS

The general $2n$-soliton static solution of the Einstein vacuum equations, in the presence of axial symmetry, is described by a diagonal metric that has, in the Weyl cylindrical coordinates, the following form[1]

$$ds^2=\frac{\mu_1\ldots\mu_{2n}}{\rho^{2n}}dt^2-\frac{\rho^{2n}}{\mu_1\ldots\mu_{2n}}\left\{C_0\rho^{6n^2}\left(\frac{\mu_1\ldots\mu_{2n}}{\rho^{2n}}\right)^{2n}\right.$$
$$\left.\times\left[\prod_{k,l=1}^{2n}(\rho^2+\mu_k\mu_l)\right]^{-1}(d\rho^2+dz^2)+\rho^2d\varphi^2\right\}, \tag{1}$$

where C_0 is an arbitrary real constant satisfying the requirement that the metric have a Lorentz signature, and the quantities μ_k are real or pairwise complex conjugate and are calculated from the formulas

$$\mu_k=W_k-z+\varepsilon_k[(W_k-z)^2+\rho^2]^{1/2},\quad k=1,2,\ldots,2n, \tag{2}$$

where W_k are arbitrarily chosen real or pairwise complex conjugate constants, and $\varepsilon_k=1$ or -1 is an arbitrary chosen sign.

Greatest interest from the physical point of view attaches to those metrics of type (1) which describe asymptotically plane waves. In accordance with the interpretation of $\{\rho,z,\varphi\}$ as cylindrical coordinates, we shall assume that the metric (1) should tend to a Lorentzian one at $(\rho^2+z^2)^{1/2}\to\infty$, and ascertain the limitations that this imposes on the choice of the constants in the metric (1). For simplicity we confine ourselves to consideration of only real μ_k, and consequently also real W_k.

As $\rho\to\infty$ and at finite z we obtain for μ_k the asymptotic equation

$$\mu_k=\varepsilon_k\rho\{1+(1/\rho)\varepsilon_k(W_k-z)+O(1/\rho^2)\} \tag{3}$$

and consequently

$$g_{00}=\frac{\mu_1\ldots\mu_{2n}}{\rho^{2n}}=\varepsilon_1\ldots\varepsilon_{2n}\left\{1+\frac{1}{\rho}\sum_{k=1}^{2n}\varepsilon_k(W_k-z)+O\left(\frac{1}{\rho^2}\right)\right\}. \tag{4}$$

Since the coefficient $1/r$ in the expansion of the Newtonian gravitational potential far from the source (r is the distance from the source) is a constant proportional to the total mass, and as $\rho\to\infty$ and at finite z the value of ρ agrees in first-order approximation with the distance to the source, the coefficient of $1/\rho$ in (4) should be a constant equal to $-2M$, where M is by definition the total mass of the source. Consequently, we get from (4)

$$\sum_{k=1}^{2n}\varepsilon_k=0,\quad \sum_{k=1}^{2n}\varepsilon_kW_k=-2M. \tag{5}$$

The first equation in (5) denotes that half of all the ε_k are equal to unity, and the other half to minus unity. The second condition allows us to calculate the total mass of the source from the given W_k and ε_k.

For convenience, we subdivide all the μ_k (and consequently all the W_k) into pairs with opposite signs ε_k, and replace the Latin indices ($k, 1, =1,2,\ldots 2n$) by the Greek indices ($\alpha,\beta,\gamma=1,\ldots,n$) that number these pairs. Thus, the aggregate of all the μ_k now turns into the aggregate $\mu_\gamma^\pm$, where the upper sign corresponds to the sign ε_k. We replace also each corresponding pair of constants $W_\gamma^\pm$ by a new pair of real constants z_γ and m_γ, and obtain

$$\mu_\gamma^\pm=W_\gamma^\pm-z\pm[(W_\gamma^\pm-z)^2+\rho^2]^{1/2},$$
$$W_\gamma^\pm=z_\gamma\mp m_\gamma, \tag{6}$$
$$M=\sum_{\gamma=1}^{n}m_\gamma.$$

As $\rho\to\infty$ the coefficient of $d\rho^2+dz^2$ in the curly brackets in the metric (1) should become equal to unity. From this condition and from expansions (3) and (4) we obtain

$$C_0=2^{2n^2}\prod_{\substack{\alpha=1\\ \beta=1}}^{n}(W_\alpha^+-W_\beta^-)^2$$
$$=2^{2n(n+1)}\left(\prod_{\gamma=1}^{n}m_\gamma^2\right)\prod_{\alpha>\beta}[(z_\alpha-z_\beta)^2-(m_\alpha+m_\beta)^2]^2. \tag{7}$$

The conditions (5) and (7) at real W_k are not only necessary but also sufficient for the metric (1) to be asymptotically flat everywhere at infinity (including the infinitely remote points of the axis $z=\pm\infty, \rho=0$).

The metric obtained in this manner has $2n$ arbitrary constants W_k. It is possible to subject these constants to one more condition, because of the leeway in the choice of the origin on the z axis, so that the general solution depends on $2n-1$ essential parameters. In addition, at fixed values of these parameters there is a leeway in the choice of the signs ε_k (subject to the only condition that the number of pluses and minuses be equal). As follows from the sequel, at a different choice of signs we obtain, generally speaking, solutions that differ in their physical interpretation.

