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[1] M. Melamud, J. Makovsky, and H. Shaked, Phys. Rev. B 3, 3873 (1971).
[2] M. Kestigian, W. J. Croft, and F. D. Leipziger, J. Chem. Eng. Data 12, 97 (1967).
[3] R. W. Kedzie, J. R. Shane, and M. Kestigian, Bull. Am. Phys. Soc. 10, 315 (1965).
[4] I. Yamamoto, J. Phys. Soc. Japan 45, 1545 (1978).
[5] B. Ya. Kotyuzhanskiĭ and L. A. Prozorova, Zh. Eksp. Teor. Fiz. 62, 2199 (1972) [Sov. Phys. JETP 35, 1151 (1972)].
[6] P. G. de Gennes, P. A. Pinkus, F. Hartmann-Bourton, and J. M. Winter, Phys. Rev. 129, 1105 (1963).
[7] K. Lee, A. M. Portis, and G. L. Witt, Phys. Rev. 132, 144 (1963); G. L. Witt and A. M. Portis, Phys. Rev. 136, A1316 (1964).
[8] A. S. Borovik-Romanov, N. M. Kreĭnes, and L. A. Prozorova, Zh. Eksp. Teor. Fiz. 45, 64 (1963) [Sov. Phys. JETP 18, 46 (1964)].
[9] A. V. Andrienko and L. A. Prozorova, Zh. Eksp. Teor. Fiz. 74, 1527 (1978) [Sov. Phys. JETP 47, 798 (1978)].
[10] V. V. Kveder, B. Ya. Kotyuzhanskiĭ, and L. A. Prozorova, Zh. Eksp. Teor. Fiz. 63, 2205 (1972) [Sov. Phys. JETP 36, 1165 (1973)].
[11] B. Ya. Kotyuzhanskiĭ and L. A. Prozorova, Zh. Eksp. Teor. Fiz. 65, 2470 (1973) [Sov. Phys. JETP 38, 1233 (1974)].
[12] V. I. Ozhogin, Zh. Eksp. Teor. Fiz. 58, 2079 (1970) [Sov. Phys. JETP 31, 1121 (1970)].
[13] V. L. Sobolev, Candidate's Dissertation, Donets Physicotechnical Institute, 1973.
[14] V. I. Ozhogin and A. Yu. Yakubovsky, Phys. Lett. 43A, 505 (1973).
[15] V. S. Lutovinov, V. A. Preobrazhenskiĭ, and S. P. Semin, Zh. Eksp. Teor. Fiz. 74, 1159 (1978) [Sov. Phys. JETP 47, 609 (1978)].
[16] B. Ya. Kotyuzhanskiĭ and L. A. Prozorova, Pis'ma Zh. Eksp. Teor. Fiz. 24, 171 (1976) [JETP Lett. 24. 149 (1976)].
[17] B. Ya. Kotyuzhanskiĭ and L. A. Prozorova, Pis'ma Zh. Eksp. Teor. Fiz. 25, 412 (1977) [JETP Lett. 25, 385 (1977)].

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Nonphonon branches of the Bose spectrum in the *B* phase of systems of the He^3 type

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We investigate all 18 branches of the Bose spectrum in the He model, which are of the form $E^2 = \Omega^2 + \alpha k^2$. The frequencies $\Omega = 0$, $(8/5)^{1/2}\Delta$, $(12/5)^{1/2}\Delta$, and 2Δ and the coefficients α are calculated for all branches. It is shown that the branches with $\Omega = 2\Delta$ have complex dispersion coefficients α.

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1. INTRODUCTION

The Bose spectrum in the superfluid phases of He^3 consists of 18 branches (18 is the number of degrees of freedom of the complex tensor order parameter $A_{ij}, i, j = 1, 2, 3$). In the B phase of He^3 there are four phonon (Goldstone) branches-one acoustic, one longitudinal-spin-wave, and two transverse-spin-wave. The remaining 14 branches have in the B phase a gap at $k = 0$ and constitute different oscillations modes of the self-consistent field. At small k these branches are of the form

$$E^2(\mathrm{k}) = \Omega^2 + \alpha k^2. \quad (1.1)$$

The frequencies Ω were calculated in Refs. 1–5. The case of nonzero k was considered in Ref. 1 with the aid of the kinetic equation and in Ref. 2 using the Bethe-Salpeter equation. The branches with $\Omega = 2\Delta$, however, as well as some branches with $\Omega = (8/5)^{1/2}\Delta$, $(12/5)^{1/2}\Delta$ were not investigated accurate to k^2.

In this paper we investigate all the nonphonon branches of the Bose spectrum at small k in the B phase of the He^3 model. The model is defined by a "hydrodynamic action" functional obtained by functional integration with respect to the Fermi fields.[6] In first-order approximation, the spectrum of the Bose excitations is given by the quadratic part of the functional. Investigation of this part makes it possible to calculate the frequencies Ω and the coefficients α in (1.1) for all 14 nonphonon branches. We take special notice of the fact that the coefficients α corresponding to the four branches with $\Omega = 2\Delta$ turn out to be complex (and different for the different branches).

The plan of the paper is the following. In Sec. 2 we

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describe briefly the He^3 model and calculate the frequencies Ω for all 14 nonphonon branches. In Sec. 3 we obtain the coefficients α for the 10 branches with $\Omega=(8/5)^{1/2}\Delta$, $(12/5)^{1/2}\Delta$. The four branches with $\Omega=2\Delta$ and the complex coefficients α called for a special investigation (Sec. 4).

