Nonlinear refraction of solitons

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We study the nonlinear evolution of a soliton with a nonplanar front in the framework of the equations found by Ostrovskiĭ and Shrira [L. A. Ostrovskiĭ and V. I. Shrira, Sov. Phys. JETP **44**, 738 (1976)]. The general solutions obtained describe self-refraction and, in particular, self-focusing of solitons in media with arbitrary laws of nonlinearity and dispersion. We discuss the asymptotic soliton behavior connected with going beyond the framework of the adiabatic approximation.

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§1. INTRODUCTION

The phenomenon of self-focusing and defocusing has recently been extensively studied for quasi-sinusoidal waves. In media with a weak dispersion the shape of a stationary wave is far from sinusoidal and, notwithstanding the obvious interest, similar phenomena for appreciably non-sinusoidal waves are relatively little studied. A number of papers[1-6] have recently been devoted to a study of the two-dimensional evolution of solitons. In Ref. 1 the Kadomtsev-Petviashvili equation, which is a two-dimensional generalization of the Korteweg-de Vries equation, was obtained. In Refs. 3 and 4 non-linear solutions of that equation were found using the methods of the inverse scattering method. However, these methods usually allow one to obtain only asymptotic solutions "in explicit form" and do not allow one at all to say anything about systems which cannot be integrated by the inverse scattering method.

A geometric optics approach was suggested for the dynamics of solitons in Ref. 2 (earlier a similar approach was suggested to be applied to the dynamics of shock waves[7]). As in many short-wavelength asymptotic schemes[8] a plane stationary (in this case solitary) wave is chosen as the zeroth approximation. In first approximation account is taken of slow changes in the parameters of a wave that remains locally close to being stationary. In an orthogonal system of coordinates (α, β) formed by successive positions of the front of the soliton ($\alpha = \text{const}$) and the normal to it ($\beta = \text{const}$), the local soliton velocity V and the angle θ that the ray makes with the x-axis are connected through the kinematic relations[2,7]

$$\theta_\beta' - \Delta_\alpha'/V = 0, \quad \theta_\alpha' + V_\beta'/\Delta = 0, \tag{1.1}$$

where Δ is the width of the ray tube. To close the system we use the law for conservation of energy along a ray tube

$$W_c \Delta = \text{const}, \tag{1.2}$$

where W_c is the total energy per unit length of the front of the soliton. Assuming that all the energy is localized in the soliton (and its shape is known), we determine thereby W_c as a function of the amplitude or the velocity. The system (1.1) [using (1.2)] is general for all single-parameter solitons (solutions of one-dimensional evolution equations[1)]), and the specific nature of the equations manifests itself in the actual form of the amplitude dependences of the velocity and of the energy of the soliton. If the soliton velocity increases when its energy density W_c increases, then defocusing occurs, and the rays diverge with increasing W_c in such a way that the local energy density W_c diminishes. If, however, the soliton velocity decreases when its energy density W_c increases, self-focusing occurs.[2] In Ref. 2 particular solutions of the system (1.1) were obtained. In the present paper we shall obtain a general solution of the system (1.1) for media with weak nonlinearities of arbitrary form. Both cases of self-refraction are considered in this paper.

§2. ANALYSIS OF THE INITIAL SYSTEM OF EQUATIONS

It is convenient to use for the description of the soliton instead of the pair of parameters V and θ the pair φ and θ, where

$$\varphi = \int \frac{dV}{c(V)\Delta(V)}, \quad c = \left| \left(-\frac{V}{\Delta\Delta_V'} \right)^{1/2} \right| = |q^{1/2}|.$$

The original set of equations takes in those variables the form

$$\theta_\alpha' + c\varphi_\beta' = 0, \quad \theta_\beta' \mp \varphi_\alpha'/c = 0, \tag{2.1}$$

where the plus sign corresponds to $q < 0$ $(q = -V/\Delta\Delta_V')$ and $q > 0$ corresponds to the minus sign. The quantity q thus determines the nature of the system (2.1): when $q > 0$ the system is hyperbolic, when $q < 0$ it is elliptical.[2)] The quantity q may change sign depending on the amplitude, and the system (2.1) can also be of a mixed nature.

