$$Z_0=\sum_{N_1}\sum_{N_2}\cdots\sum_{N_N}\exp\left\{\beta\sum_{k=1}^{N}(UN_k-VN_kN_{k+1})\right\}$$
$$=\sum_{N_1}\cdots\sum_{N_N}P(N_1,N_2)P(N_2,N_3)\ldots P(N_N,N_1)$$
$$=\sum_{N_1}P^N(N_1,N_1)=\mathrm{Sp}\,P^N=\sum_j\lambda_j^N. \qquad (A.5)$$

Here λ_j are the eigenvalues of the matrix P. For large N (number of sites) it is necessary only to use the largest eigenvalue λ_M.

For the further calculations it is necessary to introduce a unitary matrix S which diagonalizes P, i.e.,

$$S^+PS=\begin{vmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{vmatrix}=D_{ij}=\lambda_i\delta_{ij}.$$

We now write down an equation for $\langle N_i\rangle$ and the correlator $\langle N_iN_j\rangle$ (the number of sites $N\to\infty$)

$$\langle N_i\rangle=\nu=\frac{1}{Z_0}\mathrm{Sp}(P^i\bar{N}_iP^{N-i})=\frac{1}{Z_0}\mathrm{Sp}(S^+P^i\bar{N}_iP^{N-i}S)$$
$$=\frac{1}{Z_0}\mathrm{Sp}(D^iS^+\bar{N}_iSD^{N-i})=\frac{1}{Z_0}\mathrm{Sp}(D^NS^+\bar{N}S)$$
$$=\frac{1}{Z_0}\sum_i\lambda_i^N\left(\sum_m S_{mi}^2\bar{N}_m\right)\approx\frac{\lambda_M^N}{Z_0}\sum_m S_{mM}^2\bar{N}_m=\sum_m S_{mM}^2\bar{N}_m. \qquad (A.7)$$

Here

$$\bar{N}_i=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

For the correlator we get ($N\gg k$)

$$\langle N_iN_{i+k}\rangle=\frac{1}{Z_0}\mathrm{Sp}(D^{N-k}S^+\bar{N}SD^kS^+\bar{N}S)$$
$$=\frac{1}{Z_0}\sum_{i,m,p,q}\lambda_i^{N-k}\lambda_m^kS_{qi}S_{pi}S_{qm}S_{pm}\bar{N}_q\bar{N}_p\simeq\frac{1}{\lambda_M^k}\sum_m\lambda_m^k\left(\sum_p S_{pM}S_{pm}\bar{N}_p\right)^2. \qquad (A.8)$$

We find from Eq. (A.7) the matrix elements of the matrix S and substituting them into (A.8) we find the correlator $\langle N_iN_{i+k}\rangle$. Since we are interested in correlations between neighboring sites we get as a result, putting $k=1$, at $\nu\ll1$ the formula