Before we proceed to the interpretation of the various types, we investigate, following Ref. 1, the behavior of the field far from the sources. To this end we introduce for each pair $\mu_\gamma^\pm$ its own pair of spherical coordinates r_γ and θ_γ:

$$\rho=[r_\gamma(r_\gamma-2m_\gamma)]^{1/2}\sin\theta_\gamma,$$
$$z=z_\gamma+(r_\gamma-m_\gamma)\cos\theta_\gamma, \tag{8}$$

and also the "true" spherical coordinates r and θ:

$$\rho=[r(r-2M)]^{1/2}\sin\theta,\quad z=z_0+(r-M)\cos\theta, \tag{9}$$

where the constant M is the total mass introduced in (5) and (6), and z_0 will be additionally defined later from the condition that there be no dipole term in the expansion of the Newtonian potential as $r\to\infty$.

From (1), (2), and (8) we obtain for $\mu_\gamma^\pm$ and g_{00}

$$\mu_\gamma^\pm=\pm(r_\gamma-2m_\gamma)(1\mp\cos\theta_\gamma),$$
$$g_{00}=\prod_{\gamma=1}^{n}\frac{\mu_\gamma^+\mu_\gamma^-}{\rho^2}=(-1)^n\prod_{\gamma=1}^{n}\left(1-\frac{2m_\gamma}{r_\gamma}\right) \tag{10}$$

the coefficient $(-1)^n$ in the last expression is not significant, since it can be eliminated by reversing if necessary the sign in front of the entire interval (1).

Far from the sources (i.e., as $r\to\infty$) we can obtain from (8) and (9) the expansion

$$\frac{1}{r_\gamma}=\frac{1}{r}+\frac{M-m_\gamma+(z_\gamma-z_0)\cos\theta}{r^2}+\frac{q_\gamma}{2r^3}+O\left(\frac{1}{r^4}\right), \tag{11}$$

where

$$q_\gamma=2(M-m_\gamma)^2+4(M-m_\gamma)(z_\gamma-z_0)\cos\theta$$
$$+(M^2-m_\gamma^2)\sin^2\theta-(z_\gamma-z_0)^2(1-3\cos^2\theta).$$

For the Newtonian potential Φ of the field at infinity (which is connected with the component g_{00} by the formula $g_{00}=1-2\Phi$) we can obtain, using (10) and (11), an expansion that offers already some interpretation of the parameters that enter in the general solution

$$\Phi=\frac{M}{r}+\frac{Q_n}{r^3}(1-3\cos^2\theta)+O\left(\frac{1}{r^4}\right),$$
$$M=\sum_{\gamma=1}m_\gamma. \tag{12}$$

Here M is the total mass of the system and is made up of all the m_γ, which can be assumed to be the masses of the individual parts of the source system.

The absence of a dipole expansion term in (12) is due to the following choice of the constant z_0:

$$z_0=\sum_{\gamma=1}^{n}m_\gamma z_\gamma\Big/\sum_{\gamma=1}^{n}m_\gamma,$$

and corresponds to the concept of a mass-center whose position on the z axis is z_0, while the coordinates of the original part of the source with mass m_γ are z_γ.

The quadrupole moment Q_n of the $2n$-soliton field is calculated from the formula

$$Q_n=\sum_{\gamma=1}^{n}m_\gamma z_\gamma^2-Mz_0^2+\frac{1}{3}\left(\sum_{\gamma=1}^{n}m_\gamma^3-M^3\right).$$

The constructed expansion of the field far from the source, and the interpretation obtained on its basis for the parameters, do not, however, describe completely the character of the solutions, since they do not yield the structure and the properties of the source, (in particular, the presence of singularities, horizons, their single connectivity, etc.). In addition, it is not clear how to resolve the seeming contradiction between the presence, for example, of a 4-soliton solution in which two masses located on the axis with a certain distance between them are at equilibrium, on the one hand, and the law of motion of the bodies that follows from the field equations and forbids this equilibrium, on the other. The answer to these questions calls for a more detailed analysis, which will in fact be presented below.

3. CLASSIFICATION OF THE SOLITON SOLUTIONS

In the preceding section it was indicated that the considered solution with real parameters W_k can be subdivided into families of several types (between which, however, there exists a continuous transition with respect to the parameters W_k), in accordance with some formal attribute—the different choice of the signs ε_k in Eq. (2) for the quantities μ_k. It will be shown subsequently that this difference is not purely formal, and the obtained types of solutions have different physical interpretation.

From the general form of the metric (1) it is already clear (as confirmed by an investigation of the behavior of the invariants) that all the singularities of the investigated metrics, as well as the event horizons, can be located only where one of the following conditions is satisfied: $\rho=0$, $\mu_k=0$ (for certain k), and $\rho^2+\mu_k\mu_l=0$ (for certain k and l). Using expression (2) we can easily verify that each of these conditions can be satisfied only at the symmetry-axis points $\rho=0$.[1)] For our purposes it is therefore sufficient to study only the behavior of the field in the region near the axis $\rho=0$.

The behavior of each μ_k at small ρ is described by the formula

$$\mu_k=\mu_k^0(z)+\frac{\varepsilon_k\rho^2}{2|W_k-z|}+O(\rho^4),$$
$$\mu_k^0(z)=W_k-z+\varepsilon_k|W_k-z|. \tag{13}$$

This behavior of μ_k at different ε_k can be more clearly illustrated by the plots of Fig. 1. The hyperbolas $\mu_k(z)$ in Fig. 1a for different signs ε_k and at fixed $\rho\neq 0$ are crowded towards their asymptotes, and in the limit as $\rho=0$ they acquire the form of the broken lines indicated in Fig. 1b.