As an example we consider the modified Koretweg-de Vries equation:

$$u_t' \pm u^p u_x' + \delta^2 u_{xxx}''' = 0, \tag{2.2}$$

which possesses one-dimensional solitons of the form[9]

$$u = a(\text{sech}[(x - Vt)/\gamma])^{2/p}, \tag{2.3}$$

where

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$$\gamma=\frac{\delta}{a^{p/2}}\left(\frac{2(p+1)(p+2)}{p^2}\right)^{1/2}.$$

The velocity of the soliton (2.3) depends as follows on the amplitude (in a fixed frame of reference):

$$V=1\pm\mathrm{const}(p)a^p=1\pm\frac{2}{(p+1)(p+2)}a^p. \tag{2.4}$$

For the solitons (2.3) $W_c\sim a^{2-p/2}$, $\varphi\sim a^{p/2}$, $c(a)\sim a^2$; when the sign in front of the non-linear term is positive the system (1.1) is hyperbolic, if $p<4$, if the sign is negative, it is hyperbolic, if $p>4$.

We state the Cauchy problem for the system (2.1) with initial conditions: initially ($\alpha=0$) the front of the soliton is given curved according to the law $\theta=\theta_0(\beta)$ (we emphasize that the change in θ along the front here is not small, but only sufficiently slow) and the amplitude distribution along the front is $\varphi=\varphi_0(\beta)$. The hodograph transformation $\alpha=\alpha(\varphi,\theta)$, $\beta=\beta(\varphi,\theta)$ reduces (2.1) to the linear system (2.5) [it is well known that then some particular solutions of (2.1) are lost, in particular the simple waves, but they are easily found directly from (2.1)]

$$\beta_\varphi'+c(\varphi)\alpha_\theta'=0,\quad \beta_\theta'\pm c(\varphi)\alpha_\varphi'=0. \tag{2.5}$$

The lower sign corresponds here and henceforth to the elliptical case. As usually in the hodograph method, we introduce a potential function $W(\varphi,\theta)$ as follows:

$$\beta=\frac{\partial W}{\partial\theta},\quad \alpha=-\frac{\partial W}{\partial\varphi}\frac{1}{2c(\varphi)}. \tag{2.6}$$

Substitution of (2.6) into (2.5) leads to an equation for W:

$$\frac{\partial^2W}{\partial\theta^2}\mp\frac{\partial^2W}{\partial\varphi^2}\pm\left(\frac{c_\varphi'}{c(\varphi)}\right)\frac{\partial W}{\partial\varphi}=0. \tag{2.7}$$

We did not succeed in constructing the general solution of (2.7) for an arbitrary dependence $c(\varphi)$. However, for a very important class of solitary waves, namely for the weakly non-linear ones (i.e., for such waves for which the correction to the velocity is appreciably less than the velocity of the linear wave) the situation simplifies considerably. Expanding $c(\varphi)$ in a series in φ and restricting ourselves to the first terms of the expansion we get $c(\varphi)\sim\varphi^m$, $c_\varphi'/c=m/\varphi$ and Eq. (2.7) reduces thus to the well known Darboux equation

$$\frac{\partial^2W}{\partial\theta^2}\mp\frac{\partial^2W}{\partial\varphi^2}\pm\frac{2\nu}{\varphi}\frac{\partial W}{\partial\varphi}=0, \tag{2.8}$$

where $\nu=m/2$ is the Darboux index. For the solitons (2.3) $\nu=2/p$. For an algebraic soliton (solutions of the Benjamin-Ono equation) $\nu=3/2$. The initial conditions for Eq. (2.8) are specified on the curve Γ of the initial data and have the form

$$\partial W/\partial\varphi=0,\quad \partial W/\partial\theta=2\beta, \tag{2.9}$$

where the curve Γ is given in parametric form: $\theta=\theta_0(\beta)$, $\varphi=\varphi_0(\beta)$. The problem of the two-dimensional evolution of a soliton [in the framework of the system (1.1)] is thus in the general case reduced to Eq. (2.7) and under the additional assumption of weak non-linearity ($M=|V-1|\ll1$) to the well known Darboux equation. We consider the solutions of (2.8) in the hyperbolic and the elliptical cases separately.

§3. HYPERBOLIC CASE (DEFOCUSING)