$$\langle N_iN_{i+1}\rangle=\nu^2e^{-\nu/\pi}.$$

[1]P. W. Anderson, J. Phys. Chem. Solids **11**, 26 (1959).
[2]A. A. Abrikosov and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. **33**, 319 (1959) [Soviet Phys. JETP **6**, 250 (1960)].
[3]E. Abrahams and I. O. Kulik, Normal state conductivity, screening, and possible transition to the pair-correlated state in "superconducting glasses," preprint, 1977.
[4]I. O. Kulik and A. G. Fedan, Abstracts of Contributions to the XXth All-Union Meeting on Low-Temperature Physics, NT-20, M 1978, Chs. 3, 87.
[5]W. L. McLean, P. Lindefeld, and T. Worthington, Hopping conduction and superconductivity in granular aluminium, preprint, 1977.
[6]O. Fisher, M. Decroux, S. Roth, R. Chevrel, and M. Sergent, J. Phys. **A8**, L474 (1975).
[7]I. O. Kulik and R. I. Shekhter, Fiz. Nizk. Temp. **2**, 21 (1976)[Sov. J. Low Temp. Phys. **2**, 9 (1976)].
[8]L. P. Gor'kov, Zh. Eksp. Teor. Fiz. **65**, 1658 (1973) [Sov. Phys. JETP **38**, 830 (1974)].
[9]I. M. Dmitrenko and I. O. Kulik, Phys. Status Solidi B **64**, k 13 (1974).
[10]R. Fincus, P. Chaikin, and C. F. Coll, Solid State Commun. **12**, 1265 (1973)
[11]R. A. Bari, Phys. Rev. **B3**, 2662 (1971).
[12]P. W. Anderson, Phys. Rev. Lett. **34**, 953 (1975).
[13]V. J. Emery, Phys. Rev. **B14**, 2989 (1976).
[14]H. J. Lipkin, Quantum Mechanics, North Holland Publ. Co., Amsterdam, 1973.
[15]A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Quantum Field Theoretic Methods in Statistical Physics) 1962 [English translation published by Pergamon Press, Oxford].
[16]P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[17]G. F. Newell and E. W. Montroll, Rev. Mod. Phys. **25**, 353 (1953).
[18]G. Beni and P. Pincus, Phys. Rev. **B9**, 2963 (1974).

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Parameters of an electron beam in a free-electron laser under strong saturation conditions

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Bunching of the electron beam in a free-electron laser under strong saturation conditions is considered in the given-field approximation. The electron phase and the velocity distribution functions, and also the total energy transferred to the electromagnetic wave by the beam, are found.

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1. The first experiments on the amplification and generation of light by means of a relativistic electron beam passing through a region with a transverse magnetic field that was varying periodically in space were carried out recently.[1-3] A linear theory of such an amplifier or generator, known as a free-electron laser, was developed in a number of works (see, for example, Refs. 4–6). Nonlinear phenomena in similar apparatus are due to the change in the parameters of the electron beam under the action of the electromag-

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netic field (bunching of the electrons). At a sufficiently low density of the electrons in the beam, when the gain per pass is low, the change in the amplitude of the electromagnetic field in the process of electron bunching is insignificant (the given-field approximation). The nonlinear theory of the free-electron laser has been described in this approximation in a number of papers.[7,8] In these researches, the dependences of the gain and the parameters of the electron beam on the frequency and amplitude of the electromagnetic wave were investigated by means of numerical calculations.

In the interaction of an electron beam, moving in a magnetic field that is spatially periodic, with an electromagnetic wave, the electrons are accelerated or slowed, depending on the initial phase of the wave, and it is this which leads to their bunching. Significant bunching takes place over some characteristic distance which depends on the amplitude of the wave. However, in strong saturation, that is, at distances that are much greater than the bunching length, stationary modulation of the beam and stationary electron energy distribution are established. The exchange of energy between the beam and the wave ceases. The numerical calculations[7,8] apply to distances that are smaller than or comparable with the bunching distance.

In the present work, the parameters of the electron beam in the regime of strong saturation, and the total energy transferred by the beam to the electromagnetic wave, are calculated in the given-field approximation.

2. Let a monoenergetic relativistic electron beam with initial velocity v parallel to the z axis and energy

$$E=\gamma mc^2;\quad \gamma=(1-v^2/c^2)^{-1/2}$$

enter a region with a helical magnetic field $\vec{\mathscr{H}}$:

$$\mathscr{H}_{mx}=\mathscr{H}\cos qz,\quad \mathscr{H}_{my}=\mathscr{H}\sin qz, \tag{1}$$

where $q=2\pi/a$, and a is the period of change in the field. Further, let an electromagnetic wave with circular polarization propagate along the z axis:

$$\begin{aligned}&\mathscr{E}_{wx}=\mathscr{E}\cos(\omega t-kz),\quad \mathscr{E}_{wy}=\mathscr{E}\sin(\omega t-kz);\\ &\mathscr{H}_{wx}=-\mathscr{E}\sin(\omega t-kz),\quad \mathscr{H}_{wy}=\mathscr{E}\cos(\omega t-kz);\quad k=\omega/c.\end{aligned} \tag{2}$$