From (13) or from the form of the plots of Fig. 1 it follows that for each k we have $\mu_k=O(\rho^2)$ near the z-axis points is more positive than $z=W_k$ if $\varepsilon_k=1$, and more negative if $\varepsilon_k=-1$. Near the remaining points of the axis, μ_k have finite values, with $\mu_k=\mu_k^0(z)+O(\rho^2)$.

We consider a small vicinity of a certain point of the axis, with coordinate z. An important characteristic of this point is the total number of different μ_k that are of the order of $O(\rho^2)$ in the vicinity of this point. Let their number be s. Then the products of all the μ_k in the vicinity of this point will be of the order of $O(\rho^{2s})$. Since the coefficient of the metric is

$$g_{00}=\rho^{-2n}\prod_{k=1}^{2n}\mu_k,$$

it follows that in the vicinity of the chosen point, as the axis is approached (as $\rho\to 0$), g_{00} will have a finite value if $s=n$ (we call this axis point regular), and g_{00} tends to 0 if $s>n$. Consequently, the considered point of the axis is on the surface of an infinite red shift (which coincides in static fields with the horizon of events). Finally, g_{00} increases without limit if $s<n$, thus indicating the presence of some singularity in this place on the axis.[2)]

We now construct a diagram that makes it possible to represent clearly different possible types of fields. To

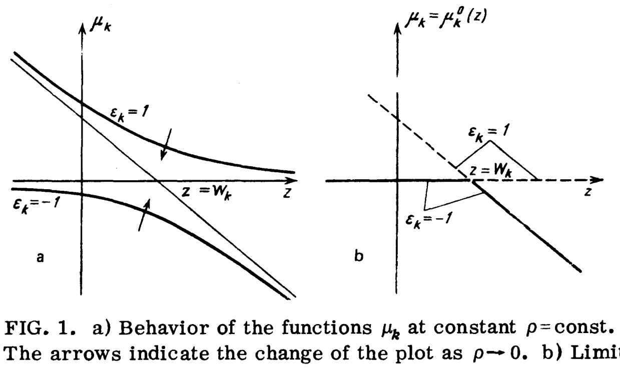

FIG. 1. a) Behavior of the functions μ_k at constant $\rho=$const. The arrows indicate the change of the plot as $\rho\to 0$. b) Limiting position of the $\mu_k(z)$ at $\rho=0$.

this end we place on the horizontal axis (the z axis) $2n$ points $z=W_k, k=1,2,\ldots,2n$ (or, in a different notation, n pairs of points $z=W_\gamma^\pm, \gamma=1,2,\ldots,n$). The vertical bars drawn from these points are directed up at the points $z=W_\gamma^+$ and down at the points $z=W_\gamma^-$, so that the direction of a bar corresponds to the choice of the sign in front of the square root in the expression for $\mu_\gamma^\pm$ [formula (6)]. Since, $\sum_{k=1}^{2n}\varepsilon_k=0$, in accordance with (4), the number of bars directed up and down from the axis is the same.

An arrow along the axis means that one of the μ_k at this location of the axis is of the order of $O(\rho^2)$. Inasmuch as according to (13) and Fig. 1 each μ_γ^+ is of the order of $O(\rho^2)$ at $z>W_\gamma^+$, and μ_γ^- is of the order of $O(\rho^2)$ at $z<W_\gamma^-$, each bar in the upper of lower half-plane of the diagram adds one arrow to the right or to the left at some point of the z axis. If s arrows are located in a given place on the axis, then g_{00} in the vicinity of these points of the axis is of the order of $O(\rho^{-2(n-s)})$.

Thus, by assigning definite (arbitrarily chosen) values of the real constants W_k and by fixing one of the possible variants of the choice of the signs ε_k, we place vertical bars on the diagram in the indicated manner, and then determine at each place of the z axis the number of arrows generated by them. Knowing the total number of arrows at a given place of the z axis, we can determine the behavior of g_{00} in the vicinity of these points and, consequently, ascertain whether these points of the axis are regular or whether they belong to the event horizon or to a singularity; this in turn makes it possible to describe the structure of the source of the field, and consequently assign a definite physical meaning to the solution.

We consider first the simplest case of 2-soliton solutions ($n=1$). Only the two types of diagram shown in Fig. 2 are possible here.

Diagram I of Fig. 2 corresponds to the Schwarzchild solution with positive mass and horizon, while diagram of type II corresponds to a solution with a bare singularity, obtained from the Schwarzchild solution with $M>0$ by reversing the sign of the mass. This follows from the fact that $W_1^+<W_1^-$ in the case of a diagram of type I, so that we can write $W_1^\pm=z_1\mp M$, where $M>0$. We then have from (10) $g_{00}=1-2M/r$. The horizon $r=2M$ corresponds, according to (9), to the z-axis segment between W_1^+ and W_1^-. In the case of the diagram II we have $W_1^-<W_1^+$, so that in the expression $W_1^\pm=z_1\mp M$ we must assume $M<0$ and the segment of the z axis between W_1^- and W_1^+ corresponds to a bare singularity ($r=0$ in the Schwarzschild solution at $M<0$).

In the more complicated case of 4-soliton solutions ($n=2$) we have six types of diagrams, which are shown in Fig. 3. The axis points where there are two arrows are regular and g_{00} has in them finite values. Where

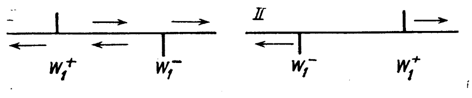

FIG. 2.

