Thin-wall current-carrying superconducting cylinder in a magnetic field

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We consider the influence of the transport current on the behavior of a thin-wall hollow superconducting cylinder in an external magnetic field. Expressions are derived for the critical current and field, at which destruction of the states of a cylinder with a specified number of flux quanta inside the cavity is destroyed. It is shown that the field dependence of I_c is oscillatory, and under certain conditions hysteresis is possible. Expressions are obtained for the oscillation of the critical temperature of the sample in an external field as a function of the transport current.

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1. The destruction of the superconductivity of cylindrical samples by transport current and by an external magnetic field has been repeatedly discussed in numerous theoretical and experimental papers. Attention to this question has been again attracted recently in connection with studies aimed at determining the structure produced when superconductivity is destroyed by current (we can mention recent discussions[1,2] of various intermediate-state models, investigations of the paramagnetic effect in hollow cylindrical samples,[3,4] and studies of the two-dimensional "mixed" state in type-I superconductors,[5] as well as observation of resistive effects[6] and investigations of other singularities in the behavior of such a system). The cited studies were made mainly on bulky samples. In our earlier paper[7] we considered in detail the penetration of an external magnetic field into the cavity of a thin-wall cylinder (with wall thickness $d \ll \lambda_L/\psi$, where λ_L is the London penetration depth and ψ is the order parameter). It was shown that the character of the penetration of the individual flux quanta into the interior of the cavity depends essentially on the screening factor $\mu \approx \frac{1}{2} r_1 d/\delta_L^2(T)$ (r_1 is the cylinder radius). At $\mu > 1$ the flux quanta penetrate jumpwise and exhibit a hysteresis behavior, while at $\mu < 1$ the field penetrates smoothly and there is no hysteresis. We discussed in Ref. 7 also some peculiarities of the observed[8,9] oscillatory dependence of the sample critical temperature on the external field.[10,11]

We study in this paper the influence of the transport current flowing along the cylinder on the penetration of an external longitudinal magnetic field into the cavity of a thinwall cylindrical sample. Just as in Ref. 7, the analysis is based on a thermodynamic approach within the framework of the Ginzburg–Landau theory.[12] It is shown that turning on the transport current I leads to a decrease and to a subsequent vanishing of the region of existence of states with a fixed number of flux quanta "frozen-in" inside the cavity. With increasing I, the hysteresis that could exist in the absence of current also vanishes. It is shown that the critical current I_c that destroys the sample superconductivity has an oscillatory dependence on the external field, owing to the successive penetration of the flux quanta into the cavity. The destruction of the superconductivity is always effected via a first-order phase transition (i.e., jumpwise at a finite value of the order parameter ψ).[1)] In the case of a transition close to second-order (which occurs at small values $\psi \ll 1$ and at $\mu < 1$), the results obtained here agree with those previously obtained in Ref. 13.

In Sec. 2 of the article we derive the thermodynamic potential that describes the behavior of a hollow current-carrying cylinder in the presence of an external field. (Usually[14] in the presence of a current the description is based directly on the Ginsburg-Landau equations corresponding to the minimum of the free energy of the system. This procedure is briefly described in the Appendix.) In Sec. 3 is considered the case of a thin ($d \ll \delta_L(T)/\psi$) cylinder, and a complete thermodynamic investigation is made of the behavior of the system, with allowance for the already mentioned screening factor μ, which is frequently disregarded but which plays an important role. In the general case the problem reduces to a solution of a system of algebraic equations of high degree, calling for numerical calculations. The results of the numerical calculations are illustrated by figures. In Sec. 4 are given analytic solutions of the problem in a number of limiting cases. The results are also discussed and compared with those of other studies of this topic.

2. Before we proceed to an investigation of the behavior of a cylinder carrying a current I and located in an external longitudinal magnetic field $H_{0z} = H_0$, we must establish the form of the functional whose minimum corresponds to the equilibrium state of the system. To this end, we write down the change of the system energy:

$$\Delta\mathscr{E} = \Delta Q - \frac{c\Delta t}{4\pi}\int [\mathbf{E}\times\mathbf{H}]d\boldsymbol{\sigma}_2 + \frac{c\Delta t}{4\pi}\int [\mathbf{E}\times\mathbf{H}]d\boldsymbol{\sigma}_1, \quad (1)$$

where ΔQ is the increment of the heat in the sample, and the last two terms yield the flux of the Poynting vector through the cylinder surface (σ_2 is the outer surface of the cylinder and σ_1 is the surface of the inner cavity). We use the relation div $\mathbf{E}\times\mathbf{H} = \mathbf{H}$ curl $\mathbf{E} - \mathbf{E}$ curl $\mathbf{H}$ and transform the surface integrals in (1) into volume integrals with the aid of the Gauss theorem. This yields (cf. Refs. 14 and 15)

$$\Delta\mathscr{E} = \Delta Q - \frac{c\Delta t}{4\pi}\int_V (\mathbf{H}\,\mathrm{rot}\,\mathbf{E} - \mathbf{E}\,\mathrm{rot}\,\mathbf{H})\,dv - \frac{\Delta t}{4\pi}\int_{V_1} H_{1z}\frac{\partial H_{1z}}{\partial t}dv$$
$$= \Delta Q - \frac{c\Delta t}{4\pi}\int_V \left(-\frac{\mathbf{H}}{c}\frac{\partial \mathbf{B}}{\partial t} - \frac{4\pi}{c}\mathbf{E}\mathbf{j}_0\right)dv - \Delta\frac{H_{1z}^2}{8\pi}V_1 \quad (2)$$