In this case the problem is similar to well studied problems of gas dynamics. The solution of Eq. (2.8) with initial conditions (2.9) in the point $M(\varphi_M,\theta_M)$ can in a well known way be expressed in terms of the Riemann function R and the initial conditions

$$W(M)=\frac{1}{2}\Big\{W_\Gamma(P)R(P)+W_\Gamma(Q)R(Q)+\int_P^Q\Big[2R\beta_0(\varphi)d\varphi-W_\Gamma(R_\varphi'd\theta+R_\theta'd\varphi)-W_\Gamma R\frac{2\nu}{\varphi}d\theta\Big]\Big\}, \tag{3.1}$$

where W_Γ is $W(\varphi,\theta)$ for $\varphi,\theta\in\Gamma$, while P and Q are the points of intersection of the curve Γ of the initial data and the characteristics drawn through the point M. The Riemann function for Eq. (2.8) has the form[10]

$$R(\zeta)=\left(\frac{\varphi_M}{\varphi}\right)^\nu P_\nu(\zeta),\quad \zeta=\frac{\varphi^2+\varphi_M^2-(\theta-\theta_0)^2}{2\varphi\varphi_M}, \tag{3.2}$$

where P_ν is a Legendre function of the first kind. Equation (2.2) has its widest application in the cases of quadratic ($p=1$) or cubic ($p=2$) non-linearity, while the case $p=\frac{1}{2}$ is also of definite interest. These values of the non-linearity index p correspond to integer values of the Darboux index ν ($p=1$, $\nu=2$; $p=2$, $\nu=1$; $p=\frac{1}{2}$, $\nu=4$). For integral values of ν the Legendre functions become Legendre polynomials P_ν ($P_1(z)=z$, $P_2(z)=\frac{1}{2}(3z^2-1),\ldots$). By differentiating (3.1) one can find, using (2.6), expressions for α and β as functions of φ and θ. One can considerably simplify Eq. (3.1), which gives the general solution of Eq. (2.8), in two important particular cases.

1) At the initial time ($\alpha=0$) a plane front is given with some distribution $\varphi=\varphi(\beta)$ of the amplitude along the front. Since $W=\theta=0$ along the Γ curve and $\beta=\beta_0(\varphi)$, we get from (3.1)

$$W(M)=\int_{\varphi_M-\theta_M}^{\varphi_M+\theta_M}\beta_0(\varphi)R\,d\varphi. \tag{3.3}$$

For the solitons of (2.3) Eq. (3.3) takes the following form, if we use (3.2)

$$W(M)=\frac{1}{2}\int_{\varphi_M-\theta_M}^{\varphi_M+\theta_M}\beta_0(\varphi)\left[\frac{3}{4}\left(1-\frac{\varphi_M^2+\theta_M^2}{\varphi^2}\right)-\frac{\varphi_M^2}{\varphi^2}\right]d\varphi;\quad p=1\ (\nu=2), \tag{3.3a}$$

$$W(M)=\frac{1}{2}\int_{\varphi_M-\theta_M}^{\varphi_M+\theta_M}\beta_0(\varphi)\left[1-\frac{\varphi_M^2+\theta_M^2}{\varphi^2}\right]d\varphi;\quad p=2\ (\nu=1). \tag{3.3b}$$

Explicit expressions for α and β, for instance, at $p=2$, are obtained from (3.3b) in accord with (2.6):

$$\tilde\alpha=C\alpha=-\frac{1}{2\varphi^2}\left\{\left[\frac{\beta_0(\varphi+\theta)}{\varphi+\theta}+\frac{\beta_0(\varphi-\theta)}{\varphi-\theta}\right]\theta-\varphi\int_{\varphi-\theta}^{\varphi+\theta}\frac{\beta_0(\varphi)}{\varphi^2}d\varphi\right\},$$

$$\beta=\frac{1}{2}\left\{\left[\frac{\beta_0(\varphi+\theta)}{\varphi+\theta}-\frac{\beta_0(\varphi-\theta)}{\varphi-\theta}\right]\theta+\theta\int_{\varphi-\theta}^{\varphi+\theta}\frac{\beta_0(\varphi)}{\varphi^2}d\varphi\right\}$$

[where $C=c(\varphi)/\varphi^2=\mathrm{const}$] and enable us to construct a family of equal level lines for the surfaces $\varphi(\alpha, \beta)$ and $\theta(\alpha, \beta)$.

2) Initially the soliton has the same amplitude φ^0 along the whole front which is curved according to the law $\theta = \theta_0(\beta)$ (phase modulation). Under those initial conditions we have from (3.1)

$$W(M)=\frac{1}{2}\left\{W(P)R(P)+W(Q)R(Q)-\int_P^Q W_\Gamma\left(R_\varphi'+\frac{2\nu}{\varphi}R\right)d\theta\right\}, \quad (3.4)$$

where P and Q are points with, respectively, the coordinates $[\varphi^0, \theta_M - (\varphi_M - \varphi^0)]$ and $[\varphi^0, \theta + (\varphi_M - \varphi^0)]$; $W_\Gamma = 2\int \beta_0(\theta)d\theta$. Equation (3.4) becomes especially simple for $\nu = 1$ and 2. As example we consider the evolution of a soliton front curved at the initial time according to the law $\theta = m \tanh\beta$ for $\nu = 1 (p = 2)$. When $\nu = 1$, Eq. (3.4) can be rewritten in the form