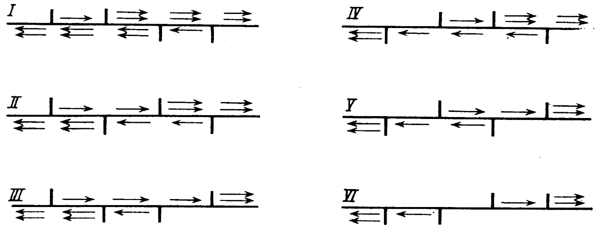

FIG. 3.

the number of arrows exceeds two, $g_{00}\to 0$ and we are dealing with a horizon of events. Where the number of arrows is less than 2, $g_{00}\to\infty$ and we encounter a bare singularity. The same types of solutions are conveniently represented in another manner—see Fig. 4. In Fig. 4 the hatches mark the location position of the horizons, and the sawtooth line marks the positions of the bare singularities. The remaining points are regular. We now examine these types separately.

Solutions of type I (Figs. 3 and 4) form a 3-parameter family whose characteristic feature is the presence of a connected horizon of events. It is interesting to note that this family, which is continuous in the parameters, includes in particular the Schwarzschild solution as well as the Tomimatsu-Sato solution[2] with zero angular momentum and with a distortion parameter $\delta=2$. In fact, as already stated, when a pair of W_k with different signs ε_k merge, the corresponding poles cancel each other and we obtain a solution with two fewer solitons. On the diagram this corresponds to coalescence and mutual cancellation of two oppositely directed bars. If a pair of bars merges on the diagrams of Fig. 2, this corresponds to $M\to 0$ and we obtain in the limit a flat space. If on diagram I of Fig. 3 the inner pair of oppositely directed bars coalesce (the inner pair W_k), then their mutual annihilation results in diagram I of Fig. 2, i.e., we have the Schwarzschild solution. If bars with identical directions coalesce pairwise in diagram I of Fig. 3, the resultant solution coincides with the Tomimatsu-Sato solution with zero angular momentum and $\delta=2$.

Solutions of type II (Fig. 3) contain two horizon events separated by a region of irregular behavior of the gravitational potential (the functions g_{00}). When the segment representing one of the horizons contracts to a point (this corresponds to the corresponding mass tending to zero), we obtain in the limit the Schwarzschild solution. It would therefore be natural to interpret solutions of this type as fields produced by two black holes that are at rest at a certain distance from each other. However, the existence of such a static solution contradicts patently the law of motion according to which these bodies

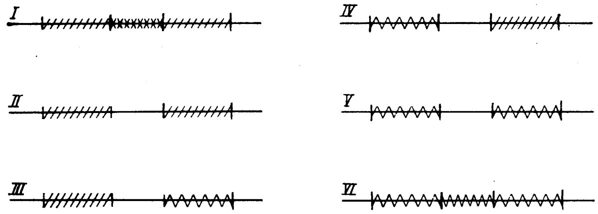

FIG. 4.

should fall on each other. The contradiction is eliminated by the fact that solutions of type II actually do not admit of such interpretation, since, just as in the known Curzon solution,[3] at the axis points located between the horizons (where the field should be regular) there the spatial metric is not locally Euclidean (the small circle surrounding the axis at this point and contracting to a point has a periphery-to-radius ratio not equal to the limit to 2π). This circumstance suggests, e.g., the presence of a "support" between the masses.[3]

Solutions of type III (Figs. 3 and 4) contain a horizon and a bare singularity. Physically they are hardly admissible, since they contain a bare singularity corresponding to a negative mass. We emphasize, however, that some of these solutions have a positive total mass, and this can correspond to a physically acceptable extended source.

Solutions of type IV are obtained from solutions of type III by reversing the direction of the z axis.

The remaining solutions (types V and VI) are the analogs of types I and II, and differ in that the masses of all the sources are negative. Reversal of the sign of each of the masses annihilates a corresponding horizon and replaces it by a bare singularity. The negativity of all the masses makes these solutions physically unacceptable.

In the general case of a $2n$-soliton solution, the number of arbitrary parameters of the solutions increases with increasing n (it is equal to $2n-1$), and the number of different types of fields that differ in the character of the source also increases. However, even at arbitrary n all the solutions can likewise by subdivided into a number of types in accordance with their physical nature.

The fields of the first type, just as in the 4-soliton case, are those having a connected horizon of events. With increasing n, the internal structure of the source becomes more complicated. Among the fields of this type are the Schwarzschild solution as well as the Tomimatsu-Sato solution with zero angular momentum and with a distortion parameter δ equal to n. The Tomimatsu-Sato solution is obtained when all the z_γ and all the m_γ are equal, i.e., when all the vertical upward bars merge on the diagram (all the W_γ^+ coincide and are equal to W^+), as do all the vertical downward bars (all the W_γ^- merge into W^-). As a result we obtain the diagram of the Schwarzchild type I (Fig. 2), the only difference being that both bars on it are now thick and are of order n (a thick bar of order n is defined as one obtained by coalescence of n ordinary equally directed vertical bars). Thus, the Tomimatsu-Sato solution with zero angular momentum depends on only one continuous arbitrary parameter (one of the two arbitrary constants W^+ and W^- can be eliminated by shifting the origin along the z coordinate), and of one discrete integer parameter n. This solution, just as the Schwarzchild solution, is of the 2-pole type (we recall that we are dealing with poles of a matrix Ψ function in the complex plane of a spectral parameter, see Ref. 1). It can be shown, however, that in contrast to the Schwarzchild case, both poles in the Tomimatsu-Sato solution are not simple but multiple. The multiplicity of both poles is the same and is precisely equal to the distortion parameter, $\delta=n$. This leads to the assumption that the general case of the Tomimatsu-Sato solution (with distortion) also apparently corresponds to a situation with two n-fold poles of the Ψ function and with a flat space-time as the background geometry.