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(V_1 is the volume of the cavity, $V=V_s+V_1$ is the total volume of the cylinder, including that of the cavity, and $H_{1z}=H_1$ is the field inside the cavity). We use here Maxwell's equations

$$\operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\quad \operatorname{rot}\mathbf{B}=\frac{4\pi}{c}\mathbf{j}=\frac{4\pi}{c}\mathbf{j}_0+\frac{4\pi}{c}\mathbf{j}_{\text{coup}},$$

$$\mathbf{B}=\mathbf{H}+4\pi\mathbf{m},\quad \operatorname{rot}\mathbf{H}=\frac{4\pi}{c}\mathbf{j}_0,\quad \operatorname{rot}4\pi\mathbf{m}=\frac{4\pi}{c}\mathbf{j}_{\text{coup}},$$

with $J_0=I/S_c$ is the density of the transport current uniformly distributed over the section of the cylinder S_c, and j_{coup} is the density of the screening currents due to the Meissner effect and leading to redistribution of the currents inside the superconductors (cf. Ref. 16).

We use next the definition $\mathbf{E}=-c^{-1}\partial\mathbf{A}/\partial t-\Delta\varphi$ ($\mathbf{A}$, φ are the potentials of the electromagnetic field) and the inequality $\Delta Q\leqslant T\Delta S$, where T is the temperature and S is the change of the entropy of the system, and rewrite (2) in the form (we assume $T=$const, $H_0=$const and $j_0=$const)

$$\Delta\left[\mathscr{E}-TS-\int\limits_V\frac{\mathbf{HB}}{4\pi}dv+\frac{H_{1z}^2}{8\pi}V_1+\int\limits_{V_s}\mathbf{j}_0\left(\frac{\mathbf{A}}{c}+\int^t\nabla\varphi\,dt\right)dv\right]\leqslant 0. \quad (3)$$

Thus, at $T=$const, $H_0=$const, $j_0=$const the minimal functional for the superconducting cylinder is

$$\Phi_s(H_0,I)=F_s-\int\limits_V\frac{\mathbf{HB}}{4\pi}dv+\frac{H_{1z}^2}{8\pi}V_1+\frac{1}{c}\int\limits_{V_s}\mathbf{j}_0\mathbf{A}_z\,dv+C_I, \quad (4)$$

where $F_s=E-TS$ is the free energy of the system, and C_I is a certain constant (dependent on the total current I), which is the result of the presence in (3) of a term with a scalar potential. (The need for taking this term into account follows from gauge-invariance considerations.)

The expression for the free energy of the superconductor takes the usual form[11,12]:

$$F_s=F_{n0}+\int\left\{\frac{B^2}{8\pi}+\alpha|\Psi|^2+\frac{\beta}{2}|\Psi|^4+\frac{1}{2m}\left|i\hbar\nabla\Psi+\frac{2e}{\hbar c}\mathbf{A}\Psi\right|^2\right\}dv, \quad (5)$$

where F_{n0} is the free energy of the metal in the normal state in the absence of a field, Ψ is the wave function (order parameter) of the superconductor, and $\alpha<0$ and $\beta>0$ are coefficients that depend on the temperature. It must be borne in mind that in the general case the vector potential $\mathbf{A}$ in (4) and (5) has two nonzero components $\mathbf{A}=\{A_\theta(r),A_z(r),A_r=0\}$. Where θ, z, and r are cylinderical coordinates. The component $A_\theta(r)$ is due to the presence of the external magnetic field $H_z|_{r=r_2}=H_0$, while the component $A_z(r)$ is due to the transport current I that flows along the cylinder and whose field in the surface is $H_\theta|_{r=r_2}=H_I=2I/cr_2$ (r_2 is the outer radius of the cylinder). Consequently, the magnetic field $\mathbf{B}=\operatorname{curl}\mathbf{A}$ also has two nonzero components, B_z and B_θ. At $I=0$ and $B_\theta=0$ Eq. (4) goes over into the expression for the thermodynamic potential of a hollow cylinder in an external magnetic field H_0, an expression used by us earlier[7]:

$$\Phi_s(H_0)=F_s(H_0)-\int\limits_V\frac{\mathbf{H}_0\mathbf{B}}{4\pi}dv+\frac{H_{1z}^2}{8\pi}V_1. \quad (6)$$