2. THE He^3 MODEL AND THE NONPHONON BRANCHES OF THE BOSE SPECTRUM

We consider the model system of the He^3 type proposed in Ref. 6. The collective Bose excitations in the system are determined by a functional of the hydrodynamic action

$$S_h=-\frac{1}{g}\sum_{p,i,a}c_{ia}^+(p)c_{ia}(p)+\frac{1}{2}\ln\det\hat{M}(c,c^+)/\hat{M}(0.0), \tag{2.1}$$

obtained after integration with respect to the Fermi fields. In (2.1), $c_{ia}(p)$ is the Fourier transform of the field $c_{ia}(\mathbf{x},\tau)$ with vector and isotopic indices i and a, respectively, and $\hat{M}$ is an operator:

$$\hat{M}=\begin{pmatrix}\dfrac{(i\omega-\xi+\mu H\sigma_3)\delta_{p_1p_2}}{Z} & \dfrac{(n_{1i}-n_{2i})\sigma_a c_{ia}(p_1+p_2)}{(\beta V)^{1/2}}\\ -\dfrac{(n_{1i}-n_{2i})\sigma_a c_{ia}^+(p_1+p_2)}{(\beta V)^{1/2}} & \dfrac{(-i\omega+\xi+\mu H\sigma_3)\delta_{p_1p_2}}{Z}\end{pmatrix} \tag{2.2}$$

Here $\xi=c_F(k-k_F)$, $n_i=k_i/k_F$, H is the magnetic field, μ is the magnetic moment of the quasiparticle, $\sigma_a(a=1,2,3)$ are Pauli matrices, Z is a normalization constant, $\beta^{-1}=T$, and $\omega=(2n+1)\pi T$ is the Fermi frequency. The negative constant g in (2.1) is proportional to the amplitude of the scattering of two fermions near the Fermi sphere under the assumption that the amplitude is equal to $g(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_3-\mathbf{k}_4)$, where $\mathbf{k}_1$ and $\mathbf{k}_2$ are the momenta of the incident fermions, and $\mathbf{k}_3$ and $\mathbf{k}_4$ those of the outgoing fermions. The method of obtaining the functional S_k is described in greater detail in Ref. 6.

The functional (2.1) contains all the information on the physical properties of the model system. We expand the functional ln det in (2.1) in the region $T_c-T\sim T_c$ in powers of the deviation of $c_{ia}(p)$ from the condensate value $c_{ia}^{(0)}(p)$, which is different for the different superfluid phases. We make the shift

$$c_{ia}(p)\to c_{ia}^{(0)}(p)+c_{ia}(p)$$

and separate from S_k the quadratic form

$$\sum_p c_{ia}^+(p)c_{jb}(p)A_{ijab}(p)+\frac{1}{2}\sum_p(c_{ia}(p)c_{jb}(-p)+c_{ia}^+(p)c_{jb}^+(-p))B_{ijab}(p). \tag{2.3}$$

It is this form which determines in first approximation the Bose spectrum obtained from the equation

$$\det Q=0. \tag{2.4}$$

Here Q is the matrix of the quadratic form and is determined by the coefficient tensors A_{ijab}, B_{ijab}.

The quadratic form (2.3) is different for different superfluid phases. Its explicit expressions for the B, A, and $2D$ phases were given in our preceding paper.[7] We need here the formula for the quadratic form (2.3) in the B phase

$$\sum_p c_{ia}^+(p)c_{jb}(p)\left[\frac{\delta_{ij}}{g}+\frac{4}{\beta V}\sum_{p_1+p_2=p}n_{1i}n_{1j}\varepsilon(-p_1)\varepsilon(-p_2)G(p_1)G(p_2)\right]$$
$$-\frac{1}{2}\sum_p[c_{ia}(p)c_{jb}(-p)+c_{ia}^+(p)c_{jb}^+(-p)]$$
$$\times\frac{4\Delta^2}{\beta V}\sum_{p_1+p_2=p}n_{1i}n_{1j}(2n_{1a}n_{1b}-\delta_{ab})G(p_1)G(p_2), \tag{2.5}$$

where

$$\varepsilon(p)=i\omega-\xi,\quad G(p)=Z(\omega^2+\xi^2+\Delta^2)^{-1}. \tag{2.6}$$

Following the substitution

$$c_{ia}(p)=u_{ia}(p)+iv_{ia}(p),\quad c_{ia}^+(p)=u_{ia}(p)-iv_{ia}(p) \tag{2.7}$$

expression (2.5) breaks up into two independent forms, one of which depends on u_{ia} and the other on v_{ia}:

$$-\sum_p(A_{ij}u_{ia}u_{ja}+B_{ijab}u_{ia}u_{jb})-\sum_p(A_{ij}v_{ia}v_{ja}-B_{ijab}v_{ia}v_{jb}). \tag{2.8}$$

The coefficient tensors $A_{ij}(p)$, $B_{ijab}(p)$ are proportional to integrals of products of fermion Green's functions. We calculate them by the Feynman procedure based on the identity

$$[(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)]^{-1}$$
$$=\int_0^1 d\alpha[\alpha(\omega_1^2+\xi_1^2+\Delta^2)+(1-\alpha)(\omega_2^2+\xi_2^2+\Delta^2)]^{-2}. \tag{2.9}$$

Using (2.9), we express $A_{ij}(p)$ in the form

$$A_{ij}(p)=-\frac{4Z^2}{\beta V}\sum_{p_1+p_2=p}n_{1i}n_{1j}\int_0^1 d\alpha$$
$$\times\left[\frac{(\xi_1+i\omega_1)(\xi_2+i\omega_2)}{[\alpha(\omega_1^2+\xi_1^2)+(1-\alpha)(\omega_2^2+\xi_2^2)+\Delta^2]^2}-\frac{1}{\omega_1^2+\xi_1^2+\Delta^2}\right], \tag{2.10}$$

with the term δ_{ij}/g of $A_{ij}(p)$ eliminated with the aid of the identity

$$\frac{\delta_{ij}}{g}+\frac{4Z^2}{\beta V}\sum_{p_1}n_{1i}n_{1j}(\omega_1^2+\xi_1^2+\Delta^2)^{-1}=0, \tag{2.11}$$

which determines the gap Δ.

