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Translated by J. G. Adashko

Closed equation for turbulent heat and mass transport

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(Submitted 16 June 1980)
Zh. Eksp. Teor. Fiz. **79**, 1797–1808 (November 1980)

A tensor nonlocal relation is derived between the mass (heat) flux and the gradient of the average density (temperature) for a turbulently flowing liquid. The laws governing the turbulent mass transport in a diffuse boundary layer near a flat solid boundary are investigated, including the section where the layer thickness is constant. Is is shown that at the start of the inlet section there is a region where the connection between the turbulent diffuse flow and gradient of the average density is essentially nonlocal. In the remaining part of the inlet section and in the region of the stabilized diffuse layer it is possible to obtain approximately a local relation between these quantities. The contribution made to the average diffusion flux on the surface by the hydrodynamic turbulent pulsations first decreases with increasing longitudinal coordinate, and then begins to increase. Longitudinal turbulent transport predominates up to the minimum point, and normal transport beyond this point. In the stabilized region, the turbulent diffusion coefficient takes on different functional forms at different distances from the surface.

PACS numbers: 47.25. – c, 47.25.Fj, 47.25.Jn, 47.10. + g

1. INTRODUCTION

In view of the wide prevalence of turbulent flows, the questions of heat and mass transport in turbulent streams attract much attention. From the theoretical point of view, the principal problem is the closing of the averaged transport equations: the density $\mathbf{J}_{turb}$ of the turbulent flux of matter or of heat must be connected with the distribution of the average density of the matter or the average temperature $\bar{T}$. As a rule, a local relation is assumed to exist between $\mathbf{J}_{turb}$ and the gradient of the average density or temperature[1-8]:

$$\mathbf{J}_{turb}=-D_{turb}\nabla\bar{c} \tag{1}$$

(to be specific, we discuss below the mass-transport problem).

The phenomenologically introduced turbulent-diffusion coefficient D_{turb} depends on the spatial coordinates, particularly on the distance to the solid surfaces. In some papers,[9] several phenomenological quantities are introduced in the form of a tensor $\hat{D}_{turb}$ that generalizes relation (1). To find the coefficient (or tensor) D_{turb}, it is customary to use the Reynolds analogy between D_{turb} and the turbulent viscosity coefficient ν_{turb}:[1-9]

$$D_{turb}(\mathbf{r})\sim\nu_{turb}(\mathbf{r}). \tag{2}$$

For the last quantity, a power-law variation is usually postulated near the boundaries of solids:

$$\nu_{turb}\propto y^k, \tag{3}$$

where a value 3 or 4 is assumed for the exponent k.

2. AVERAGED MASS-TRANSPORT EQUATIONS

The purpose of the present paper is to derive a relation between $\mathbf{J}_{turb}$ and $\nabla\bar{c}$ on the basis of the initial (non-averaged) equation of convective diffusion in incompressible liquids[1-7]:

$$\partial c/\partial t+\mathbf{v}\nabla c=D\Delta c. \tag{4}$$

Here $\mathbf{v}(\mathbf{r}, t)$ is the instantaneous distribution of the velocities of the liquid, D is the molecular-diffusion coefficient, and $c(\mathbf{r}, t)$ is the field of the impurity densities. The latter is assumed to be too small to influence the hydrodynamic characteristics of the flow.

We resolve the velocity of the liquid and the density into averaged and pulsating components

$$\begin{gathered}\mathbf{v}(\mathbf{r},t)=\mathbf{u}(\mathbf{r},t)+\mathbf{v}'(\mathbf{r},t),\quad c(\mathbf{r},t)=\bar{c}(\mathbf{r},t)+c'(\mathbf{r},t),\\ \mathbf{u}(\mathbf{r},t)=\langle\mathbf{v}(\mathbf{r},t)\rangle,\quad \bar{c}(\mathbf{r},t)=\langle c(\mathbf{r},t)\rangle.\end{gathered} \tag{5}$$