The equation of motion of the electron in the total field,

$$\frac{d\mathbf{p}}{dt}=e\vec{\mathscr{E}}_w+\frac{e}{c}\left[\frac{d\mathbf{r}}{dt},(\vec{\mathscr{H}}_m+\vec{\mathscr{H}}_w)\right], \tag{3}$$

can be reduced to a one-dimensional equation describing motion along the z axis. For this we must determine the transverse components of the velocity from (3) and substitute them in the expression for dp_z/dt. If the relative change in the energy of the electron due to its interaction with the field is small, then this one-dimensional equation reduces to the pendulum equation[6,7,8]

$$d^2\varphi/dt^2+\Omega^2\sin\varphi=0, \tag{4}$$

where

$$\begin{aligned}&\varphi=(k+q)z-\omega t,\\ &\Omega^2=2e\mathscr{E}\mathscr{H}(e/\gamma mc)^2.\end{aligned} \tag{5}$$

Formula (5) is valid if $\gamma\gg 1$. The condition of smallness of the relative change of the energy of the electron, which was used for derivation of Eq. (4), is equivalent to the conditions $v_1^2/c^2\ll 1-v/c$, $\Delta v_z/c\ll 1-v/c$, where v_1 is the transverse velocity of the electron in the field and Δv_z is the change in its longitudinal velocity. This imposes the following limitation on the values of the fields $\mathscr{E}$ and $\mathscr{H}$:

$$e\mathscr{E}\mathscr{H}\ll(m\omega c/e\gamma)^2. \tag{6}$$

In the condition (6) is satisfied, the motion of the electron in the set of coordinates moving with velocity v is nonrelativistic. In practice, condition (6) is satisfied with a large margin.

For an electron crossing the boundary of the region with the helical magnetic field ($z=0$) at the instant $t=t_0$, the initial conditions to Eq. (4) have the form

$$\varphi(t_0)=-\omega t_0=\varphi_0,\quad (d\varphi/dt)_{t_0}=(\omega_0-\omega)(1-v/c),$$

where $\omega_0=qv(1-v/c)^{-1}$ is the resonant frequency of the electromagnetic wave.[1]

It is convenient to introduce in Eq. (4) the dimenionless variable $\tau=\Omega(t-t_0)$. We then obtain the equation

$$d^2\varphi/d\tau^2+\sin\varphi=0 \tag{7}$$

with initial conditions

$$\varphi(0)=\varphi_0,\quad (d\varphi/d\tau)_{\tau=0}=\Delta_0, \tag{8}$$

where $\Delta_0=(\omega_0-\omega)/2\gamma^2\Omega$ is the dimensionless detuning of the frequency.

We note that since the relative change in the velocity of the electron is small, we have, accurate to terms of order $\Omega/\omega\ll 1$, $\tau=z/l$ where $l=v/\Omega$ is the characteristic length, over which electron bunching occurs, and z is the distance traversed by the electron in the interaction region. Thus the quantity τ is a saturation parameter.[7]

3. The parameters of the electron beam are characterized by the distribution function

$$f(\varphi,\Delta,\tau)=\int_{-\infty}^{\infty}d\varphi_0\,\delta(\varphi-\varphi(\tau,\varphi_0,\Delta_0))\,\delta\left(\Delta-\frac{d\varphi}{d\tau}(\tau,\varphi_0,\Delta_0)\right), \tag{9}$$

where $\varphi=(k+q)z-\omega t$, $\Delta=[(k+q)\dot{z}-\omega]/\Omega$, $\tau=z/l$. The dimensionless quantity Δ is connected with the change δE in the energy of the electron in the following fashion:

$$\frac{\delta E}{E}=\frac{\gamma^2\Omega}{\omega}(\Delta_0-\Delta). \tag{10}$$

A positive δE means that the electron loses energy to the electromagnetic field. The distribution function f is normalized so that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi\int_{-\infty}^{\infty}d\Delta\,f(\varphi,\Delta,\tau)=1. \tag{11}$$