Considering the limit $T\to 0$, we change in (2.10) to integration near the Fermi sphere in accordance with the rule

$$(\beta V)^{-1}\sum_{p_1}\to k_F^2(2\pi)^{-4}c_F^{-1}\int d\omega_1\,d\xi_1\,d\Omega_1,$$

where $\int d\Omega_1$ is an integral with respect to the angle variables. We then calculate directly the integrals with respect to ω_1 and ξ_1. We arrive at the formula

$$A_{ij}(p)=\frac{Z^2k_F^2}{4\pi^3c_F}\int_0^1 d\alpha\int d\Omega_1 n_{1i}n_{1j}\left[\ln\left(1+\frac{\alpha(1-\alpha)(\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2)}{\Delta^2}\right)\right.$$
$$\left.+\frac{\Delta^2+2\alpha(1-\alpha)(\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2)}{\Delta^2+\alpha(1-\alpha)(\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2)}\right]. \tag{2.12}$$

A similar procedure yields for B_{ijab},

$$B_{ijab}(p)=\frac{Z^2k_F^2}{4\pi^3c_F}\int_0^1 d\alpha\int d\Omega_1 n_{1i}n_{1j}(2n_{1a}n_{1b}-\delta_{ab})\cdot\frac{\Delta^2}{\Delta^2+\alpha(1-\alpha)(\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2)}. \tag{2.13}$$

Our aim being an investigation of the nonphonon spectrum, we consider $A_{ij}(p)$, $B_{ijab}(p)$ at small k, but for nonzero ω. We put first $\mathbf{k}=0$ in (2.12) and (2.13). Then the integrals with respect to the angle variables and with with respect to the parameter α separate and can be easily calculated:

$$\int d\Omega_1 n_{1i} n_{1j} = {}^4/{}_3 \pi \delta_{ij},$$

$$\int d\Omega_1 n_{1i} n_{1j} (2n_{1a} n_{1b} - \delta_{ab}) = {}^4/{}_{15}\pi (2\delta_{ai}\delta_{bj} + 2\delta_{aj}\delta_{bi} - 3\delta_{ab}\delta_{ij}); \quad (2.14)$$

$$\int_0^1 d\alpha \left[\ln\left(1+\frac{\alpha(1-\alpha)\omega^2}{\Delta^2}\right) + \frac{\Delta^2+2\alpha(1-\alpha)\omega^2}{\Delta^2+\alpha(1-\alpha)\omega^2} \right] = f(\omega),$$

$$\int_0^1 \frac{\Delta^2 d\alpha}{\Delta^2+\alpha(1-\alpha)\omega^2} = g(\omega), \quad (2.15)$$

where

$$f(\omega) = (\omega^2+2\Delta^2) h(\omega), \quad g(\omega) = 2\Delta^2 h(\omega),$$

$$h(\omega) = [\omega(\omega^2+4\Delta^2)^{1/2}]^{-1} \ln \frac{(\omega^2+4\Delta^2)^{1/2}+\omega}{(\omega^2+4\Delta^2)^{1/2}-\omega}. \quad (2.16)$$

Substituting (2.14)–(2.16) in (2.12) and (2.13) we obtain the first-approximation formulas for $A_{ij}(p)$, $B_{ijab}(p)$:

$$A_{ij}(p) = \frac{Z^2 k_F^2}{3\pi^2 c_F} \delta_{ij} f(\omega), \quad (2.17)$$

$$B_{ijab}(p) = \frac{Z^2 k_F^2}{15\pi^2 c_F} (2\delta_{ai}\delta_{bj} + 2\delta_{aj}\delta_{bi} - 3\delta_{ab}\delta_{ij}) g(\omega). \quad (2.18)$$

Substituting them in (2.8), we arrive at the quadratic form

$$-\frac{Z^2 k_F^2}{15\pi^2 c_F} \sum_p \{5f(\omega)(u_{ia}^2+v_{ia}^2) + g(\omega)[2(u_{ia}u_{ai} - v_{ia}v_{ai}) + 2(u_{aa}u_{ii} - v_{aa}v_{ii}) - 3(u_{ia}^2 - v_{ia}^2)]\}. \quad (2.19)$$

From this equation we easily obtain the frequencies of all 14 nonphonon branches.

The expression under the sign of summation with respect to p breaks up into several independent forms with the variables

$$(u_{12}, u_{21}), \quad (u_{23}, u_{32}), \quad (u_{31}, u_{13}), \quad (u_{11}, u_{22}, u_{33}),$$
$$(v_{12}, v_{21}), \quad (v_{23}, v_{32}), \quad (v_{31}, v_{13}), \quad (v_{11}, v_{22}, v_{33}). \quad (2.20)$$

The form with the variables (u_{12}, u_{21}) is given by

$$[5f(\omega) - 3g(\omega)](u_{12}^2 + u_{21}^2) + 4g(\omega) u_{12} u_{21}. \quad (2.21)$$

Equating its determinant to zero, we get

$$5[f(\omega) - g(\omega)][5f(\omega) - g(\omega)] = 0. \quad (2.22)$$

If the first factor in (2.22) is equal to zero, we arrive at the equation $\omega^2 h(\omega) = 0$, which yields the branch $E^2 = 0$. This is one of the phonon branches equal to zero at $\mathbf{k}=0$. The vanishing of the second factor in (2.22) leads to the equation $(5\omega^2 + 8\Delta^2)h(\omega) = 0$ and yields the branch $E^2 = 8\Delta^2/5$. The forms with the variables (u_{23}, u_{32}), (u_{31}, u_{13}) have at $\mathbf{k}=0$ the same coefficients as (2.21). They yield two more branches $E^2=0$ and two branches $E^2 = 8\Delta^2/5$.