Here and elsewhere, the brackets $\langle\ldots\rangle$ denote averaging, while the prime denotes pulsating quantities. Averaging of (4) leads to the fundamental equation of convective diffusion in turbulent flow:

$$\partial\bar{c}/\partial t+\mathbf{u}\nabla\bar{c}+\nabla\mathbf{J}_{\mathrm{turb}}=D\Delta\bar{c}. \tag{6}$$

Equation (6) is not closed, since the paired correlator of the pulsations of the density and of the velocity

$$\mathbf{J}_{\mathrm{turb}}(\mathbf{r},t)=\langle c'(\mathbf{r},t)\mathbf{v}'(\mathbf{r},t)\rangle \tag{7}$$

(the density of the turbulent diffusion flow) is not expressed in terms of the profile of the average density $\bar{c}(\mathbf{r},t)$.

3. CONNECTION BETWEEN THE TURBULENT FLUX OF MATTER AND THE GRADIENT OF THE AVERAGE DENSITY

Using the equation for the density pulsations

$$\hat{L}c'\equiv\partial c'/\partial t+\mathbf{u}\nabla c'-D\Delta c'=-\mathbf{v}'\nabla\bar{c}-\nabla(\mathbf{v}'c')+\nabla\mathbf{J}_{\mathrm{turb}}, \tag{8}$$

we obtain an equation for the mixed paired correlator of the pulsations of the density and velocity at different space-time points:

$$\begin{aligned}\hat{L}\langle c'(\mathbf{r},t)v_j'(\mathbf{r}',t')\rangle=&-V_{ji}(\mathbf{r}',t';\mathbf{r},t)\nabla_i\bar{c}(\mathbf{r},t)\\&-\nabla_i\langle c'(\mathbf{r},t)v_j'(\mathbf{r}',t')v_i'(\mathbf{r},t)\rangle.\end{aligned} \tag{9}$$

Here and below we use the velocity-pulsation correlators

$$V_{i_1\ldots i_n}(\mathbf{r}_1,t_1;\ldots;\mathbf{r}_n,t_n)=\langle v_{i_1}'(\mathbf{r}_1,t_1)\ldots v_{i_n}'(\mathbf{r}_n,t_n)\rangle. \tag{10}$$

The boundary conditions for Eq. (9) follow from the weakening of the correlations as $|t-t'|\to\infty$ or $|\mathbf{r}-\mathbf{r}'|\to\infty$, and also from the conditions for the density. If, e.g., the density is given on the boundary B, then

$$\langle c'(\mathbf{r},t)v_j'(\mathbf{r}',t')\rangle=0 \quad \text{at} \quad \mathbf{r}\in B. \tag{11}$$

Equation (9) is not closed, since it contains a third-order correlator. However, in the investigation of mass transport near a solid surface, this term can be neglected in first-order approximation. The primary reason is that inside the viscous sublayer the correlators V, and consequently also the mixed correlators, decrease in power-law fashion [see formulas (25)–(28) below]. In addition, even on the outer boundary of the viscous sublayer the higher correlators are small relative to the parameter

$$\langle v_n'^2\rangle^{1/2}/\bar{v}_\tau, \tag{12}$$

which amounts to 0.03–0.05.

If we neglect the third-order correlator in (9), we obtain for the turbulent mass flux the formula

$$J_j(\mathbf{r}_1,t_1)=-\int\Delta_{ji}(\mathbf{r}_1,t_1;\mathbf{r},t)\nabla_i\bar{c}(\mathbf{r},t)\,d\mathbf{r}\,dt, \tag{13}$$

where the "density of the coefficient of turbulent diffusion" is equal to

$$\Delta^{(0)}_{i_1i_0}(\mathbf{r}_1,t_1;\mathbf{r}_0,t_0)=G(\mathbf{r}_1,t_1;\mathbf{r}_0,t_0)V_{i_1i_0}(\mathbf{r}_1,t_1;\mathbf{r}_0,t_0). \tag{14}$$

Here G is the Green's function of the operator $\hat{L}$ (8) with boundary conditions of the type (11).