The idea that the Tomimatsu-Sato solution corresponds to multiple poles of the Ψ function in the complex plane of the spectral parameter was first advanced by M. Francavigla.[3)]

The second type includes solutions with several somewhat separated horizons. To interpret these solutions as fields produced by a system of isolated bodies it is also necessary to postulate the presence of "supports."

Two other types of field are obtained when the signs of all the masses are reversed, as a result of which the horizons corresponding to positive masses are replaced by bare singularities with negative mass, so that these solutions are unphysical.

Finally, we include in the last type all the remaining solutions, which constitute fields produced by different combinations of sources of the types listed above, but with smaller values of n. As a rule, solutions of this type call for the presence of "supports." Exceptions are possible here however, and are of physical interest. The reason is that a system of bodies containing both positive and negative masses can be at equilibrium without requiring, obviously, the presence of any "supports." The attraction of the positive masses should in this case be offset by the repulsion by the negative masses. It is important to emphasize that on introducing "negative masses" we have in mind only the circumstance that some of the parameters m_γ in the solution can be negative. Actually the physical mass of a source is defined only for a sufficiently remote observer and is equal to the sum of all the m_γ. If this sum is positive, the corresponding solution can correspond to a physically reasonable situation. At the most, such a solution can describe the external field of a real body inside of which the metric has a different character.

The condition for the absence of any "supports," i.e., the condition for regularity of the metric outside the horizons and of the true singularities (in particular, the condition that the space metric be locally Euclid) leads to definite relations between the values of their masses and their relative distances. These relations can be interpreted as the general-relativity analog of the condition for the equilibrium of a system of bodies. We consider next a concrete example of such a solution and determine the corresponding equilibrium conditions.

Among the 6-soliton solutions there is one possessing the diagram shown in Fig. 5. This solution contains two horizons corresponding to positive masses (for simplicity we assume these masses to be identical and equal to M), and also a bare singularity of negative-mass $-m$, located halfway between them.

We derive first the condition for local Euclidean spatial metric at the regular points of the z axis (i.e., out-

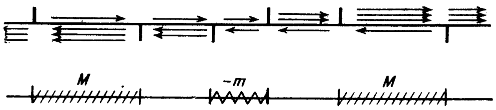

FIG. 5.

side the horizons and singularities) in a general $2n$-soliton case, and then apply this condition, which is the condition for the equilibrium of a system of sources, to the solution shown in the diagram of Fig. 5. We consider for this purpose a certain regular point of the z axis. At this point, g_{00} has a finite value and consequently in the vicinity of this point the number s of the quantities μ_k that are of the order of $O(\rho^2)$ is n (the number of arrows on the diagram at this point is also equal to n). We designate the parameters W_k corresponding to these μ_k by W_k° (which can include both W^+ and W^-), and remaining ones by W_k^*. The same symbols will be used also for the corresponding μ_k. We note that for semi-infinite regular sections of the z axis, the set of all the W_k° consists only of W_γ^+ on the positive section of the axis and of W_γ^- on the negative section of this axis.

When a certain small circle surrounding the z axis near a chosen regular point contracts to a point, the limit P_0 of the ratio of its periphery to the radius, multiplied by 2π, is given in accord with (1) by the expression

$$P_0^2=\lim_{\rho\to0}(g_{\varphi\varphi}/\rho^2g_{\rho\rho})=\lim_{\rho\to0}\left[C_0\rho^{6n^2}\left(\frac{\mu_1\ldots\mu_{2n}}{\rho^{2n}}\right)^{2n}\right]^{-1}\prod_{k,l=1}^{2n}(\rho^2+\mu_k\mu_l), \tag{14}$$

where C_0 is defined in (7).

As $\rho\to0$ we have, according to (13),

$$\mu_k^\circ=\rho^2\left\{\frac{1}{2(z-W_k^\circ)}+O(\rho^2)\right\},$$
$$\mu_k^\cdot=2(W_k^\cdot-z)+\frac{\rho^2}{2(W_k^\cdot-z)}+O(\rho^4). \tag{15}$$

In the vicinity of the regular points of the axis, where μ_k° is equal to n, we have

$$g_{00}=\frac{\mu_1\ldots\mu_{2n}}{\rho^{2n}}=\frac{\prod(W_k^\cdot-z)}{\prod(W_l^\circ-z)}+O(\rho^2), \tag{16}$$

where the product is calculated in the denominator over all W_k for which μ_k is $O(\rho^2)$ in the vicinity of the considered point of the axis, and in the numerator over all the remaining W_k.

For P_0^2 we can obtain from (14)–(16)

$$P_0^2=\frac{\prod(W_p^\cdot-W_q^\circ)^2}{\prod(W_k^+-W_l^-)^2}. \tag{17}$$

The product in the numerator is taken here over all p that number W_p^*, and at each fixed p—over all q that number W_q°. In the denominator, the product is calculated over all k that number W_k^+, and for each k—over all l that number W_l^-.