$$W(M)=\frac{1}{\varphi^0}\left\{\varphi_M[W(P)+W(Q)]-\int_P^Q W_\Gamma(\theta)d\theta\right\}, \quad (3.4a)$$

whence we find expressions for α and β:

$$\begin{aligned}\tilde{\alpha}&=\alpha C=-\frac{\varphi^0}{2\varphi}[\beta_0(x)|_{\theta-\Delta\varphi}^{\theta+\Delta\varphi}],\\ \beta&=\frac{1}{2}[\beta_0(\theta+\Delta\varphi)+\beta_0(\theta-\Delta\varphi)+W_\Gamma(x)|_{\theta-\Delta\varphi}^{\theta+\Delta\varphi}]\end{aligned} \quad (3.5)$$

(where $\Delta\varphi = \varphi - \varphi^0$) which take for the given $\theta_0(\beta)$ dependence the form

$$\begin{aligned}\tilde{\alpha}&=\frac{\varphi^0}{4\varphi}\ln\frac{(1-\Delta\varphi)^2-\theta^2}{(1+\Delta\varphi)^2-\theta^2},\\ \beta&=\frac{1}{2\varphi^0}\left[\varphi\ln\frac{(1+\theta)^2-\Delta\varphi^2}{(1-\theta)^2-\Delta\varphi^2}-\frac{1-(\theta+\Delta\varphi)^2}{1-(\theta-\Delta\varphi)^2}+x\ln\frac{1+x}{1-x}\Big|_{\theta-\Delta\varphi}^{\theta+\Delta\varphi}\right].\end{aligned} \quad (3.6)$$

In Fig. 1a we show for a converging wave a picture of the equal-level lines of φ and θ in the α, β plane. The horizontal sections of the curve give the amplitude and angle distributions along the front (α = const), and the vertical sections give those along the ray (β = const). A more visualizable, but less general graphical form of representing the solution is provided by the pictures of the fronts and rays in the x, y-plane at different times (Fig. 1b). For clarity we used in Fig. 1b scale distortion. It is clear from Fig. 1 that due to focusing the amplitude in the central region of the front increases with increasing α, since an increase in the amplitude of the wave leads to an increase of its velocity, and the rays also diverge. The amplitude remains finite in the focus. Moreover, when $\alpha = \alpha_*$ there occurs a singularity in the form of a kink in the front and a jump in the amplitude.

3) Initial distributions which are not unique. It is well known that Riemann's Eq. (3.1) is applicable if the characteristics drawn through the point M intersect the Γ curve only once. This is equivalent to uniqueness of the inverse functions $\beta_0(\varphi)$ and $\beta_0(\theta)$, or, what amounts to the same, to monotonicity of the initial distributions $\theta_0(\beta)$ and $\varphi_0(\beta)$. In those cases when the functions $\theta_0(\beta)$, $\varphi_0(\beta)$ are not monotonic but have one or more extrema, one must split the Γ curve into sections of monotonicity [the solution of the problem in those regions is given by Eq. (3.1)] and then solve the Goursat problem (with initial data on the characteristics) using

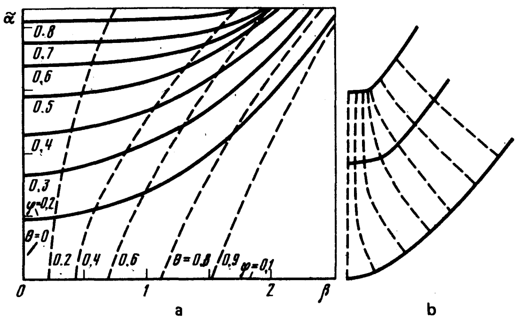

FIG. 1. Evolution of a convergent wavefront in the hyperbolic case ($\varphi^0 = 0.1$): a) full drawn lines—φ = const.; dashed lines—0 = const; b) solid lines—wavefronts, dashed lines—rays.

the results of the solution of the Cauchy problem as boundary conditions. For the solution of the Goursat problem the Riemann function method is not applicable and one can use another form of the general solution of the Darboux Eq. (2.8) (valid only for integer ν):

$$W(\varphi, \theta)=\int \varphi\, d\varphi \ldots \int \varphi\, d\varphi[G(\theta+\varphi)+\Phi(\theta-\varphi)]. \quad (3.7)$$

The integration is performed here ν times, G and Φ are arbitrary functions determined from the boundary conditions. Such a form is not convenient for a solution of the Cauchy problem, as it is difficult to express the arbitrary functions G and Φ in terms of the initial data. [The solution in the form (3.7) was obtained in Ref. 6 for the case $\nu = 2 (p = 1)$.] In those cases when the initial distribution contains sections of a plane front with constant amplitude, the solution obtained by the hodograph transformation method joins up with the simple waves.