To take into account the contribution made to $\mathbf{J}_{\mathrm{turb}}$ by the third-order correlators, it is necessary to solve an equation similar to (9) for mixed third-order correlators. As a result

$$\Delta=\Delta^{(0)}+\Delta^{(1)}+\ldots, \tag{15}$$

where $\Delta^{(0)}$ is defined by (14) and

$$\begin{aligned}\Delta^{(1)}_{i_1i_0}(\mathbf{r}_1,t_1;\mathbf{r}_0,t_0)=&-\int G(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)\\&\times\nabla_{i_2}G(\mathbf{r}_2,t_2;\mathbf{r}_0,t_0)V_{i_1i_2i_0}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2;\mathbf{r}_0,t_0)\,d\mathbf{r}_2\,dt_2.\end{aligned} \tag{16}$$

It follows from (15) that the connection between the turbulent diffusion flux $\mathbf{J}_{\mathrm{turb}}$ and the gradient of the average density $\nabla\bar{c}$ is in the general case nonlocal both with respect to the spatial and with respect to the temporal variables: $\mathbf{J}_{\mathrm{turb}}(\mathbf{r},t)$ is determined not only by the value of $\nabla\bar{c}$ at the same space-time point $(\mathbf{r},t)$, but also by the form of $\nabla\bar{c}$ at preceding instants of time t' in a certain region surrounding the point $\mathbf{r}$. The size of the $(t,\mathbf{r})$ region where the effect of the form of $\nabla\bar{c}$ on $\mathbf{J}_{\mathrm{turb}}$ is substantial is connected with the characteristic scales of the hydrodynamic correlators $V_{i_1\ldots i_n}$ and of the Green's function G. Therefore, in the general case, the hypothesis that a local connection exists between $\mathbf{J}_{\mathrm{turb}}$ and $\nabla\bar{c}$, i.e.,

$$J_i(\mathbf{r},t)=-D_{ij}(\mathbf{r},t)\nabla_j\bar{c}(\mathbf{r},t), \tag{17}$$

does not hold.

Relation (17) is approximately valid only in those regions where $\nabla\bar{c}(\mathbf{r},t)$ varies with respect to all the variables more slowly than the density of the coefficient of turbulent diffusion $\Delta_{ij}(\mathbf{r},t;\mathbf{r}',t')$. In this case

$$D_{ij}(\mathbf{r},t)\approx\int\Delta_{ij}(\mathbf{r},t;\mathbf{r}',t')\,d\mathbf{r}'\,dt'. \tag{18}$$

The quantities Δ_{ij} and D_{ij} have tensor properties, so that in the general case the scalar relation (1) cannot be used even if (17) is satisfied.

Expression (15) is a series in terms of hydrodynamic correlators of increasing order $V_{i_1\ldots i_n}$. It will be used below to study the heat and mass transport in a viscous sublayer near a solid surface. It will be shown that the series (15) converges rapidly: in the most important region—within the confines of the diffuse boundary layer—the second term of the series is smaller than the first by more than one order of magnitude.

In the customarily used semi-empirical or phenomenological theories of mass and heat transport it is assumed that the turbulent diffusion coefficient in (1) or (17) is determined solely by hydrodynamic characteristics, and does not depend on the coefficient of molecular diffusion D. The cause for this point of view is that the turbulent transport is effected by velocity pulsations $\mathbf{v}'$ of the liquid. However, besides the fact that the ensuing density pulsations c' are carried mechanically by the liquid, fluctuating molecular diffusion fluxes $D\nabla c'$ are also produced, and in the general case they alter the field $c'(\mathbf{r},t)$, and consequently influence the turbulent transport. In accord with this point of view, the density of the coefficient of turbulent diffusion Δ_{ij} in the exact formula (13), as well as the tensor of the turbulent diffusion in the approximate formula (17), depends not only on the hydrodynamic characteristics of the flow, mainly the average velocity of the liquid $\mathbf{u}(\mathbf{r},t)$ and the velocity-pulsation correlators $V_{i_1\ldots i_n}$, but also on the coefficient of molecular diffusion D. In the treatment of mass transport near a solid surface it will be shown below that the turbulent-diffusion coefficient depends substantially on D.