We consider now the form with (u_{11}, u_{22}, u_{23}):

$$[5f(\omega) + g(\omega)](u_{11}^2 + u_{22}^2 + u_{33}^2) + 4g(\omega)(u_{11}u_{22} + u_{22}u_{33} + u_{33}u_{11}). \quad (2.23)$$

Equating its determinant to zero we obtain

$$[5f(\omega) - g(\omega)]^2 5[f(\omega) + g(\omega)] = 0. \quad (2.24)$$

The equation $(5f-g)^2 = 0$ gives two more branches $E^2 = 8\Delta^2/5$, while $f+g=0$ is equivalent to the equation $(\omega^2 + 4\Delta^2)h(\omega) = 0$ and gives the branch $E^2 = 4\Delta^2$. The u variable give thus five branches $E^2 = 8\Delta^2/5$, three branches $E^2 = 0$, and one branch $E^2 = 4\Delta^2$.

We examine now the v branches. The forms with the variables (v_{12}, v_{21}), (v_{23}, v_{32}), and (v_{31}, v_{13}) are the same at $\mathbf{k}=0$. The coefficients of these forms differ from the corresponding coefficients of the u forms by the substitution $g(\omega) \to -g(\omega)$. Therefore the counterpart of Eq. (2.22) is of the form

$$5[f(\omega) + g(\omega)][5f(\omega) + g(\omega)] = 0. \quad (2.25)$$

The equality $f+g=0$ yields three branches $E^2 = 4\Delta^2$ (one each for the three forms), while $5f+g=0$ is equivalent to $(5\omega^2 + 12\Delta^2)h(\omega) = 0$ and yields the three branches $E^2 = 12\Delta^2/5$.

The equation corresponding to the form with (v_{11}, v_{22}, v_{33}), is obtained from (2.24) by making the substitution $g(\omega) \to -g(\omega)$ and its form is

$$[5f(\omega) + g(\omega)]^2 5[f(\omega) - g(\omega)] = 0. \quad (2.26)$$

The equation $(5f+g)^2 = 0$ gives two more branches $E^2 = 12\Delta^2/5$, and $f-g$ gives $E^2 = 0$, corresponding to a phonon (acoustic) branch that vanishes at $\mathbf{k}=0$. The v variable thus yield five branches $E^2 = 12\Delta^2/5$, three branches $E^2 = 4\Delta^2$, and one branch $E^2 = 0$.

We write down all the obtained frequencies with together with their corresponding variables:

$$\Omega^2=0: \quad u_{12}-u_{21}, \quad u_{23}-u_{32}, \quad u_{31}-u_{13}, \quad v_{11}+v_{22}+v_{33};$$
$$\Omega^2=4\Delta^2: \quad v_{12}-v_{21}, \quad v_{23}-v_{32}, \quad v_{31}-v_{13}, \quad u_{11}+u_{22}+u_{33};$$
$$\Omega^2=8\Delta^2/5: \quad u_{12}+u_{21}, \quad u_{23}+u_{32}, \quad u_{31}+u_{13}, \quad u_{11}-u_{22}, \quad u_{11}+u_{22}-2u_{33};$$
$$\Omega^2=12\Delta^2/5; \quad v_{12}+v_{21}, \quad v_{23}+v_{32}, \quad v_{31}+v_{13}, \quad v_{11}-v_{22}, \quad v_{11}+v_{22}-2v_{33}. \quad (2.27)$$

We note that the branches with $\Omega = 0$ and $\Omega = 2\Delta$ are "dual" in the sense that the variables corresponding to them differ by the substitution $u_{ia} \rightleftarrows v_{ia}$. In this sense, the branches with $\Omega = (8/5)^{1/2}\Delta$ and $\Omega = (12/5)^{1/2}\Delta$ are dual. We note also that the sum of the squares of the frequencies of dual branches is equal to $4\Delta^2$.

3. BRANCHES WITH $\Omega = (8/5)^{1/2}\Delta$ AND $(12/5)^{1/2}\Delta$ AT SMALL k

We calculate now the nonphonon branches of the spectrum with corrections $\sim k^2$. The most labor-consuming is the analysis of the branches with $\Omega = 2\Delta$, in which the coefficients of k^2 are complex. We consider therefore first the branches other than $\Omega = 2\Delta$, in which the the corrections to the spectrum can be obtained by expanding the coefficient tensors $A_{ij}(p)$, $B_{ijab}(p)$ in powers of k^2, confining ourselves to terms $\sim k^2$. We have

$$\int_0^1 d\alpha \left[\ln\left(1+\frac{\alpha(1-\alpha)(\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2)}{\Delta^2}\right) + \frac{\Delta^2+2\alpha(1-\alpha)[\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2]}{\Delta^2+\alpha(1-\alpha)[\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2]} \right]$$
$$= f(\omega) + \frac{df(\omega)}{d\omega^2} c_F^2 (\mathbf{n}_1\mathbf{k})^2 + O(k^4),$$
$$\int_0^1 \frac{\Delta^2 d\alpha}{\Delta^2+\alpha(1-\alpha)(\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2)} = g(\omega) + \frac{dg(\omega)}{d\omega^2} c_F^2 (\mathbf{n}_1\mathbf{k})^2 + O(k^4). \quad (3.1)$$