§4. ELLIPTICAL CASE (SELF-FOCUSING)

The solution of the elliptical Eq. (2.8) (this case corresponds to the lower sign) with the hyperbolic conditions (2.9) is, strictly speaking, an example of an incorrect problem. However, similar problems have been successfully solved recently.[11,12] We show that if the initial conditions are sufficiently smooth—in a sense that will be exactly defined below—the problem has a unique solution. The solution remains smooth only for a certain time, and the moment when the singularities appear and their nature are of particular physical interest. We continue the function $W(\varphi, \theta)$ into the complex region with respect to one of the variables, and the other independent variable will play the role of a parameter. The substitution of variables $z \to -iz$ changes the elliptical equation to a hyperbolic one with the well known Riemann function $R(\xi)$. Let, to fix the ideas, β_0 be $\beta_0(\theta)$. Analytically continuing $\beta_0(\theta)$ into the complex region $\beta_0(\theta) \to \beta_0(\theta + i\theta_2)$ and thus defining the function $W(\theta, \theta_2, \varphi)$ on the curve of the initial data, we find the complete solution using the Riemann formula (3.1). The value of the function W at the point $M(\varphi_M, \theta_M)$ is thus, as in the hyperbolic case, an integral along the curve of the initial data continued into the complex plane on the section bounded by the points of intersection with the characteristics passing through the point M. The condition for the existence of a solution is the condition for the existence of an analytical continuation of the function $\beta_0(z)$. The presence of discontinuities of the higher derivatives of the functions $\beta_0(\varphi)$ and $\beta_0(\theta)$ makes an analytical continuation impossible, while as in the hyperbolic case a solution exists even when there are first-order discontinuities of the first derivative. The non-existence of a solution means that the given initial distribution will evolve non-analytically right from the start, the soliton will disintegrate, and our approach is inapplicable. As above we consider separately the cases of initial amplitude and of phase modulation.

1) In the first case the amplitude distribution $\varphi_0(\beta)$ is given on the initially plane front. We analytically continue the function $\beta_0(\varphi) \to \beta_0(\varphi + i\varphi_2)$ and, substituting this expression into (3.1) [and using (3.2)] we get again Eq. (3.3) with that difference that the integration is taken from the point $P(\varphi_M - i\theta_M)$ to the point $Q(\varphi_M + i\theta_M)$ (Fig. 2). If the function $\varphi_0(\beta)$ is non-monotonic (let it have, to fix the ideas, a single maximum value, equal to φ_{max}), the function $\beta_0(z)$ has a branch point $z = \varphi_{max}$. The function $\beta_0(z)$ is analytic in the φ, φ_2 plane with a cut along the real axis from φ_{max} to ∞. The integration path must in that case not intersect the cut.

In the case of an initial phase modulation a soliton with constant amplitude φ_0 is specified, with a front that is curved according to the law $\theta = \theta_0(\beta)$. Its further evolution is described by Eq. (3.4), where the integration goes from the point $P[\theta_M - i(\varphi_M - \varphi^0), \varphi_M]$ to the point $Q[\theta_M + i(\varphi_M - \varphi^0), \varphi_M]$. If $\theta = \theta_0(\beta)$ is non-monotonic, the integration contour is chosen using the same considerations as in the analogous case of amplitude modulation. As an example we consider the evolution of the soliton (2.3) (for $p = 2$) for the same initial conditions as in the example of § 3, i.e., $\theta = m \tanh\beta$ and $\varphi = \varphi^0$. Changing variables $\theta = \theta/m$ and analytically continuing $\beta_0 = \operatorname{arctanh}\theta$, we get, after a double integration over θ_2 in accordance with (2.6) and (3.4), the answer in elementary functions:

$$\tilde{\alpha} = C\alpha = \frac{1}{2}\frac{\varphi^0}{\varphi}\left[\operatorname{arctg}\frac{\varphi-\varphi^0}{1+\theta} + \operatorname{arctg}\frac{\varphi-\varphi^0}{1-\theta}\right], \qquad \beta = \left\{\frac{1}{2}\ln x - x \operatorname{arctg}\left(\frac{\varphi-\varphi^0}{x}\right)\right\}\Bigg|_{1-\theta}^{1+\theta}. \qquad (4.1)$$