In order for the spatial metric to be local Euclidean at the regular points of the z axis, it is necessary to have $P_0=1$. It follows from (17) that $P_0^2=1$ on both semi-infinite regular sections of the z axis, inasmuch as in these sections the set of all the W_k° coincides with the set of all W_k^+ or with the set of all W_k^-, while w_k^* coincides respectively with W_k^- or W_k^+. In the intervals between the bodies, however, the equality $P_0=1$ does not hold in general. Postulating the satisfaction of this equality for each interval between the bodies on the axis, we obtain the sought equilibrium conditions:

$$\prod_{p,q}(W_p^\cdot-W_q^\circ)^2=\prod_{k,l}(W_k^+-W_l^-)^2. \tag{18}$$

The diagrams constructed above make it easy to subdivide all the W_k into W_k° and W_k^*, and also into W_k^+ and W_k^-. In fact, at each point $z=w_k$ we draw a vertical bar whose direction upwards (downwards) indicates that this W_k is $W_k^+(W_k^-)$. Next, to the right (in the case of W_k^+) or to the left (in the case of W_k^-) of each vertical bar we add at every point of the z axis one horizontal arrow each, designating thereby that the corresponding μ_k is of the order $O(\rho^2)$. Thus, after first choosing some point on the axis, we ascertain which W_k are due to the arrow corresponding to this z. These W_k make up in fact the set W_k° for the chosen point. The remaining W_k belong to the set W_k^*.

We calculate first the value of P_0^2 for the 4-soliton solution with two horizons (type II on Fig. 3 or 4). The points $z=W_k$ are subdivided into pairs $W_\gamma^\pm$ in the following manner ($W_1,\ldots,W_4$ are numbered in the order in which they follow each other along the z axis from left to right):

$$W_1=W_1^+,\quad W_2=W_1^-,\quad W_3=W_2^+,\quad W_4=W_2^-. \tag{19}$$

Introducing for each pair $W_\gamma^\pm$ new constants z_γ and m_γ, which characterize respectively the position of each of the horizons and the corresponding mass, we obtain

$$W_\gamma^\pm=z_\gamma\mp m_\gamma. \tag{20}$$

In the interval between the horizons on the diagram II of Fig. 3 there are two arrows, one of which due to W_1 and the other to W_4. Therefore $W_1=W_1^+$ and $W_4=W_2^-$ form the set W_k°, and the remaining $W_2=W_1^-$ and $W_3=W_2^+$ make up the set W_k^* for this interval. From (17), taking (19) and (20) into account, we obtain after cancelling identical factors

$$P_0^2=\frac{(W_2-W_4)^2(W_3-W_1)^2}{(W_1-W_4)^2(W_3-W_2)^2}=\left[\frac{a^2-(m_1-m_2)^2}{a^2-(m_1+m_2)^2}\right]^2\geqslant1,$$

where $a=z_2-z_1$ is the distance between the masses m_1 and m_2. Equality to unity takes place only when one of the masses vanishes, so that in the presence of two masses, $m_1>0$ and $m_2>0$, we have $P_0^2\neq1$ on the axis between them, and there is no local Euclidean behavior.

We now consider similarly the 6-soliton solution illustrated in Fig. 5. In the sequence along the z axis we have

$$W_1=W_1^+,\quad W_2=W_1^-,\quad W_3=W_2^-,\quad W_4=W_2^+,\quad W_5=W_3^+,\quad W_6=W_3^-,$$
$$W_1^\pm=z_1\mp M,\quad W_2^\pm=z_2\pm m,\quad W_3^\pm=z_3\mp M. \tag{21}$$

We place the zero of the coordinate z at the center of the singularity with negative mass and obtain thereby $z_2=0$. We denote the distances between the bodies by a, so that $z_3=-z_1=a$. Because of the symmetry in the entire system, the equilibrium conditions (the local Euclidean conditions) for both intervals between the bodies coincide. For the sake of argument we consider the right-hand interval. The set W_k° contains here W_1, W_4, and W_6, while the set W_k^* the points W_2, W_3, and W_5. Therefore the equilibrium conditions (18), with allowance for (21), take the form

$$(W_2-W_6)^2(W_3-W_6)^2(W_3-W_1)^2(W_5-W_4)^2$$
$$=(W_1-W_6)^2(W_4-W_6)^2(W_5-W_2)^2(W_5-W_3)^2$$

or

$$a^2[a^2-(m+M)^2]=(a^2-M^2)[a^2-(m-M)^2].$$

From this we easily calculate the distance between the bodies as a function of their masses, at which the following equilibrium takes place.

$$(a/M)^2=(M-m)^2/M(M-4m). \tag{22}$$

It follows from this expression that equilibrium is possible if $0<m/M<1/4$. At $m=0$, Eq. (22) yields $a=M$. This corresponds to vanishing of the bare singularity and to coalescence of the outermost horizons into one (the mass of the corresponding source is then $2M$). As m increases from zero to $M/4$, the value of a increases monotonically and tends to infinity as $m\to M/4$. This is in full agreement with the Newtonian limit, in which such a configuration is in a (neutral) equilibrium if and only if $m=M/4$, and the distance a can be arbitrarily large.

We note finally that the requirement $M>4m$, which follows from the equilibrium condition (22), ensures automatically a positive total mass of the source, equal to $2M-m$, for the considered 6-soliton solution.

4. INVARIANCE OF THE CURVATURE OF THE SOLITON SOLUTIONS

We have considered so far the behavior of the soliton solutions far from the sources, and also near the symmetry axis, and have shown that the metric coefficients of the soliton metrics (in the diagonal case) can have singularities only on the symmetry axis. We examine now the behavior of the invariants of the curvature tensor of these solutions.