We can thus obtain the corrections to the quadratic form

(2.19). Calculating the integrals, corresponding to this correction, with respect to the angle variables,

$$\int d\Omega_1 n_{1i}n_{1j}(\mathbf{n}_1\mathbf{k})^2 = {}^4/_{15}\pi(k^2\delta_{ij}+2k_ik_j), \tag{3.2}$$

$$\int d\Omega_1 (2n_{1a}n_{1b}-\delta_{ab})n_{1i}n_{1j}(\mathbf{n}_1\mathbf{k})^2 = {}^4/_{105}\pi[k^2(2\delta_{ai}\delta_{bj}+2\delta_{aj}\delta_{bi}-5\delta_{ab}\delta_{ij}) +4k_ak_b\delta_{ij}-10k_ik_j\delta_{ab}+2k_ak_i\delta_{bj}+2k_bk_j\delta_{ai}+2k_ak_j\delta_{bi}+2k_bk_i\delta_{aj}],$$

we write the correction to (2.19) in the form

$$-\frac{k_F^2Z^2c_F}{15\pi^2}\sum_p\Big\{\frac{df(\omega)}{d\omega^2}[k^2(u_{ia}^2+v_{ia}^2)+2k_ik_j(u_{ia}u_{ja}+v_{ia}v_{ja})] + \frac{1}{7}\frac{dg(\omega)}{d\omega^2}[k^2(2(u_{aa}u_{bb}-v_{aa}v_{bb})+2(u_{ia}u_{ai}-v_{ia}v_{ai})-5(u_{ia}^2-v_{ia}^2)) +k_ik_j(4(u_{ai}u_{aj}-v_{ai}v_{aj})-10(u_{ia}u_{ja}-v_{ia}v_{ja}) +8(u_{ij}u_{aa}-v_{ij}v_{aa})+8(u_{ia}u_{aj}-v_{ia}v_{aj}))]\Big\}. \tag{3.3}$$

Since the B phase is isotropic and there is no preferred direction in it, it suffices to consider excitations that propagate in an arbitrary direction, say along the third axis. Following the substitutions $k_1 = k_2 = 0$ and $k_3 = k$, the correction (3.3) breaks up into a sum of forms with the same variables (2.20) as the main form (2.19). Adding (2.19) and (3.3), we obtain (at $k_1 = k_2 = 0$, $k_3 = k$) under the $\sum_p$ sign the sum of the following forms:

$$(w_{12}^2+w_{21}^2)\Big[5f(\omega)\mp 3g(\omega)+c_F^2k^2\Big(\frac{df}{d\omega^2}\mp\frac{5}{7}\frac{dg}{d\omega^2}\Big)\Big] \pm 4w_{12}w_{21}\Big(g(\omega)+\frac{1}{7}c_F^2k^2\frac{dg}{d\omega^2}\Big),$$

$$w_{31}^2\Big[5f(\omega)\mp 3g(\omega)+c_F^2k^2\Big(3\frac{df}{d\omega^2}\mp\frac{15}{7}\frac{dg}{d\omega^2}\Big)\Big] + w_{13}^2\Big[5f(\omega)\mp 3g(\omega)+c_F^2k^2\Big(\frac{df}{d\omega^2}\mp\frac{1}{7}\frac{dg}{d\omega^2}\Big)\Big] \pm 4w_{31}w_{13}\Big(g(\omega)+\frac{3}{7}c_F^2k^2\frac{dg}{d\omega^2}\Big), \tag{3.4}$$

$$(w_{11}^2+w_{22}^2)\Big[5f(\omega)\pm g(\omega)+c_F^2k^2\Big(\frac{df}{d\omega^2}\mp\frac{1}{7}\frac{dg}{d\omega^2}\Big)\Big] + w_{33}^2\Big[5f(\omega)\pm g(\omega)+c_F^2k^2\Big(3\frac{df}{d\omega^2}\pm\frac{9}{7}\frac{dg}{d\omega^2}\Big)\Big]$$

$$\pm 4w_{11}w_{22}\Big(g(\omega)+\frac{1}{7}c_F^2k^2\frac{dg}{d\omega^2}\Big)\pm 4w_{33}(w_{11}+w_{22})\Big(g(\omega)+\frac{3}{7}c_F^2k^2\frac{dg}{d\omega^2}\Big).$$

We must put here $w_{ia} = u_{ia}$ and take the upper signs $\pm,\mp$, or else put $w_{ia} = v_{ia}$ and take the lower signs. The forms with the variables w_{32}, w_{23} are obtained from the second form of (3.4) by making the replacement $(w_{31}, w_{13}) \to (w_{32}, w_{23})$.

It is easy to obtain from (3.4) all the spectrum branches of interest to us (except $\Omega = 2\Delta$) with corrections $\sim k^2$. Consider, for example, the first of the forms (3.4). Putting $w_{12} = u_{12}$, $w_{21} = u_{21}$ (taking the upper signs in $\pm,\mp$), we equate to zero the determinant of the form

$$5\Big[f(\omega)-g(\omega)+\frac{c_F^2k^2}{5}\frac{d}{d\omega^2}(f(\omega)-g(\omega))\Big] \times\Big[5f(\omega)-g(\omega)+c_F^2k^2\frac{d}{d\omega^2}(f(\omega)-{}^3/_7 g(\omega))\Big]=0. \tag{3.5}$$

Equating to zero the first factor in (3.5) yields

$$\omega^2h(\omega)+\frac{c_F^2k^2}{5}\frac{d}{d\omega^2}\omega^2h(\omega)=0,$$

or

$$\omega^2+\frac{c_F^2k^2}{5}+\omega^2\frac{c_F^2k^2}{5}\frac{d\ln h(\omega)}{d\omega^2}=0.$$

The last term here is of higher order, since $d\ln h(\omega)/d\omega^2$ is finite as $\omega\to 0$. As a result we obtain the phonon branch of the longitudinal spin oscillations:

$$E^2=c_F^2k^2/5. \tag{3.6}$$

All the remaining phonon branches can also be obtained in the approach described here when considering the forms (u_{13}, u_{31}), (u_{23}, u_{32}), (v_{11}, v_{22}, v_{33}).