In what follows, we shall be interested in the distribution functions of the electrons in the phase $n(\varphi)$ and in the velocity $g(\Delta)$, and also in the mean value of the change $\overline{\delta E}$ in the energy of the electron at large values of the saturation parameter τ. These quantities have been studied numerically in earlier papers at $\tau\lesssim 1$.[7,8]

At $\tau\gg 1$, the problem is materially simplified, since the distribution function (9) becomes then independent

of τ. Actually, finding the distribution function of the electrons reduces to the following problem: There exists an ensemble of pendulums, the initial angular displacements of which are uniformly distributed over the circle $0 < \varphi_0 < 2\pi$, while the initial angular velocity of all the pendulums, Δ_0 is the same. It is required to find the distribution of these pendulums with respect to the angular displacement φ and angular velocity Δ at a subsequent instant of time. This distribution is given by the formula (9). It is clear that after each pendulum has completed a large number of oscillations ($\tau \gg 1$), a certain stationary distribution is established. This stationary distribution function $f(\varphi, \Delta)$ can be obtained by averaging the integrand in formula (9) over τ for a time $T(\varepsilon)$, where $\varepsilon = \Delta_0^2/2 - \cos\varphi_0$ is the energy of the pendulum with initial angular displacement φ_0.

It is convenient to represent the distribution function in the form of a sum of two terms: $f = f_1 + f_2$, so that f_1 describes the distribution of the trapped electrons ($\varepsilon < 1$) and f_2 the distribution of the untrapped electrons ($\varepsilon > 1$). In terms of the problem of the pendulums, a trapped electron corresponds to an oscillating pendulum, and an untrapped one correspond to a pendulum which rotates vertically around the support. It is obvious that the trapped electrons exist only at $\Delta_0 < 2$.

Averaging the integrand in (9) over τ, as indicated above, it is not difficult to obtain the following asymptotic expressions (at $\tau \gg 1$) for the functions f_1 and f_2:

$$f_1(\varphi, \Delta) = 2 \int_{\Delta_0^2/2-1}^{1} \frac{d\varepsilon}{T(\varepsilon)} \frac{\delta(\Delta^2 - 2(\varepsilon + \cos\varphi))}{[1 - (\varepsilon - \Delta_0^2/2)^2]^{1/2}}, \tag{12}$$

$$f_2(\varphi, \Delta) = 2^{1/2} \int_{\varepsilon_0}^{\Delta_0^2/2+1} \frac{d\varepsilon}{T(\varepsilon)} \frac{\delta(\Delta - (2(\varepsilon + \cos\varphi))^{1/2})}{[(1 - (\varepsilon - \Delta_0^2/2)^2)(\varepsilon + \cos\varphi)]^{1/2}}. \tag{13}$$

The lower limit of integration in (13) depends on the value of Δ_0: $\varepsilon_0 = 1$ at $\Delta_0 < 2$; $\varepsilon_0 = \Delta_0^2/2 - 1$ at $\Delta_0 > 2$.

The period $T(\varepsilon)$ is determined by the usual expression

$$T(\varepsilon) = 2^{1/2} \int_{x_0}^{1} dx [(1 - x^2)(\varepsilon + x)]^{-1/2}, \tag{14}$$

where $x_0 = -\varepsilon$ at $\varepsilon < 1$ and $x_0 = -1$ at $\varepsilon > 1$. We note that the quantity $T(\varepsilon)$ represents the actual period of motion at $\varepsilon > 1$ and half the period of motion at $\varepsilon < 1$. In formulas (12) and (13), we have change from integration over φ_0 to integration over the energy $\varepsilon = \Delta_0^2/2 - \cos\varphi_0$.

The formulas (12) and (13) can also be obtained directly from formula (9) by taking the limit as $\tau \to \infty$. At arbitrary values of τ the distribution function (9) can be represented in the form of a sum over the residues of the δ functions at the points φ_{0n}.[7] At $\tau \to \infty$ the sum can be replaced by an integral, and this also leads to the formulas (12) and (13).