It is easily seen that if the changes in the system take place under the condition that A is constant, then the Poynting-vector flux in (1) is zero (inasmuch as $\mathbf{E}=0$ in this case). If H_I and H_0 are constant on the boundary, the Poynting vector is already different from zero (inasmuch as in this case $\mathbf{A}\neq$const and $\mathbf{E}\neq 0$). Therefore the functionals (4) and (6) differ from the free energy by an amount equal to the electromagnetic energy that has penetrated into the sample through its boundaries. We note that by varying the free energy with respect to A at constant Ψ we obtain in the usual manner[12] the Maxwell equation

$$\operatorname{rot}\mathbf{B}=\operatorname{rot}\operatorname{rot}\mathbf{A}=\frac{4\pi}{c}\mathbf{j},$$

$$\mathbf{j}=\mathbf{j}_s=-\frac{ie\hbar}{m}(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*)-\frac{4e^2}{mc}|\Psi|^2\mathbf{A}, \quad (7)$$

and variation of (5) with respect to Ψ^* at constant $\mathbf{A}$ yields the equation

$$\alpha\Psi+\beta|\Psi|^2\Psi-\left(\nabla+\frac{2ie}{\hbar c}\mathbf{A}\right)^2\Psi=0. \quad (8)$$

It is easy to show that variation of the integral terms in (4) and (6) with respect to $\mathbf{A}$ (at constant H_I and H_0) yields a zero contribution. Therefore regardless of which of the functionals, (4), (5), or (6) is taken as the basis, in all cases the minimization condition leads to the equations (7) and (8) of the Ginzburg-Landau theory. The difference between these functionals becomes significant, however, when the energies of the corresponding states are compared.

It will be convenient to change to relative variables (cf. Ref. 12), putting

$$H_{\text{cm}}^2=\frac{4\pi|\alpha|^2}{\beta},\quad \Psi_0^2=\frac{|\alpha|}{\beta},\quad \psi^2=\frac{|\Psi|^2}{\Psi_0^2}, \quad (8a)$$

with

$$\delta_L^2(T)=\frac{mc^2\beta}{16\pi e^2|\alpha|},\quad \xi^2(T)=\frac{\hbar^2}{2m|\alpha|},\quad \varkappa=\frac{\delta_L(T)}{\xi(T)},$$

where $\xi(T)$ is the temperature dependent coherence length of the superconductor, and $\varkappa$ is a parameter of the Ginzburg-Landau theory. In terms of the relative variables, the expression for the difference between the thermodynamic potentials of the superconducting and normal states takes the form (cf. Ref. 7)

$$f=\frac{\Phi_s-\Phi_n}{V_sH_{\text{cm}}^2/8\pi}=\psi^4-2\psi^2-\frac{4\pi}{V_sH_{\text{cm}}^2}\mathbf{M}\mathbf{H}_0+n\frac{\Phi_0}{8\pi}\frac{H_1-H_0}{V_sH_{\text{cm}}^2/8\pi}$$
$$+\frac{1}{V_sH_{\text{cm}}^2/8\pi}\int\left(\frac{B_\theta^2}{8\pi}-\frac{B_\theta H_\theta}{4\pi}+\frac{2\delta_L^2}{\varkappa^2}\left|\nabla_z\psi-\frac{2ie}{\hbar c}A_z\psi\right|^2\right)dv$$
$$+\frac{1}{V_sH_{\text{cm}}^2/8\pi}\int\frac{\mathbf{j}_0\mathbf{A}_z}{c}dv+c_I, \quad (9)$$

where Φ_s is defined in (4), $\Phi_n=F_{n0}-VH_0^2/8\pi+E_I$, E_I is the energy connected with the transport of the current in the conductor, c_I is a constant ($c_{I=0}=0$), $\mathbf{M}$ is the magnetic moment of the hollow cylinder in the field H_0, H_1 is the magnetic field inside the cylinder cavity, and A_z is the potential produced by the transport current. The terms due to the transport currents are explicitly separated in (9) (in the second and third lines). At $I=0$ and $B_0=0$ Eq. (9) goes over into the expression used by us in Ref. 7.

Expression (9) obtained above for the thermodynamic potential is valid at arbitrary dimensions of the system. To obtain an explicit expression for f in terms of the cylinder parameters it is necessary to substitute in (9) the explicit expressions for $\mathbf{M}$, H_1 and A_z obtained by solving Eqs. (7) for a hollow cylinder with appropriate boundary conditions. The solution of such an electrodynamic problem at constant Ψ, expressed in terms of Bessel functions of imaginary argument, is given, for example, in Ref. 17. We are interested here in the case of a thin-wall cylinder with $d \ll \delta$, where $d = r_2 - r_1$ is the thickness of the cylinder wall, and $\delta = \delta_1/\psi$. Expanding the Bessel functions in terms of the small parameter $d/\delta \ll 1$ and retaining terms of order d^3 inclusive, we obtain after rather cumbersome calculations the following expression in place of (9):

$$f=\psi^4-2\psi^2+\frac{2A(\Phi/\Phi_0^*-n)^2\psi^2}{1+{}^1/_2\mu\psi^2}+C\left(\frac{\Phi}{\Phi_0^*}\right)^2\psi^2-\frac{D}{\psi^2}\left(\frac{\Phi_I}{\Phi_0}\right)^2+c_I, \quad (10)$$

where

$$A=\frac{\xi^2(T)}{r_1^2}\left(1+\frac{1}{2}\frac{d}{r_1}\right), \quad C=\frac{2}{3}\frac{d^2}{r_1^2}\frac{\xi^2(T)}{r_1^2}\frac{1}{1+d/r_1}, \quad D=\frac{8\varkappa^4\xi^6(T)}{r_1^4 d^2},$$