We analyze separately the solutions for converging and diverging wave fronts.

a) The picture of the evolution of a converging wave front is shown in Fig. 3. Because of the linear focusing the amplitude in the central, curved part of the front increases initially; this leads to a lagging of that part and to an even larger bending of the front. The rate of the turning of the ray and of the growth in the amplitude increases with increasing α. When $\alpha = \alpha_*$ and

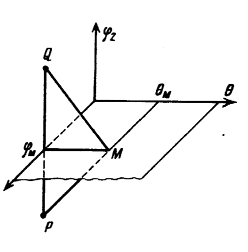

FIG. 2. Integration path in the $\varphi, \varphi_2, \theta$ space.

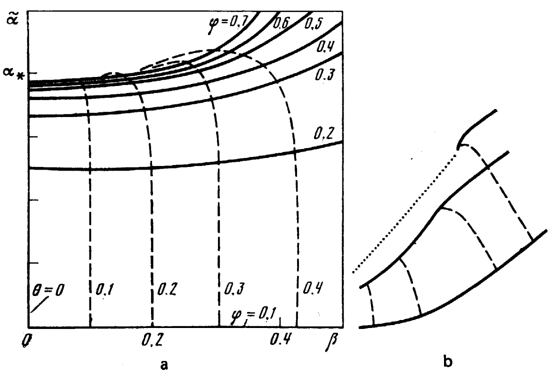

FIG. 3. Evolution of a convergent wavefront in the elliptical case ($\varphi^0 = 0.1$). The notation is the same as in Fig. 1, the dotted line is the region of the discontinuity.

$\beta = 0$ there appears a singularity in the solution: the derivatives $\varphi'_\alpha, \varphi'_\beta, \theta'_\alpha, \theta'_\beta$ become infinite. Assuming that the appearance of a singularity at the point $(\alpha_*, 0)$ does not "spoil" at once the whole solution (in the framework of our equations this assumption is valid) we can, by using (4.1), determine the moment when the singularity appears for each given β, i.e., on a fixed ray. It turns out that the region of the discontinuity (as we shall call the region where singularities occur in the solution and the adiabatic approach is, in general, inapplicable) widens with increasing α, but not on the entire front, but only in the region bounded by the asymptotes $\beta = \pm\beta_l$ ($\beta_l = \ln[2/(\varphi - \varphi^0)]$). The self-focusing affects the change in the soliton amplitude in an essential way until it leads to its disintegration.

b) The picture of the evolution is somewhat different in the case of a diverging wavefront, as can be seen from Fig. 4. Because of the linear divergence, the amplitude in the central part decreases and the speed correspondingly increases. The front is deformed in such a way that its peripheral sections become curved, after which the soliton amplitude on these sections starts to grow with increasing velocity until singularities appear at the points $(\alpha_*, \pm\infty)$. The discontinuity region is bounded by the asymptotes $\beta = \pm\beta_l$. Characteristic for the examples considered is the presence of a long incubation period during which the self-focusing effect develops weakly (the rays turn through an angle

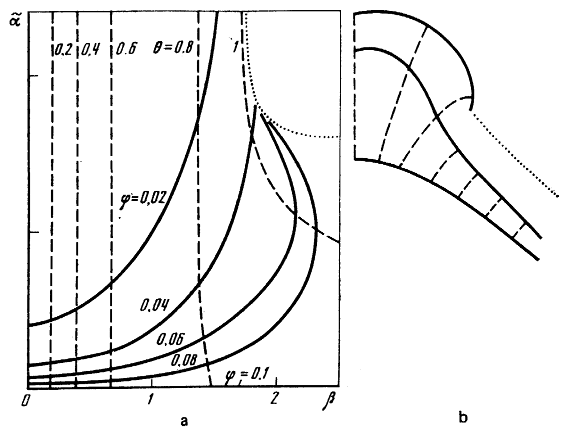

FIG. 4. Evolution of a divergent wavefront in the elliptical case. The notation is the same as in Fig. 3.

of the order φ_0 when $\alpha \sim \alpha_*$).

The examples considered give a qualitative representation of the self-focusing of solitons in media with varying non-linearity and dispersion laws, which is described by the general solution (3.1).