4. The distribution function $f(\varphi, \Delta)$ found above enables us to determine all the characteristics of the electron beam in the regime of strong saturation. The character of the modulation of the electron beam is determined by the phase distribution function $n(\varphi)$, which can be obtained by integrating the distribution

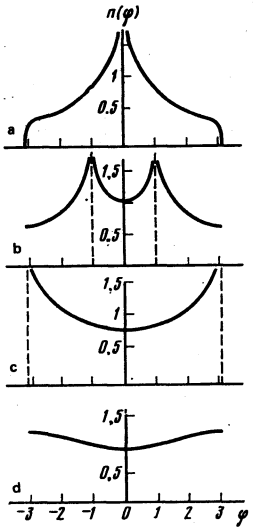

FIG. 1. Modulation of the electron beam in the strong saturation regime at different detunings: a—$\Delta_0 = 0$, b—$\Delta_0 = 1$; c—$\Delta_0 = 2$; d—$\Delta_0 = 3$.

function $f(\varphi, \Delta)$ over Δ. Using formulas (12) and (13), we obtain

$$n(\varphi) = 2^{1/2} \int_{\varepsilon_1}^{1+\Delta_0^2/2} \frac{d\varepsilon}{T(\varepsilon)} \left[\left(1 - \left(\varepsilon - \frac{\Delta_0^2}{2}\right)^2\right)(\varepsilon + \cos\varphi) \right]^{-1/2}. \tag{15}$$

The lower limit of integration ε_1 is determined by the condition that the radicand in the integral (15) be positive: $\varepsilon_1 = -\cos\varphi$ at $\cos\varphi < 1 - \Delta_0^2/2$ and $\varepsilon_1 = \Delta_0^2/2 - 1$ at $\cos\varphi > 1 - \Delta_0^2/2$.

Figure 1 shows the graphs of the function $n(\varphi)$ at different values of the detuning Δ_0, obtained according to (15) by numerical integration. At $\Delta_0 = 0$ (Fig. 1a) the function $n(\varphi)$ has at the point $\varphi = 0$ a singularity of the form $\ln|1/\varphi|$. Actually, if $\varepsilon = -1$ the integral (15) diverges logarithmically at $\Delta_0 = 0$ and $\varphi = 0$, so that this singularity is caused by pendulums with initial phases close to zero. Moreover, as is seen from Fig. 1a, the function $n(\varphi)$ vanishes at $\varphi = \pm\pi$. Near these points

$$n(\varphi) \sim |\ln(\pi - |\varphi|)|^{-1};$$

this is connected with the increase in the period of the oscillations as $\varepsilon \to 1$:

$$T(\varepsilon) \sim \ln|1/(1-\varepsilon)|.$$

At $0 < |\Delta_0| < 2$ (Fig. 1b) the distribution function $n(\varphi)$ has a logarithmic singularity at values of φ such that $\cos\varphi = 1 - \Delta_0^2/2$. At these values of φ the integral (15) diverges logarithmically on the lower limit at $\varepsilon = \Delta_0^2/2 - 1$, also because of the contribution of pendulums with zero initial phase.

The cause of the singularities of the function $n(\varphi)$ can be explained in the following manner. We consider a group of pendulums with initial phases in the range $\delta\varphi_0$, close to some value of φ_0, and with initial angular velocities Δ_0. The greatest contribution to the density $n(\varphi)$ of the group of pendulums is made near the turning points φ_0 such that $\cos\varphi = \cos\varphi_0 - \Delta_0^2/2$, since the pendulums spend the greatest time near these points. The

interval $\delta\varphi$ of the turning points is generally speaking proportional to the initial interval $\delta\varphi_0$:

$$\sin\varphi\delta\varphi=\sin\varphi_0\delta\varphi_0.$$

The case $\varphi_0=0$ is an exception; here the turning points of the chosen group of pendulums are bunched together: $\delta\varphi\sim(\delta\varphi_0)^2$. Thus, at $\cos\varphi=1-\Delta_0^2/2$ the value of the density $n(\varphi)$ will be greatest because of the pendulums with initial phases close to zero.