$$\mu=\frac{r_1 d}{\delta_L^2(T)}\left(1+\frac{3}{2}\frac{d}{r_1}\right), \quad \Phi_0^*=\Phi_0\Big/\left(1+\frac{d}{r_1}\right), \quad \Phi_0=\frac{hc}{2e},$$

$$\Phi=\pi r_1^2 H_0, \quad \Phi_I=\pi r_1^2 H_I, \quad H_I=2I/cr_1.$$

The quantity μ in (10) (the so-called screening factor) plays an important role in cylindrical systems (see Refs. 17 and 7), n is an integer that shows how many flux quanta are "frozen" inside the cylinder, the term $\sim A(\Phi/\Phi_0^* - n)^2$ describes the oscillatory character of the penetration of the external field inside the cavity, the parabolic term $\sim C\Phi^2$ describes the destruction of the superconductivity of the cylinder with increasing Φ (see Ref. 7 for details). Expression (10) differs from that given in Ref. 7 in the last two terms, which are due to the presence of the transport current in the system. This is our principal equation, on which the arguments that follow are based.

Figure 1 shows several plots of $f(\psi^2)$ in accord with

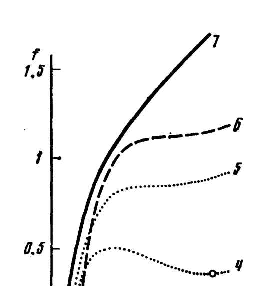

FIG. 1. Dependence of the thermodynamic potential f on ψ^2 in a state with $n = 0$ at the following values of ϕ_I and ϕ: 1 – $\phi_1 = 0, \phi = 0$; 2 – $\phi_I = 0{,}1, \phi = 0, 1$; 3 – $\phi_I = 0{,}1, \phi = 0, 5$; 4 – $\phi_I = 0, 2, \phi = 0, 1$; 5 – $\phi_I = 02, \phi = 0, 5$; 6 – $\phi_I = 0, 3, \phi = 0, 1$; 7 – $\phi_I = 0, 1, \phi = 1$. The points of minimum f, which correspond to the values $\psi = \psi_0$, are marked by circles. The cylinder parameters in all the figures are: $\xi_0 = 2.10^{-5}$ cm, $d = 1{,}4.10^{-5}$ cm, $r_1 = 7.10^{-4}$ cm, $x = 0{,}2 (T = 2.10^{-4}$ K).

Eq. (10) for different $\phi = \Phi/\Phi_0^*$ and $\phi_I = \Phi_I/\Phi_0$. In view of the presence of the indeterminate constant c_I in (10), the curves of Fig. 1 belonging to different ϕ_1 are shifted relative to one another in such a way that the minima of the functions $f(\psi^2)$ corresponding to larger values of Φ_I lie respectively higher along the coordinate axis. (This corresponds to the fact that turning on the current makes the state of the system energywise less favored.) Attention is called to the fact that as $\psi \to 0$ the curves $f(\psi^2) \to -\infty$. Let us explain the cause of this behavior.

This behavior is due to the presence in (10) of the term $\sim -D/\psi^2$, which is due in turn to the term $\sim j_0 A_z$ in (9). In fact, if we eliminate A_z from (9) with the aid of Eq. (7), $A_z = -4\pi c^{-1}\delta_L^2 j_z/\psi^2$, and substitute $j_z \approx j_0$ (the transport current in a thin-wall cylinder is uniformly distributed over the thickness), then we obtain the term $\sim -D/\psi^2$ in (10). It is easily seen from (7) that the divergence of the type $\sim -D/\psi^2$ as $\psi \to 0$ is due to the fact that we assume in (7) that the entire current $\mathbf{j} = \mathbf{j}_s$ is superconducting. On the other hand, the condition I = const as $\psi \to 0$ leads to the unphysical requirement $\mathbf{j}_s \to \mathbf{j}_0$ and $A_z \to \infty$ in (7). In fact, one should write in (7) $\mathbf{j} = \mathbf{j}_s + \mathbf{j}_n$, where $j_s \to 0$ as $\psi \to 0$ and the entire current becomes normal, $\mathbf{j} = \mathbf{j}_n = \mathbf{j}_0$. Then $\mathbf{A} \to$ const and the divergence in (7) is eliminated (as $\psi \to 0$, obviously, $f \to 0$, corresponding to the normal state). For a correct allowance for the normal component of the current, however, it is necessary to have a theory that describes the process of the mutual transformation of the superconducting and normal currents, i.e., that takes into account the nonequilibrium processes in the superconductor. It was not the purpose of the present study to develop such a formalism. We shall therefore not use below expression (10) for the thermodynamic potential f in the immediate vicinity of the point $\psi = 0$, where this expression becomes unsuitable. It is clear that at the points of minimum f (at $\psi \neq 0$) the system is entirely in the superconducting state ($\mathbf{j}_n = 0$), and the description of the system near the minimum points with the aid of the equilibrium expression (19) for f is then correct.