The Riemann curvature tensor, which coincides in vacuum with the Weyl conformal curvature tensor, has in the case of vacuum two complex (or four real) independent algebraic invariants:

$$I_1=\overset{+}{R}_{ijkl}\overset{+}{R}{}^{ijkl},\quad I_2=\overset{+}{R}_{ij}{}^{kl}\overset{+}{R}_{kl}{}^{mn}\overset{+}{R}_{mn}{}^{ij},$$

where

$$\overset{+}{R}_{ijkl}=R_{ijkl}+i\overset{*}{R}_{ijkl},\quad \overset{*}{R}_{ijkl}={}^1/{}_2\varepsilon_{kl}{}^{mn}R_{ijmn},$$

R_{ijkl} is the Riemann tensor.

For static fields with axial symmetry these invariants can be expressed in terms of two real functions, $|P^2|$ and Q, using the formulas

$$I_1=8(|P|^2+3Q^2),\quad I_2=48(|P|^2-Q^2)Q, \tag{23}$$

where P is a certain complex function.

Calculation of the functions P and Q for $2n$-soliton metrics leads to the expressions

$$P=\frac{1}{8\rho^2g_{\rho\rho}}\{-2iB+A(A+1)(A+2)\}=\frac{1}{8\rho^2g_{\rho\rho}}P_1,$$
$$Q=\frac{1}{8\rho^2g_{\rho\rho}}\{A\bar{A}+A+\bar{A}\}=\frac{1}{8\rho^2g_{\rho\rho}}Q_1. \tag{24}$$

The complex functions A and B are of the form

$$A=\sum_{k=1}^{2n}\frac{\varepsilon_k(W_k-z+i\rho)}{[(W_k-z)^2+\rho^2]^{1/2}},$$

$$B=\rho\sum_{k=1}^{2n}\frac{\varepsilon_k(W_k-z+i\rho)}{(W_k-z-i\rho)[(W_k-z)^2+\rho^2]^{3/2}}. \tag{25}$$

From (23)–(25) it follows that the invariants I_1 and I_2 can become infinite only where at least one of the following conditions holds:

$$\rho^2g_{\rho\rho}=0,\quad A\to\infty,\quad B\to\infty.$$

The last two conditions, however, cannot be satisfied, since the functions A and B are bounded at all ρ and z. In fact, each term in the expression for A is a complex function with modulus unity:

$$\sum_{k=1}(\cos\alpha_k+i\sin\alpha_k),$$

where

$$\cos\alpha_k=\frac{\varepsilon_k(W_k-z)}{[(W_k-z)^2+\rho^2]^{1/2}}$$
$$\sin\alpha_k=\frac{\varepsilon_k\rho}{[(W_k-z)^2+\rho^2]^{1/2}},$$

from which it follows in fact that A is bounded. In the same notation, the function B takes the form

$$B=\sum_{k=1}^{n}\sin\alpha_k(\cos\alpha_k+i\sin\alpha_k)^2,$$

from which it is obvious that the function B is bounded.

Thus, the singularity in the invariance of the curvature should take place only under the condition $\rho^2g_{\rho\rho}=0$ which, as can be easily verified by using the explicit form of the metric (1), can be satisfied only on the symmetry axis. This is the necessary condition for the presence of a true singularity. Therefore to ascertain whether a singularity is present in this solution, it suffices to study the behavior of the invariance of the curvature near the symmetry axis points, i.e., as $\rho\to 0$.

We consider an arbitrary point of the axis with coordinate $z\neq W_k$. As $\rho\to 0$ putting $\rho\ll|W_k-z|$, we can expand the functions A and B in powers of ρ in the vicinity of this point. At $\rho=0$ we have

$$A=2(n-s),\quad B=0,$$

where s is the number of different μ_k (they were designated μ_k° above), which are of the order $O(\rho^2)$ in the vicinity of the considered point of the axis. These values are obtained for A and B from (25) if it is recognized that the corresponding μ_k° terms in (25) are equal to -1 at $\rho=0$ (a total of s terms), while the remaining $2n-s$ terms are equal to unity. The corresponding terms of the expansions for the functions A and B as $\rho\to 0$ and at $z\neq W_k$ are obtained directly from (25). We finally get the following expansions:

$$A=2(n-s)+i\rho A_1(z)+\rho^2A_2(z)+\ldots,$$
$$B=\rho A_1(z)-4i\rho^2A_2(z)+\ldots$$
$$A_1(z)=\sum_{k=1}^{2n}\frac{\varepsilon_k}{|W_k-z|},$$
$$A_2(z)=\sum_{k=1}^{2n}\frac{-\varepsilon_k}{2(W_k-z)|W_k-z|}. \tag{26}$$

For P_1 and Q_1 defined in (24) we have

$$P_1=4(n-s)(2n-2s+1)(n-s+1)+i\rho A_1\cdot 12(n-s)(n-s+1)$$
$$+\rho^2\{6A_2[2(n-s)(n-s+1)-1]-3A_1^2(2n-2s+1)\}+\dots,$$
$$Q_1=4(n-s)(n-s+1)+\rho^2[2A_2(2n-2s+1)+A_1^2]+\dots.$$

The coefficients of the metric (1) as $\rho\to 0$ and at $z\neq W_k$ have the following expansions:

$$g_{00}=\left(\frac{\rho}{2}\right)^{-2(n-s)}\frac{\prod(W_k^*-z)}{\prod(W_l^\circ-z)}[1+O(\rho^2)],$$
$$g_{\rho\rho}=\rho^{2(n-s)(n-s+1)}\frac{[1+O(\rho^2)]}{K(z)}, \quad (27)$$

where

$$K(z)=2^{2(n-s)(n-s+1)}\left[\frac{\prod(W_k^*-z)}{\prod(W_l^\circ-z)}\right]^{2(n-s)+1}\frac{\prod(W_m^*-W_n^\circ)^2}{\prod(W_p^+-W_q^-)^2}. \quad (28)$$

The product $\Pi(W_l^\circ-z)$ is calculated here over all W_l° for which μ_1 at a given z and $\rho\to 0$ is of the order of $O(\rho^2)$, and the product $\Pi(W_k^*-z)$ is calculated over all the remaining W_k (i.e., W_k^*).