At the separation boundary $\Delta_0=2$ (Fig. 1c) the singularities of the function $n(\varphi)$ turn out to occur at values $\varphi=\pm\pi$ and have a weaker character:

$$n(\varphi)\sim\ln|\ln(\pi-|\varphi|)|.$$

Of course, the indicated singularities of the function $n(\varphi)$ take place only at $\tau=\infty$. At large but finite values of the saturation parameter τ the function (φ) will have a maximum $\sim\ln\tau$ at the corresponding points. The value of $n(\varphi)$ calculated from formula (15) at a distance x from the singular point is actually achieved at $\tau\sim x^{-2}$.

At $\Delta_0>1$ (Fig. 1d) all the pendulums rotate vertically about their support points and the function $n(\varphi)$ has no singularities. At large values of the detuning Δ_0, it follows from formula (15) that $n(\varphi)=1-\Delta_0^{-2}\cos\varphi$.

5. We get the velocity distribution function from formulas (12) and (13), averaging them over the phase φ. The function $g(\Delta)$ obtained in this way and normalized to unity can be conveniently divided into parts $g_1(\Delta)$ and $g_2(\Delta)$, which describe the trapped and untrapped electrons, respectively. It is evident that the function $g_1(\Delta)$ differs from zero at $|\Delta_0|<2$ and $|\Delta|<2$, and is an even function of Δ. The function $g_2(\Delta)$ differs from zero if the signs of the quantities Δ and Δ_0 are the same.

In the following, we set $\Delta_0>0$ for definiteness. In the opposite case in the formulas below for $g_2(\Delta)$ it is necessary to replace Δ by $-\Delta$. We obtain the following expressions for the distribution functions $g_{1,2}$ from formulas (12) and (13):

$$g_{1,2}(\Delta)=\frac{2}{\pi}\int_{A_{1,2}}^{B_{1,2}}\frac{d\varepsilon}{T(\varepsilon)}\left[1-\left(\varepsilon-\frac{\Delta_0^2}{2}\right)^2\right]^{-1/4}\left[1-\left(\varepsilon-\frac{\Delta^2}{2}\right)^2\right]^{-1/4}. \quad (16)$$

The limits of integration $A_{1,2}$ and $B_{1,2}$ are different for the functions g_1 and g_2, and depend on the relation of Δ and Δ_0. They are determined by the requirement that both radicands in formula (16) be positive, and also by the conditions $\varepsilon<1$ for g_1 and $\varepsilon>1$ for g_2: thus, the range of variation of the quantity Δ is divided into intervals that depend on Δ_0. The limits of integration in formula (16) for each of the intervals are shown in the Table.

TABLE I. Limits of integration in formula (16) as functions of Δ and Δ_0.

	$\Delta_0<2$			$\Delta_0>2$	
	$\Delta<\Delta_0$	$\Delta_0<\Delta<2$	$2<\Delta<(4+\Delta_0^3)^{1/3}$	$(\Delta_0^3-4)^{1/3}<\Delta<\Delta_0$	$\Delta_0<\Delta<(4+\Delta_0^3)^{1/3}$
B_1	1	1	–	–	–
A_1	$\Delta_0^2/2-1$	$\Delta^2/2-1$	–	–	–
B_2	$\Delta^2/2+1$	$\Delta_0^2/2+1$	$\Delta_0^2/2+1$	$\Delta^2/2+1$	$\Delta_0^2/2+1$
A_2	1	1	$\Delta^2/2-1$	$\Delta_0^2/2-1$	$\Delta^2/2-1$

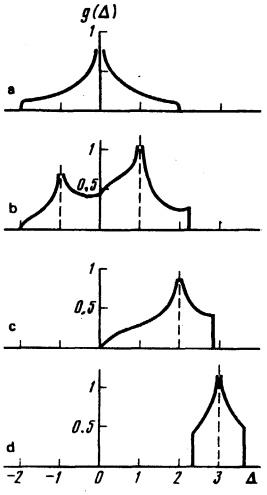

FIG. 2. Electron velocity distribution function in the strong saturation regime at different detunings: a—$\Delta_0=0$; b—$\Delta_0=1$; c—$\Delta_0=2$; d—$\Delta_0=3$.