We are thus interested in the behavior of the system near the minima of the potentials f (10). We write down first the extremum condition for the potential $\partial f/\partial\psi = 0$, which takes according to (10) the form[2)]

$$\frac{A(\phi-n)^2}{(1+{}^1/_2\mu\psi^2)^2}=1-\psi^2-\frac{1}{2}C\phi^2-\frac{1}{2}\frac{D}{\psi^4}\phi_I^2, \quad (11)$$

where we put $\phi = \Phi/\Phi_0^*$, $\phi_I = \Phi_I/\Phi_0$. Relation (11) determines the value of ψ_0^2 corresponding to the minimum of the potential f, as a function of the external field H_0 ($\Phi = \pi r_1^2 H_0$) and the field of the current H_I ($\Phi_I = \pi r_1^2 H_I$).

It is clear from Fig. 1 that the minimum of the potential f vanishes with increasing ϕ (or ϕ_I) at the point ψ_{00}, which corresponds at the condition $\partial^2 f/\partial\psi^2 = 0$. This point[3)] determines the boundary of the region outside of which the potential f has no minimum, i.e., the superconducting state (at any rate, that corresponding to the solution ψ = const) turns out to be impossible. We present here the formula obtained from the condition $\partial^2 f/\partial\psi^2 = 0$ upon substitution of formula (11):

$$1+\mu\left(\frac{3}{2}\psi^2-1+\frac{C\phi^2}{2}\right)-\frac{D\phi_I^2}{\psi^6}=0. \quad (12)$$

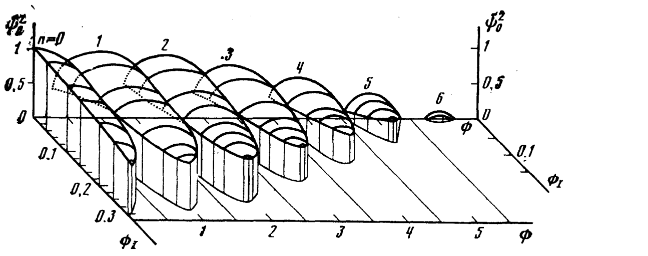

FIG. 2. The modulus of the order parameter, corresponding to the minimum of f, vs. ϕ and ϕ_I, plotted in accord with (11) at $T = 2\times10^{-4}$ K. The boundaries of the $\psi_0^2(n)$ surfaces correspond to the values of ψ_{00}.

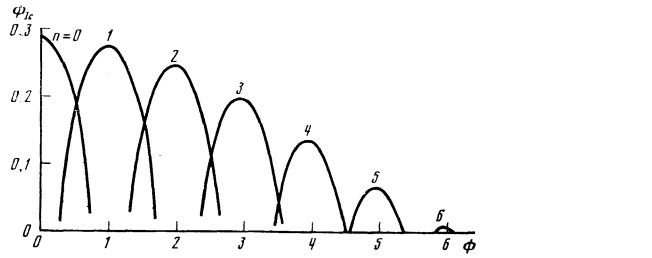

FIG. 4. Dependence of the critical current on ϕ at $T = 2\times10^{-4}$ K. The overlap of curves with different n means the possibility of hysteresis.

Simultaneous solution of (11) and (12) yields the values of ψ_{00}^2 and ϕ_{Ic} corresponding to the boundary of the pure superconducting state. The system (11), (12) calls for solution of algebraic equations of high degree, something impossible to perform analytically in the general case. Figures 2–4 show the functions $\psi_0^2(\phi, \phi_I)$ obtained numerically from (11) at the minimum of the potential f, as well as the plots of the critical current $I_c(\phi)$ corresponding to the end points of the pure superconducting state.

An interesting feature of the curves on Figs. 2–4 is the oscillatory dependence of the critical current on the external magnetic field, due to the penetration of the individual flux quanta inside the cylinder cavity. Another feature of these curves is the ambiguity of the function $I_c(\phi)$ (the curves of Fig. 4 overlap at $\mu > 1$), thus pointing to a possible hysteresis of the critical current. These features are apparently fully observable in experiment. To our knowledge the corresponding experiments with very thin hollow cylinders have not yet been performed.

Figure 5 shows the dependence of the field H_1 inside the cavity as a function of the external field H_0 and of the transport current I.

4. If the left-hand side of (11) is regarded as a small quantity, then it is easy to obtain analytic expressions for the solutions of the system (11), (12):

$$\psi_{00}^2 = \frac{2}{3} - \frac{1}{3}C\phi^2 + \frac{2A(\phi-n)^2}{3(1+{}^1/_3\mu-{}^1/_6\mu C\phi^2)^3},$$
$$\phi_{Ic}^2 = \frac{2}{D}\psi_{00}^4\left[1-\psi_{00}^2-\frac{1}{2}C\phi^2-\frac{A(\phi-n)^2}{(1+{}^1/_2\mu\psi_{00}^2)^2}\right]. \qquad (13)$$

Formulas (13) are valid at $\phi - n \ll 1$ and describe the behavior of the curves ϕ_{00}^2 and ϕ_{Ic} (see Figs. 3 and 4) near the points $\phi = n$. In addition, these formulas are valid for arbitrary values of $(\phi - n)$, but at $\mu\psi_{00}^2 \gg 1$.