From (24)–(27) we obtain expressions for the functions P and Q

$$P=\rho^{-2(n-s)(n-s+1)-2}K(z)\{4(n-s)(2n-2s+1)(n-s+1)$$
$$+12i\rho A_1(z)(n-s)(n-s+1)+\rho^2(6A_2(2(n-s)(n-s+1)-1)$$
$$-3A_1^2(2n-2s+1)]+\dots\}[1+O(\rho^2)],$$
$$(29)$$
$$Q=\rho^{-2(n-s)(n-s+1)-2}K(z)\{4(n-s)(n-s+1)$$
$$+\rho^2[2A_2(2n-2s+1)+A_1^2]+\dots\}[1+O(\rho^2)],$$

where $A_1(z)$ and $A_2(z)$ are defined in (26), and $K(z)$ in (28). We recall also that s is equal to the number of arrows on the diagram in the considered place on the axis.

An unbounded growth of the functions P and Q when a certain point of the axis is approached means that in this place the curvature invariants I_1 and I_2 become infinite, and consequently a true singularity is present there. We now examine the question of the presence of true singularities in soliton solutions, considering separately different values of s.

1) $s=n$. These points of the z axis were called regular, since the metric coefficients assume in their vicinity finite (nonzero) values (although local Euclidean behavior can exist at the points themselves). For P and Q we get from (29) in this case

$$P=-3K(z)(2A_2+A_1^2)[1+O(\rho)]<\infty,$$
$$Q=K(z)(2A_2+A_1^2)[1+O(\rho)]<\infty,$$

i.e., the curvature invariants are regular at these points.

2) $s<n$. According to (27) at these points $g_{00}\to\infty$, and we obtain for P and Q

$$P=\rho^{-2(n-s)(n-s+1)-2}K(z)\times 4(n-s)(n-s+1)(2n-2s+1)[1+O(\rho)]\to\infty,$$
$$Q=\rho^{-2(n-s)(n-s+1)-2}K(z)\times 4(n-s)(n-s+1)[1+O(\rho)]\to\infty.$$

In this case we encounter a true singularity.

3) $s=n+1$. At these points, according to (27), $g_{00}\to 0$, i.e., a horizon of events takes place. Then

$$P=K(z)(-6A_2+3A_1^2)[1+O(\rho)]<\infty,$$
$$Q=K(z)(-2A_2+A_1^2)[1+O(\rho)]<\infty.$$

The horizon turns out to be regular. (An example here can be the Schwarzchild solution shown on diagram I of Fig. 2.)

4) $s>n+1$. In this case $g_{00}\to 0$, i.e., we likewise are dealing with a horizon, but the values of P and Q increase here without limit

$$P=\rho^{-2(n-s)(n-s+1)-2}K(z)\times 4(n-s)(n-s+1)(2n-2s+1)[1+O(\rho)]\to\infty,$$
$$Q=\rho^{-2(n-s)(n-s+1)-2}K(z)\times 4(n-s)(n-s+1)[1+O(\rho)]\to\infty,$$

i.e., the horizon is singular in this case.

We note in conclusion that an analysis of the behavior of the invariants I_1 and I_2 far from the sources leads to the following expressions:

$$I_1=24\frac{M^2}{r^6}\left[1+O\left(\frac{1}{r}\right)\right],\qquad I_2=-48\frac{M^3}{r^9}\left[1+O\left(\frac{1}{r}\right)\right].$$

Since the fields under consideration are asymptotically plane, it follows, as expected, that the principal terms of the expansions of the invariants of the curvature are determined only by the total mass of the source

$$M=\sum_{\gamma=1}^{n}m_\gamma$$

and agree with the corresponding expressions for the invariants of the Schwarzchild field.

[1])An exception is the case when $W_k=W_l$ and $\varepsilon_k=-\varepsilon_l$, wherein $\rho^2+\mu_k\mu_l=0$ for all ρ and z. In this case, however, the $2n$-soliton solution reduces to a different one with two fewer solitons, so that when the parameters W_k of two pole trajectories μ_k with equal signs ε_k coincide, these poles (and the solitons corresponding to them) annihilate each other.

[2])We note here that the transition to the Weyl coordinates ρ and z is frequently effected with the aid of a transformation that degenerates on the exis, so that although all the singularities of the metrics do lie on the $\rho=0$ axis, they are not necessarily segments, and can consititue surfaces with different geometries.

[3])Oral communication from M. Francaviglia at the second meeting in the memory of M. Grossmann (Trieste, August 1979).

[1]V. A. Belinskii and V. E. Zakharov, Zh. Eksp. Teor. Fiz. **77**, 3 (1979) [Sov. Phys. JETP **50**, 1 (1979)].

[2]A. Tomimatsu and H. Sato, Phys. Rev. Lett. **29**, 1344 (1972).

[3]J. L. Synge, Relativity, the General Theory, Wiley-Interscience, 1960.

Translated by J. G. Adashko