Figure 2 gives the graphs of the function $g(\Delta)=g_1+g_2$ at various values of the detuning Δ_0, obtained by numerical integration from formula (16). As follows from formula (16), the function $g_1(\Delta)$ has a logarithmic singularity at $\Delta=\pm\Delta_0$ while the function $g_2(\Delta)$ has one at $\Delta=\Delta_0$. These singularities are analogous to those considered above for the function $n(\varphi)$ and are due to the contribution of pendulums with initial phases that are close to zero.

6. The energy balance between the electron beam and the electromagnetic field is determined by the mean value $\overline{\delta E}$ of the energy supplied to the wave by the electron. According to formula (10), this quantity is proportional to the difference $\bar{\Delta}-\Delta_0$. Thus, finding the total energy furnished to the wave by the electron in the strong saturation regime reduces to averaging of the quantity Δ with the help of the distribution function $g(\Delta)$.

It is obvious that $\bar{\Delta}=0$ for the trapped electrons, so that the quantity $\bar{\Delta}$ is determined by the distribution function $g_2(\Delta)$ of the untrapped electrons. However, it is more convenient to use the distribution function (13) directly, with the help of which it is not difficult to obtain

$$\bar{\Delta}=\pm2\int_{\varepsilon_0}^{1+\Delta_0^3/2}\frac{d\varepsilon}{T(\varepsilon)}\left[1-\left(\varepsilon-\frac{\Delta_0^2}{2}\right)^2\right]^{-1/4}, \quad (17)$$

where, just as in (13), $\varepsilon_0=1$ at $\Delta_0<2$, $\varepsilon_0=\Delta_0^2/2-1$ at $\Delta_0>2$, and the sign in formula (17) must be chosen to be the same as the sign of the quantity Δ_0.

At large values of detuning, $|\Delta_0|\gg1$, using the expression $T(\varepsilon)=2\pi(2\varepsilon)^{-1/2}(1+3/16\varepsilon^2)$, which is valid at $\varepsilon\gg1$, we find the following from formula (17):

$$\Delta_0-\bar{\Delta}=\Delta_0^{-3}, \quad |\Delta_0|\gg1. \quad (18)$$

In the opposite limiting case of small detunings, $|\Delta_0|\ll1$, the principal contribution to the integral is made by the region $\varepsilon-1\ll1$; here $T(\varepsilon)\approx-\ln|\varepsilon-1|$. In this case, we get from formula (17):

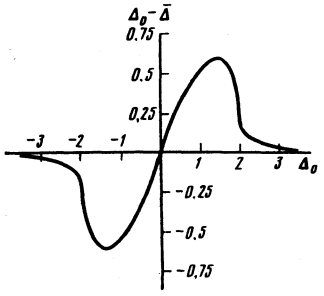

FIG. 3. Dependence of the mean energy $\bar{\sigma}E/E$ (in units of $\gamma^2\Omega/\omega$) transferred by the electron to the field on the detuning, calculated from formulas (10) and (17).

$$\Delta_0-\bar{\Delta}=\Delta_0\left(1-\frac{1}{\ln(1/|\Delta_0|)}\right),\quad |\Delta_0|\ll 1. \tag{19}$$

Formula (19) reflects the fact that in the case of small detunings, a large part of the electrons are trapped by the wave and a small fraction (of the order of Δ_0) of the untrapped electrons have a small [of the order of $1/\ln(1/|\Delta_0|)$] mean velocity relative to the wave.