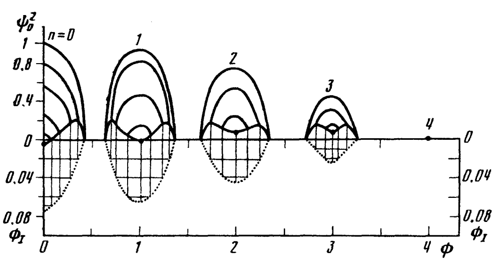

FIG. 3. The same as in Fig. 2, but at $T = 0.8\times10^{-4}$ K. The thin vertical lines are the ordinates of the end points of the surfaces ψ_0^2. The dashed curves show the dependence of the critical current on ϕ.

We note that formulas (13) admit of a transition to the limiting case of a flat film in a magnetic field parallel to its surface. To this end it is necessary to put $\phi = \Phi/\Phi_0 = n$ in (13) (cf. Ref. 7; this corresponds to equality of the magnetic fields on the two sides of the film), and neglect the small terms $d/r_1 \ll 1$ ($r_1 \to \infty$). Formulas (13) then go over into those obtained by Ginzburg[14] for the case of a plane film.

Solutions of (11) and (12) can also be obtained in some limiting cases. Thus, by obtaining from (11) the condition $dj/d\psi = 0$ (this corresponds to the maximum-current criterion, cf. Ref. 12), we obtain the equation

$$-1+\frac{3}{2}\psi^2+\frac{1}{2}C\phi^2+\frac{A(\phi-n)^2}{(1+{}^1/_2\mu\psi^2)^3}=0. \qquad (14)$$

We note that this equation is exactly equivalent to the condition $\partial^2 f/\partial\psi^2 = 0$, i.e., the maximum current is reached at the inflection point of the function $f(\psi)$. Assuming that $\mu\psi_{0c}^2 \gg 1$ (here $\psi_{0c}^2 = \frac{2}{3} - \frac{1}{3}C\phi^2$), we readily obtain from (11) and (14) the formula

$$\left(\frac{2}{3}+\frac{1}{3}\mu-\frac{1}{6}\mu C\phi^2\right)^3 = \mu A(\phi-n)^2+\frac{d}{r_1}\frac{1}{\varkappa^2}\phi_{Ic}^2. \qquad (15)$$

This formula, at a given temperature, establishes the

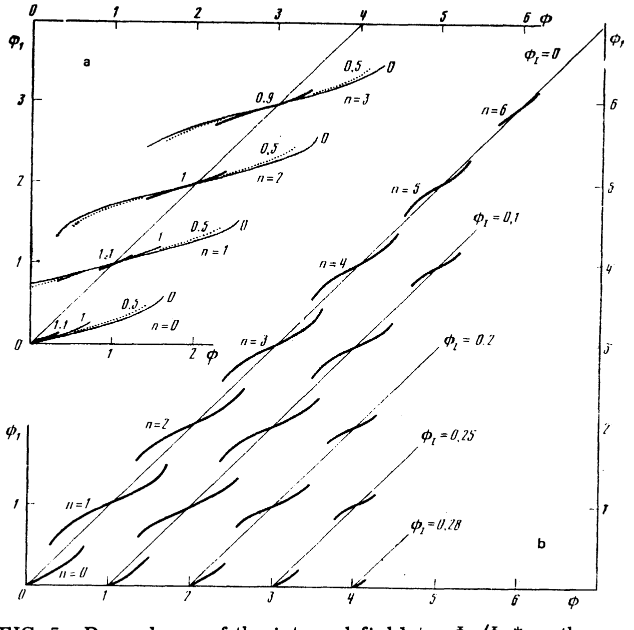

FIG. 5. Dependence of the internal field $\phi_1 = \Phi_1/\Phi_0{}^*$ on the external field $\phi = \Phi/\Phi_0{}^*$ in states with different values of $\phi_I = \Phi_I/\Phi_0$ (see the numbers on the curves). a) $T = 5\times10^{-4}$ K, b) $T = 2\times10^{-4}$ K. In Fig. b, the curves corresponding to different ϕ_I are shifted relative to one another along the abscissa axis (all should start out from the origin, as in Fig. a.).

connection, at the critical point ($\partial f/\partial\psi=0$, $\partial^2 f/\partial\psi^2=0$) between n (the number of frozen-in quanta), ϕ (the external field), and ϕ_{Ic} (the field of the current). When plotted in the coordinates (ϕ, ϕ_{Ic}), so long is the parabolic term $C\phi^2$ is small, we have a family of ellipses (cf. the curves on Figs. 3 and 4). It can then be shown (see also Fig. 2) that the value of ψ_{00}^2 depends little on the current or the field. At $\phi_I=0$, Eq. (15) goes over into Eq. (17) of Ref. 7.