Equating to zero the second factor in (3.5), we obtain the equation

$$(5\omega^2+4\Delta^2)h(\omega)+\frac{c_F^2k^2}{7}\frac{d}{d\omega^2}(7\omega^2+8\Delta^2)h(\omega)=0,$$

which yields the spectrum branch

$$E^2=\frac{8\Delta^2}{5}+\frac{c_F^2k^2}{5}\Big(1-\frac{16\Delta^2}{35}\frac{d\ln h(\omega)}{d\omega^2}\Big)\Big|_{\omega^2=-8\Delta^2/5}.$$

Using the formula

$$\frac{d\ln h(\omega)}{d\omega^2}=-\frac{1}{2\omega^2}-\frac{1}{2(\omega^2+4\Delta^2)}-\Big[\omega\sqrt{\omega^2+4\Delta^2}\ln\frac{(4\Delta^2+\omega^2)^{1/2}+\omega}{(4\Delta^2+\omega^2)^{1/2}-\omega}\Big]^{-1}$$

and substituting $\omega = i(8/5)^{1/2}\Delta$, we obtain

$$d\ln h(\omega)/d\omega^2={}^5/_{48}\Delta^{-2}(1-2\sqrt{6}/\mathrm{arctg}\,2\sqrt{6}).$$

This leads to the spectrum branch corresponding to the variable $u_{12}+u_{21}$:

$$E^2=\frac{8\Delta^2}{5}+\frac{c_F^2k^2}{105}\Big[20+\frac{2\sqrt{6}}{\mathrm{arctg}\,2\sqrt{6}}\Big]. \tag{3.7}$$

Similar calculations are easily made for the remaining branches, with the exception of those with $\Omega = 2\Delta$. The simplest way of obtaining the sought formulas is to substitute in the quadratic forms (3.4) values of w_{ia} such that only the variable of interest remains, and then equate the form to zero. For example, to obtain the phonon branch (3.6) we substitute in the first of the forms (3.4) $w_{12} = u_{12} = u_{21}$, while in the investigation of the nonphonon branch we substitute $w_{12} = u_{12} = u_{21}$.

We write down the branches of the Bose spectrum together with their corresponding variables. These are the four phonon branches:

$$E^2=c_F^2k^2/5;\quad u_{12}-u_{21};$$
$$E^2=2c_F^2k^2/5;\quad u_{13}-u_{31},\quad u_{23}-u_{32}; \tag{3.8}$$
$$E^2=c_F^2k^2/3;\quad v_{11}+v_{22}+v_{33};$$

five u-branches with $\Omega(8/5)^{1/2}\Delta$:

$$E^2=\frac{8\Delta^2}{5}+\frac{c_F^2k^2}{105}\Big[20+\frac{2\sqrt{6}}{\mathrm{arctg}\,2\sqrt{6}}\Big];\quad u_{12}+u_{21},\quad u_{11}-u_{22};$$

$$E^2=\frac{8\Delta^2}{5}+\frac{c_F^2k^2}{210}\Big[85-\frac{2\sqrt{6}}{\mathrm{arctg}\,2\sqrt{6}}\Big];\quad u_{13}+u_{31},\quad u_{23}+u_{32}; \tag{3.9}$$

$$E^2=\frac{8\Delta^2}{5}+\frac{c_F^2k^2}{105}\Big[50-\frac{2\sqrt{6}}{\mathrm{arctg}\,2\sqrt{6}}\Big];\quad u_{11}+u_{22}-2u_{33}$$

and five v-branches with $\Omega = (12/5)^{1/2}\Delta$:

$$E^2=\frac{12\Delta^2}{5}+\frac{c_F^2k^2}{105}\Big[20-\frac{2\sqrt{6}}{\pi-\mathrm{arctg}\,2\sqrt{6}}\Big];\quad v_{12}+v_{21},\quad v_{11}-v_{22};$$

$$E^2=\frac{12\Delta^2}{5}+\frac{c_F^2k^2}{210}\Big[85+\frac{2\sqrt{6}}{\pi-\mathrm{arctg}\,2\sqrt{6}}\Big];\quad v_{13}+v_{31};\quad v_{23}+v_{32}; \tag{3.10}$$

$$E^2=\frac{12\Delta^2}{5}+\frac{c_F^2k^2}{105}\Big[50+\frac{2\sqrt{6}}{\pi-\mathrm{arctg}\,2\sqrt{6}}\Big];\quad v_{11}+v_{22}-2v_{33}.$$

4. BRANCHES WITH $\Omega = 2\Delta$ AT SMALL k

We have obtained all the Bose-spectrum branches except those with $\Omega = 2\Delta$, by expanding the coefficients of the tensors A_{ij}, B_{ijab} at small k (3.1). This procedure, however, cannot be used for branches with $\Omega = 2\Delta$, since the function $h(\omega)$ (2.16), and with it also the functions $f(\omega)$ and $g(\omega)$, has a singularity $\sim(\omega^2+4\Delta^2)^{-1/2}$ at $\omega^2 \to -4\Delta^2$. Therefore the branches with $\Omega = 2\Delta$ call for a special investigation.