Figure 3 shows a graph of the dependence of the mean energy $\overline{\delta E}$ transferred to the electromagnetic field on the detuning Δ_0. The maximum in the graph corresponds to $\Delta_0\approx 1.4$ with $\Delta_0-\bar{\Delta}\approx 0.6$, so that the maximum relative change in the energy of the electron in the strong saturation regime is

$$\overline{\delta E}/E=0.6\gamma^2\Omega/\omega. \tag{20}$$

We note that at small values of the saturation parameter $\tau=z/l\ll 1$, the width of the amplification band is decreased with increase in z: $\delta\omega/\omega\sim a/z$ (a is the pitch of the helical magnetic field).[5-7] In the strong saturation regime, the length z in this relation is replaced by the bunching length l (in our notation, this corresponds to $\Delta_0\sim 1$).

The given-field approximation that we have used is valid under the condition that the relative change in the energy of the electromagnetic wave is small: $n\overline{\delta E}\ll\mathscr{E}^2$, where n is the concentration of electrons in the beam. Using formulas (5) and (20), we can rewrite this condition in the form $\mathscr{E}\gg\mathscr{K}^{1/3}(ena)^{2/3}$. If this inequality is satisfied, the bunching length l is small in comparison with the amplification length that enters into the linear theory.[6]

[1]L. R. Elias, W. M. Fairbank, J. M. Madey, H A. Schwettman and T. J. Smith, Phys. Rev. Lett. **36**, 717 (1976).
[2]D. A. Deacon, L. R. Elias, J. M. Madey, G. J. Raiman, H. A. Schwettman and T. I. Smith, Phys. Rev. Lett. 38, 892 (1977).
[3]D. B. McDermott, T. C. Marshall, S. P. Schlesinger, R. K. Parker and W. L. Granatstein, Phys. Rev. Lett. **41**, 1368 (1978).
[4]J. M. Madey, J. Appl. Phys. **42**, 1906 (1971).
[5]F. A. Hopf, P. Meystre, M. O. Scully, and W. H. Louisell, Phys. Rev. Lett. **37**, 1215 (1976),
[6]L. A. Vaĭnshteĭn, Zh. Tekhn. Fiz. **49**, 1129, 1137 (1979) [Sov. Phys. Techn. Phys. **24**, 625, 629 (1979)].
[7]W. H. Louisell, J. F. Lam, D. A. Copeland and W. B. Colson, Phys. Rev. **A19**, 288 (1979).
[8]C. W. Planner, Phys. Lett. **67A**, 263 (1978).
[9]W. B. Colson, Phys. Quant. Electron. 5 (1977).

Angular distributions of resonant gamma-ray scattering by ^{57}Fe nuclei in hydrated sulfates of iron

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The angular distributions of the resonant scattering of 14.4 keV gamma rays by ^{57}Fe nuclei in the polycrystalline iron compounds ($FeSO_4\cdot H_2O$ and $FeSO_4\cdot 7H_2O$) were measured for the individual components of the quadrupole doublets of the hyperfine structure. The gamma quanta scattered with and without recoil were separated by using the method of the "black" absorber placed between the scatterer and the detector. The measured angular distributions of the resonant scattering differ from the "hard core" distribution, a fact attributed to the anisotropy of the Mössbauer-effect probability. The values of the anisotropy ε are -0.20 ± 0.05 and 0.10 ± 0.06 for $FeSO_4\cdot H_2O$ and $FeSO_4\cdot 7H_2$, respectively, at positive values of the electric field gradient.

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INTRODUCTION

It is known[1] that the quadrupole doublets of the Mössbauer transitions $\frac{3}{2}\to\frac{1}{2}$(^{57}Fe, ^{119}Sn) are symmetrical, i.e., both lines have the same intensity, if the Mössbauer-effect probability f' is isotropic. If, however, f' depends on the angle between the directions of the crystal axis and of the emission of the gamma quantum, $f'=f'(\theta_f)$, then an asymmetry of the intensities can appear in polycrystalline samples (the Gol'danskiĭ-

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