§5. DISCUSSION OF THE RESULTS. ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS

1. *Region of applicability of the initial system.* The initial set of Eqs. (1.1) was obtained in the geometric-optics approximation. In the linear theory the geometric-optics approximation is applicable provided $\Lambda^2 \gg R\lambda$, where λ is the wavelength, Λ a characteristic scale on which θ and φ change along the front, and R a characteristic radius of curvature of the front (we consider the worst case when $R \gg \Lambda$). For solitons this condition is insufficient. In order that one can assume the wave to be locally close to a soliton it is necessary that the non-one-dimensional corrections by which the exact field equations differ from the locally one-dimensional equations of the type (2.3) are small. However, the one-dimensional evolution equations themselves are, as a rule, obtained under the assumption of weak non-linearity ($M = |V-1| \ll 1$) and dispersion. This leads to an additional condition of the kind $\lambda_c \ll RM$ (the condition for an adiabatic evolution of the soliton) which is compatible with the linear conditions, if $1 \gg M \gg (\lambda_c/\Lambda)^2$. For instance, for a soliton on water this means that $(\Lambda H)^2 \gg h^4$, where h is the depth of the liquid and H the height of the solitary wave. The region of applicability of the solutions found earlier[3)] is thus defined by the inequality

$$\Lambda \gg \lambda_c / M^{1/2}. \tag{5.1}$$

Both in the hyperbolic and in the elliptic cases condition (5.1) ceases, as a rule, to be satisfied only close to singularities. For estimates, therefore, it is convenient to assume the characteristic distances at which a singularity manifests itself in the solution to be the scale of the applicability of the solution.

2. *Hyperbolic case.* As in a non-linear hyperbolic system the effect of the non-linearity is not at all compensated, the appearance of singularities of the kind of an infinite derivative ("toppling") in the solution is inevitable. Assuming that the existence of a "shock wave" is possible which propagates along the soliton boundary conditions were given in Refs. 2, 5, 6 which connect the magnitude of the jump in the amplitude and the angle (i.e., the kink in the front) and the speed of its propagation along the soliton front. In the framework of the initial set (1.1) it is completely impossible to say anything about the further behavior of the solution. Very roughly one can estimate the characteristic distance at which the singularity appears as Λ/ε, where $\varepsilon \sim (\varphi_{\max} - \varphi_{\min})/\varphi^0$. Qualitatively the evolution of the soliton is determined by the relation between the following effects, each of which manifests itself in its own characteristic range of α:

a) convergence and divergence of the wavefront or of its separate sections which take place already in the linear approximation;

b) scattering of perturbations along the soliton front determined by the characteristics of the set (2.1) [characteristic scale $\Lambda/c(\varphi^0)$];

c) "toppling" of the scattered perturbations, leading to the formation of shock waves and loss of energy by the soliton through the emission of linear waves lagging behind it, and in some cases to a disintegration of the soliton. Small perturbations ($\varepsilon \ll 1$) of the soliton manage to be scattered far before toppling occurs in the framework of (2.1). When $\varepsilon \sim 1$ toppling proceeds at the same time scale as "scattering." Convergence of a section of the front decreases the scattering time and the toppling time, while divergence increases them. Several peculiar effects connected with converging and diverging fronts are described in Ref. 2.

3. *Asymptotic behavior of the solutions in the hyperbolic case.* In the region where condition (5.1) is not satisfied the change in the soliton parameters proceeds non-adiabatically and is accompanied by the emission of energy in non-soliton form. Taking into account the thus produced high-frequency dissipation one can attempt to "correct" the initial set in such a way that it describes the evolution of the soliton after the "collapse" in the framework of (2.1). An exact (in the framework of the Kadomtsev-Petviashvili equation) dispersion relation was obtained in Ref. 3 for small perturbations propagating along the front of the soliton. Up to terms $\propto k^2$ (k is the wave number along β) this relation has the form

$$\Omega = c(\varphi^0)k - i\mu k^2, \tag{5.2}$$

where Ω is the frequency of a small-amplitude wave with wavevector $\mathbf{k}$, $\mu = (2/3)^{1/2}$. The second, "dissipative" term in the dispersion relation (5.2) is caused by the emission of energy in non-soliton form by the soliton. The dispersion relation (5.2) corresponds to the linear Burgers equation. Taking it into account that the small perturbation of the soliton is finite leads to the Burgers equation

$$\varphi_\alpha \pm c(\varphi^0)\left(\varphi_\beta' + \frac{2\nu}{\varphi^0}(\varphi - \varphi^0)\varphi_\beta'\right) - \mu\varphi_{\beta\beta}'' = 0. \tag{5.3}$$