The critical point at which superconductivity is destroyed can be reached not only by increasing the current I (or the field H_0), but also by changing the temperature of the sample. It is easy to obtain from (15) the dependence of the effective transition temperature T^* (cf. Ref. 7) on the magnetic field and on the current:

$$\frac{T_c-T^*}{T_c}=0.55\varkappa^2\frac{\xi_0^2}{r_1 d}\left\{a_1\left[\left(\frac{\Phi}{\Phi_0}-n\right)^2+\left(\frac{\Phi_I}{\Phi_0}\right)^2\right]^{1/3}+c_1\left(\frac{\Phi}{\Phi_0}\right)^2-2\right\}, \quad (16)$$

where

$$a_1=3(d^2/\varkappa^2 r_1)^{1/3},\quad c_1=d^3/3\varkappa^2 r_1^3.$$

It is assumed here that $d/r_1\ll 1$ and the formula $\xi^2(T)=0.55\xi_0^2(1-T/T_c)^{-1}$, which is valid for pure superconductors,[11] is used. At $\Phi_I=0$ Eq. (16) goes over into Eq. (23) of Ref. 7. Equation (16) describes the influence of the transport current on the effect of the oscillation of the critical transition temperature in a magnetic field (see Refs. 7–11 for details).

We obtain analogously from (11) and (14) an equation valid at the critical point ($\partial f/\partial\psi=0$, $\partial^2 f/\partial\psi^2=0$) in another limiting case ($\mu<1$, $\mu\psi_{00}^2\ll 1$):

$$1-\frac{1}{2}C\phi^2-A(\phi-n)^2=3\phi_I^{2/3}\frac{\xi^2\varkappa^{4/3}}{r_1^{4/3}d^{2/3}}. \quad (17)$$

At $\phi_I=0$ this formula goes over into expression (15) of Ref. 7. We note that if we neglect the parabolic term $(\frac{1}{2})C\phi^2$, then (17) coincides with the result obtained in Ref. 13.

As seen from an analysis of (11) and (14), as well as from Fig. 3, the condition $\psi_{00}^2\ll 1$ is realized only at $\phi_I\ll 1$ (it can be shown that at the critical point $\phi_I\sim\psi_{00}^3$, so that (17) is valid only at small $\phi_I\ll 1$). At finite values of ϕ_I the superconductivity is destroyed via a first-order phase transtion (at finite values of ψ_{00}^2, see Fig. 2), and in place of (17) we must use Eq. (15). On the other hand in the case of small currents, $\phi_I\ll 1$, it is easy to obtain from (17) an expression for the effective transition temperature (in pure superconductors):

$$\frac{T_c-T^*}{T_c}=0.55\frac{\xi_0^2}{r_1^2}\left\{(\phi-n)^2+\frac{1}{3}\frac{d^2}{r_1^2}\phi^2+3\phi_I^{2/3}\left(\frac{\varkappa^2 r_1}{d}\right)^{2/3}\right\}. \quad (18)$$

If we assume, in accord with the experiments,[8,9] a measuring current $I\sim 10\ \mu A$ ($j_0\sim 10^3\,A/cm^2$), then $\phi_I\sim 10^{-3}$, i.e., the last (current) term in (18) is smaller by two orders of magnitude than the first (field) term, $\phi-n\sim 1$). Thus, under the conditions of the experiments,[8,9] the influence of the measuring current is small. It is of interest, however, to perform the corresponding experiments under conditions of a strong measuring current ($\phi_I\sim 1$, in which case the current density is $j_0\sim 10^4-10^6\ A/cm^2$) and to carry out a comparison with formula (16) for samples with $\mu\gg 1$. According to this formula, the effect of the oscillations of $T^*(\Phi)$ should be observed also at high current densities.

In fact, in the absence of a magnetic field (if $\mu\gg 1$ and $r_1\to\infty$) formula (15) coincides with the expression for the critical current of a thin film.[14] It is clear that a cylindrical sample (just as a thin film) can be superconducting not only at low but also at high current density, provided that the temperature of the sample is not too close to T_c. Therefore the sample-resistance oscillations produced when the magnetic field is turned on (similar to those observed in Refs. 8 and 9 at small currents near T_c) should in principle be observed also at large currents (far from T_c). We note incidentally that with increasing distance from T_c the screening factor μ increases, and this leads to a decrease of the oscillations. It is therefore necessary in the experiment to seek a certain optimal region in which these oscillations can be noticeable.

It is also of definite interest to use the formulas derived above to study the destruction, by current, of states with definite numbers n of flux quanta frozen in the cylinder in the absence of an external field. To this end it is necessary to put $\phi=0$ in (15)–(18). Equations (15) and (17) then yield expressions for the critical current in the state n at $T=$const, while (16) and (18) give values of T^* at given values of n and of the current ϕ_I.

In conclusion, a few words concerning the regions on Figs. 2 and 3 where there are no pure superconducting solutions with $\psi\neq 0$. According to the thermodynamic theory developed above, a normal state should be realized in these regions. This theory, however, operates with an order parameter ψ, which is assumed to be independent of the coordinates. Therefore the absence of solutions $\psi=$const still does not mean that no inhomogeneous resistive states can be realized and can be energy wise more profitable than the pure normal state. For example, in the case of a bulky cylinder a state that is inhomogeneous over the thickness or a mixed state can be produced.[1–4] One cannot exclude the possible realization of an inhomogeneous (along the cylinder) resistive state also in a thin-wall cylinder, as a result of the appearance of helical currents similar to those that are apparently responsible[18] for the paramagnetic effect in thick-wall hollow cylinders. It is desirable in this connection to perform special experiments with thin-wall cylinders.