We start with the branch corresponding to the variable $u_{11}+u_{22}+u_{33}$. We separate from the quadratic form (2.5) the terms corresponding to the indicated variable, putting $u_{ia}(p) = c(p)\delta_{ia}$, $v_{ia} = 0$. In this case

$$\delta_{ia}\delta_{jb}n_{1i}n_{1j}=\delta_{ia}\delta_{jb}(2n_{1a}n_{1b}-\delta_{ab})n_{1i}n_{1j}=1$$

and (2.5) is transformed into

$$\sum_p c^2(p)A(p,u_{11}+u_{22}+u_{33}), \qquad (4.1)$$

where

$$A(p,u_{11}+u_{22}+u_{33}) = -\frac{4Z^2}{\beta V}\sum_{p_1+p_2=p}\left[\frac{(\xi_1+i\omega_1)(\xi_2+i\omega_2)-\Delta^2}{(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)}-\frac{1}{\omega_1^2+\xi_1^2+\Delta^2}\right]. \qquad (4.2)$$

Using the Feynman procedure to calculate $A(p, u_{11}+u_{22}+u_{33})$ and integrating with respect to ω_1 and ξ_1, we obtain

$$A(p,u_{11}+u_{22}+u_{33}) = \frac{k_F^2Z^2}{4\pi^3c_F}\int F\,d\Omega_1, \qquad (4.3)$$

where

$$F=\int_0^1 d\alpha\left[\ln\frac{\Delta^2}{\Delta^2+\alpha(1-\alpha)(\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2)}-2\right]. \qquad (4.4)$$

We obtain similarly the coefficient functions corresponding to the variables $v_{12}-v_{21}$, $v_{13}-v_{31}$, $v_{23}-v_{32}$. They can be written in the form

$$A(p,v_{ik}-v_{ki}) = \frac{k_F^2Z^2}{4\pi^3c_F}\int (n_{1i}^2+n_{1k}^2)F\,d\Omega_1. \qquad (4.5)$$

We calculate the integral (4.4) for F:

$$F=-\frac{2a}{b}\,\mathrm{arctg}\,\frac{b}{a}=-\frac{a}{b}\left(\pi-2\,\mathrm{arctg}\,\frac{a}{b}\right), \qquad (4.6)$$

where

$$a^2=\Delta^2+{}^1/_4(\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2),\quad b^2=-{}^1/_4(\omega^2+c_F^2(\mathbf{n}_1\mathbf{k})^2). \qquad (4.7)$$

As $\omega^2 \to -4\Delta^2$ the quantity b^2 is positive and close to Δ^2. As will be shown below, $a^2 = 0(k^2)$, so that $2\arctan(a/b) \ll \pi$ and in first-order approximation $F \approx \pi a/\Delta$. This leads to an equation that determines in first order the dispersion of the branch $u_{11}+u_{22}+u_{33}$:

$$\int a\,d\Omega_1=0. \qquad (4.8)$$

We direct k along the third axis and denote $\cos\theta_1 = x$. Then $(\mathbf{n}_1\mathbf{k}) = kx$ and (4.8) is transformed into

$$\int_{-1}^{1} dx(4\Delta^2-E^2+c_F^2k^2x^2)^{1/2}=0. \qquad (4.9)$$

Putting

$$z^2=(4\Delta^2-E^2)/c_F^2k^2, \qquad (4.10)$$

we obtain the equation

$$\int_{-1}^{1} dx(x^2+z^2)^{1/2}=0, \qquad (4.11)$$

or

$$(1+z^2)^{1/2}+z^2\ln\{z^{-1}[1+(1+z^2)^{1/2}]\}=0. \qquad (4.12)$$

Putting also

$$t=2\ln\{z^{-1}[1+(1+z^2)^{1/2}]\}, \qquad (4.13)$$

we can rewrite (4.12) in the simple form

$$t+\mathrm{sh}\,t=0. \qquad (4.14)$$

If t is a nontrivial root of this equation we obtain, substituting $z = 1/\sinh(t/2)$ in (4.10)

$$E^2=4\Delta^2-c_F^2k^2/\mathrm{sh}^2(t/2). \qquad (4.15)$$

Equation (4.14) and the dispersion law (4.15) turned out to be the same as for the single nonphonon branch of the spectrum in the Fermi-gas model with scalar point interaction, investigated in Ref. 8. It is indicated in Ref. 8 that a physical meaning can be possessed by the branch (4.15), the first to appear upon analytic continuation with respect to the variable E from the upper half-plane to the unphysical sheet. This branch is obtained if t is replaced by the smallest (in absolute value) nontrivial ($\neq 0$) root of (4.14), which is equal to

$$t_1\approx 2.251+i\,4.212. \qquad (4.16)$$

The obtained branch was called in Ref. 8 "resonant excitation." It corresponds to the pole $c_{ia}(p)$, of the Green's function of the Bose fields, which is located near the branch point $E^2 = 4\Delta^2$.

We can treat similarly the remaining three branches with $\Omega = 2\Delta$. They correspond to the equations

$$\int_{-1}^{1}(x^2+z^2)^{1/2}(1\mp x^2)dx=0, \qquad (4.17)$$

or

$$\mp 2(1+z^2)^{1/2}+(4\mp z^2)\left[(1+z^2)^{1/2}+z^2\ln\frac{z+(1+z^2)^{1/2}}{z}\right]=0, \qquad (4.18)$$

in which it is necessary to take the minus sign for the variable $v_{12}-v_{21}$ and the plus sign for $v_{13}-v_{31}$, $v_{23}-v_{32}$. Changing to the variable t (4.13), we obtain in place of (4.18)

$$\frac{\mathrm{ch}\,t-2}{2\,\mathrm{ch}\,t-1}\mathrm{sh}\,t+t=0,\quad v_{12}-v_{21}; \qquad (4.19)$$

$$\frac{3\,\mathrm{ch}\,t-2}{2\,\mathrm{ch}\,t-3}\mathrm{sh}\,t+t=0,\quad v_{13}-v_{31},\ v_{23}-v_{32}. \qquad (4.20)$$

As a result, the dispersion laws for all branches with $\Omega = 2\Delta$ are given by formula (4.15), where t are the nontrivial roots of Eqs. (4.14) for the branch $u_{11}+u_{22}+u_{33}$, (4.19), for $v_{12}-v_{21}$, and (4.20) for $v_{13}-v_{31}$, $v_{23}-v_{32}$. The branches with direct physical meaning are those appearing first in the analytic continuation from the physical sheet. For Eq. (4.19), the sought nontrivial solution with minimum modulus is of the form

$$t_2\approx 2.93+i\,4.22, \qquad (4.21)$$

TABLE I.