Since under a non-adiabatic transformation the soliton must lose energy, one may assume that the dispersion relation (5.2) and thereby Eq. (5.3) are (accurate to the value of μ) true also for different shapes of solitons. (Apparently one can show this rigorously using an asymptotic procedure similar to the one used in Ref. 1.) The shape of the soliton differs under a non-adiabatic (but nevertheless sufficiently smooth) evolution somewhat from that of a stationary solitary wave. (The recently developed perturbation methods based upon the inverse scattering theory enabled one to find these deformations for some exactly integrable systems, in particular for the Korteweg-de Vries equation.[13]) However, since only integral characteristics appear in Eq. (5.2) (we restrict ourselves everywhere to the first terms in the expansion in φ^0) we may assume that all functions such as $c(\varphi^0)$, $a(\varphi^0)$, $V(\varphi^0)$ remain unchanged with the same accuracy. Equation (5.3) enables us to reach important conclusions about the asymptotic behavior of the solution. Any (sufficiently small in comparison to φ^0) initial perturbations initially scatter along the front (analogous to the break-up of the group velocity in one-dimensional problems[7]) and the next stage of their evolution is described by Eq. (5.3). The exact solution of Eq. (5.3) is known (e.g., Ref. 7). The asymptotic behavior of its solutions depends weakly on the initial conditions. If the initial perturbation was localized on the soliton front, its area $A[A = \int_{-\infty}^{\infty} \varphi(\beta)d\beta]$ is conserved, the amplitude ($|\varphi_{max} - \varphi_{min}|$) decreases $\propto (A/t)^{1/2}$ but the base of the profile increases $\propto (At)^{1/2}$. A perturbation with zero initial area is damped in proportion to t^{-1}. When one perturbation overtakes another, they merge.

The Burgers equation (5.2) is valid for small perturbations of the soliton ($\varepsilon \ll 1$). If, however, the perturbations are not small, but localized on the front, Eq. (5.2) can be used for the description of the final stage of the evolution when the amplitudes of the resulting shock waves have become sufficiently small. However, in that case we cannot connect the asymptotic behavior of the solution with the initial conditions, having only a qualitative idea about the evolution of the solution. Zakharov's asymptotic solution[3] ($\nu = 2, p = 1$) corresponds in that case to a completely damped shock wave.

An isolated case is the situation of "negative viscosity" (i.e., $\mu < 0$) which occurs when $dW_c/da < 0$ and $dV/da < 0$, e.g., for the solitons of (2.3) when the sign of the non-linear term is negative and $p > 4$. In that case the instability is explosive[14] and our approach is inadequate.

4. *Elliptical case*. A plane soliton is unstable and the non-linear stage of the evolution of the instability is given by the general Eq. (3.1). The evolution of the soliton leads to the appearance in the solution of "spike" type of singularities (similar to the singularities arising in problems on the one-dimensional self-modulation of quasi-sinusoidal waves in the hydrodynamic approximation).[7,11,12] There are in this case no simple general formulae to estimate the limits of applicability of the solution and it is necessary, after having found the solution, to determine the region of its applicability using condition (5.1).

§6. CONCLUSION

Application of the geometric optics approach enabled us to reduce the problem of describing the two-dimensional dynamics of quasi-stationary solitons to the solution of the linear Darboux equation, the general solutions of which, which describe both self-focusing and defocusing (in the weak non-linearity approximation), are given in this paper. In a well defined sense the problem is exhausted. However, there is, apparently, no general answer to the resulting problem of what happens further when the solutions which we have found cease to be applicable, the quasi-stationarity of the soliton is violated, and the non-soliton part of the solution ceases to be negligibly small. For the solution of actual problems it is necessary to study the appropriate local two-dimensional equations of the Kadomtsev-Petviashvili type, which is a very complicated problem. However, for the case of small perturbations on the soliton front the Burgers equation (4.3) enabled us to describe the stage of the evolution after the "collapse" of the solution in the framework of the initial hyperbolic system. The Burgers equation also describes qualitatively the asymptotic behavior of a wider class of initial perturbations. This asymptotic behavior is "intermediate" in being compared with the asymptotic behavior of the solution obtained by the inverse scattering method for the Kadomtsev-Petviashvili equation.[3]

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1) Everywhere in this paper we assume the medium to be isotropic.

2) We note that more general kind of system which describes three-dimensional motion of solitons is also determined by the sign of q (sign q = sign (dW_c/dV)

3) We emphasize that in the weak non-linearity approximation ($M \ll 1$) the general solution of the set (2.1) is found without additional assumptions.

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