APPENDIX

To obtain relation (11) we can start directly from the Ginzburg-Landau equations (7) and (8). In fact, multiplying (8) by Ψ^* and adding to the complex conjugate of the product, we obtain after integrating over the volume

$$\int(2\alpha|\Psi|^2+2\beta|\Psi|^4)\,dv-\frac{\hbar^2}{2m}\int(\Psi^*\nabla^2\Psi+\Psi\nabla^2\Psi^*)\,dv$$
$$+\int\left[\frac{2ie\hbar}{mc}A_\theta(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*)_\theta+\frac{4e^2}{mc^2}|\Psi|^2(A_\theta^2+A_z^2)\right]dv=0. \quad (A1)$$

Putting $\Psi=f_0(r)e^{in\theta}$ and assuming f_0 to be a constant independent of the coordinates, we express A_0 in the last term of (A1) interms of j_θ by means of (7). Using next the Gauss integral theorem and expressing the terms with j_θ in terms of the magnetic field $\mathbf{B}$ (with the aid of the equation curl $\mathbf{B}=4\pi c^{-1}\mathbf{j}$), we obtain after simple transformations in place of (A1)

$$\int (\alpha f_0^2+\beta f_0^4)\,dv+\frac{1}{2}\mathbf{H}_0\mathbf{M}+\frac{1}{8\pi}H_0V-\frac{1}{8\pi}H_1^2V_1$$
$$-\frac{1}{8\pi}\int_{V_s} B_z^2\,dv+\frac{\hbar cn}{8e}(H_1-H_0)+\int_{V_s}\frac{2e^2}{mc^2}A_z^2f_0^2\,dv=0. \qquad \text{(A2)}$$

(Obviously, relations (A1) and (A2) are equivalent to the extremum condition on the functional $\delta F=0$ (or $\delta\Phi_s=0$), from which Eqs. (7) and (8) follows.

It is necessary next to use the expressions for the field H_1 and the moment $\mathbf{M}$ of the hollow cylinder in terms of Bessel functions (these expressions are given in Ref. 7), as well as the expressions for $A_z(r)$ and $A_\theta(r)$ (cf. Ref. 17):

$$A_z(r)=H_1\frac{\delta[I_0(\xi)K_1(\xi_1)+K_0(\xi)I_1(\xi_1)]}{I_1(\xi_1)K_1(\xi_2)-I_1(\xi_2)K_1(\xi_1)},\quad B_z(r)=\frac{1}{r}\frac{d}{dr}(rA_\theta),$$
$$A_\theta(r)=\frac{\hbar cn}{2e\delta\xi}+\delta\frac{aI_1(\xi)+bK_1(\xi)}{K_0(\xi_1)I_0(\xi_2)-I_0(\xi_1)K_0(\xi_2)}$$
$$a=H_0K_0(\xi_1)-H_1K_0(\xi_2),\quad b=H_0I_0(\xi_1)-H_1I_0(\xi_2).$$

Here K_n and I_n are Bessel functions of imaginary argument, $\xi=r/\delta$, $\xi_1=r_1/\delta$, $\xi_2=r_2/\delta$, $\delta=\delta_L/\psi$; ψ is the modulus of the order parameter (in relative units, see the text). Expanding the Bessel functions in the small parameter $d/\delta\ll 1$ ($d=r_2-r_1$) and retaining terms of order $(d/\delta)^3$ we arrive after cumbersome calculations again at relation (11).

[1)]Actually the destruction of superconductivity by a current takes place in some finite region of the value of I near I_c, and is due to the appearance of the resistive state, i.e., to the gradual restoration of the normal resistance. This process can not be described within the simple Ginzberg-Landau thermodynamic theory.

[2)]Relation (11) can be obtained also directly from (7) and (8) by using the explicit solutions[17] for the potentials $A_-(r)$ and $A_z(r)$. The corresponding calculation is given in the Appendix.

[3)]It can be shown that at this point $f''_\psi>0$, i.e., we have an inflection point. Since it is clear that the inflection point must be located at $f>0$, where $f\sim\Phi_s(H_0,\ I)-\Phi_n(H_0,\ I)$ [see (9)], it follows that this point is obviously in the region of metastability of the superconducting state. It can be verified that it is necessary in this case to put $c_f>0$ in (9).

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Elastic properties of crystal surfaces

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The general properties of the surface stress tensor, describing elastic properties of crystal surfaces, are determined. The boundary conditions are obtained for the bulk stress tensor on the surface of a crystal of arbitrary shape. The elastic interaction between point and line defects on crystal surfaces is considered.

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It is well known that the thermodynamic properties of a liquid surface are governed entirely by one quantity which is the work done in reversible changes of the surface area. As pointed out long ago by Gibbs,[1] in the case of a solid we have to distinguish the work done in forming the surface and in deforming it. Thus, in describing the properties of crystal surfaces we have to introduce not only the surface energy but also the surface stress tensor. We shall determine the general properties of this tensor and find the boundary conditions replacing in our case the familiar Laplace formula for the capillary pressure.

In the second section we shall consider the elastic interaction of surface defects over distances which are large with the atomic separations. As in the case of

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