Variables	Square of Bose spectrum	
	Our data	Wolfle's data
$u_{11}+u_{22}+u_{33}$	$4\Delta^2+(0.237-i0.295)c_F^2k^2$	$4\Delta^2+O(c_F^2k^2)$
$u_{12}-u_{21}$	$c_F^2k^2/5$	$c_F^2k^2/5$
$u_{13}-u_{31}$, $u_{23}-u_{32}$	$2c_F^2k^2/5$	$2c_F^2k^2/5$
$u_{12}+u_{21}$	$8\Delta^2/5+0.224c_F^2k^2$	$8\Delta^2/5+0.2c_F^2k^2$
$u_{13}+u_{31}$, $u_{23}+u_{32}$	$8\Delta^2/5+0.388c_F^2k^2$	$8\Delta^2/5+0.4c_F^2k^2$
$u_{11}-u_{22}$	$8\Delta^2/5+0.224c_F^2k^2$	$8\Delta^2/5+O(c_F^2k^2)$
$u_{11}+u_{22}-2u_{33}$	$8\Delta^2/5+0.442c_F^2k^2$	$8\Delta^2/5+O(c_F^2k^2)$
$v_{11}+v_{22}+v_{33}$	$c_F^2k^2/3$	$c_F^2k^2/3$
$v_{12}-v_{21}$	$4\Delta^2+(0.111-i0.169)c_F^2k^2$	$4\Delta^2+O(c_F^2k^2)$
$v_{13}-v_{31}$, $v_{23}-v_{32}$	$4\Delta^2+(0.353-i0.335)c_F^2k^2$	$4\Delta^2+O(c_F^2k^2)$
$v_{12}+v_{21}$	$12\Delta^2/5+0.164c_F^2k^2$	$12\Delta^2/5+O(c_F^2k^2)$
$v_{13}+v_{31}$, $v_{23}+v_{32}$	$12\Delta^2/5+0.418c_F^2k^2$	$12\Delta^2/5+O(c_F^2k^2)$
$v_{11}-v_{22}$	$12\Delta^2/5+0.164c_F^2k^2$	$12\Delta^2/5+0,2c_F^2k^2$
$v_{11}+v_{22}-2v_{33}$	$12\Delta^2/5+0.502c_F^2k^2$	$12\Delta^2/5+0.467c_F^2k^2$

and for Eq. (4.21) of the form

$$t_3\approx 1.94+i\,4.12. \tag{4.22}$$

The coefficients of $c_F^2k^2$ are complex for all the branches with $\Omega=2\Delta$, as already noted above. The physical cause is the possibility of the decay of the Bose excitation into two fermions.

5. CONCLUSION

The method used in the present paper makes it possible to calculate all the branches of the Bose spectrum at small momenta k, accurate to terms $\sim k^2$. The results at $k=0$ (the frequencies of the collective modes) coincide with those obtained in Refs. 1–5 by essentially different methods. A comparison of the present results with one of the most complete investigations, that of Wolfle,[1] is shown in the table. To facilitate the comparison, we replaced in the table the coefficients of the type $(20+2\sqrt{6}/\mathrm{arctg}\,2\sqrt{6})/105$ in front of $c_F^2k^2$ by their numerical values. It is seen that the branches calculated by Wolfle accurate to terms $\sim k^2$ practically coincide with those obtained here. We note at the same time that Wolfle calculated, up to k^2, only 9 out of 18 branches (Nagai[2] obtained 14 branches). The dispersions of the branches with $\Omega=2\Delta$ were not obtained in Refs. 1 and 2, and those of two branches with $\Omega=(8/5)^{1/2}\Delta$ and three with $\Omega=(12/5)^{1/2}\Delta$ were furthermore not obtained in Ref. 1.

We note in conclusion that the branches with $\Omega=(8/5)^{1/2}\Delta$, $(12/5)^{1/2}\Delta$ are readily observed and can be used to determine the temperature dependence of the gap $\Delta=\Delta(T)$. The experimental situation corresponds to finite values of the momentum $\mathbf{k}$, so that calculation of the dispersion $E=E(\mathbf{k})$ is important for comparison with experiment.

[1]P. Wolfle, Physica (Utrecht **90B**, 96 (1977).
[2]K. Nagai, Prog. Theor. Phys. **54**, 1 (1975).
[3]L. Tewordt and D. Einzel, Phys. Lett. **56A**, 97 (1976).
[4]K. Maki, J. Low Temp. Phys. **24**, 755 (1976).
[5]V. R. Chechetkin, Fiz. Nizk. Temp. **2**, 1365 (1976) [Sov. J. Low Temp. Phys. **2**, 665 (1976)].
[6]V. Alonso and V. N. Popov, Zh. Eksp. Teor. Fiz. **73**, 1445 (1977) [Sov. Phys. JETP **46**, 760 (1977)].
[7]P. N. Brusov and V. N. Popov, Zh. Eksp. Teor. Fiz. 78, 234 (1980) [Sov. Phys. JETP **51**, 117 (1980)].
[8]V. A. Andrianov and V. N. Popov, Teor. Mat. Fiz. **28**, 340 (1976).

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