Relation (13) enables us to close Eq. (6) for the average density $\bar{c}(\mathbf{r}, t)$:

$$\partial\bar{c}/\partial t+\mathbf{u}\nabla\bar{c}=D\Delta\bar{c}+\nabla_i\int\limits_{-\infty}^{t}\int\Delta_{ij}(\mathbf{r}, t; \mathbf{r}', t')\,\nabla_j\bar{c}(\mathbf{r}', t')\,d\mathbf{r}'\,dt'. \qquad (19)$$

In regions where the approximate local relation (17) is valid, Eq. (19) becomes differential:

$$\partial\bar{c}/\partial t+\mathbf{u}\nabla\bar{c}=D\Delta\bar{c}+\nabla_i[D_{ij}(\mathbf{r}, t)\,\nabla_j\bar{c}],$$

where the tensor coefficient of turbulent diffusion D_{ij} is determined by (18).

4. MASS TRANSPORT NEAR A SOLID BOUNDARY AT LARGE PRANDTL NUMBERS

In many cases, the thermal or diffusion molecular Prandtl numbers for liquids are quite large: $\mathrm{Pr}\gg 1$. In this case the main resistance to heat or mass transport is concentrated in the interior of the viscous sublayer. In this region, the hydrodynamic characteristics decrease monotonically with decreasing distance y to the plane solid surface. We investigate below the rules of mass transport for a one-dimensional stationary process, when the average quantities $\mathbf{u}=u(y)\mathbf{i}$ and $\bar{c}=\bar{c}(y)$ depend only on y, while the paired correlators and other two-point characteristics depend on y, y' and $x-x'$, $z-z'$, $t-t'$, etc.

In the considered case, the Green's function G satisfies the equation

$$\left[\frac{\partial}{\partial t}+u(y)\frac{\partial}{\partial x}-D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\right]G(y, y', x-x', z-z', t-t') = \delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'), \qquad (13')$$

with the conditions $G\to 0$ as $y\to 0$, $y\to\infty$, $|x|\to\infty$, $|z|\to\infty$, and $|t|\to\infty$. At small values of the molecular-diffusion coefficient D, the function G has far from the boundary a symmetrical narrow maximum at $x=x'+u(y)(t-t')$, $y=y'$, $z=z'$, with a width $(Dt)^{1/2}$. For the characteristic times T_{corr}, within which the hydrodynamic correlators change as functions of $t-t'$, this width $(DT_{\mathrm{corr}})^{1/2}$ is much less[10,11] than the characteristic scales of variation of these quantities in $x-x'(L_{\mathrm{corr}}^{(x)})$ and $z-z'(L_{\mathrm{corr}}^{(z)})$. Therefore in all the products of G and V in (14) and (16) we can make in the arguments of the correlators the replacements $x_n\to x'-u(y')(t'-t_n)$ and $z_n\to z'$, so that all the integrals with respect to x and z_i in (13), (15), and (16) can be calculated.

At not too small distances from the boundary ($y\gg(DT_{corr})^{1/2}$) we can analogously calculate the integrals with respect to y_i, taking the correlators and $\nabla\bar{c}(y)$ outside the integral sign at $y_i=y'$. Then the following relation is approximately valid

$$J_{\mathrm{turb}}(y)=-\frac{d\bar{c}(y)}{dy}\int\limits_{-\infty}^{0}V_{yy}[u(y)\tau, y, 0, \tau; 0, y, 0, 0]\,d\tau$$
$$+\nabla_j\left\{\frac{d\bar{c}(y)}{dy}\int\limits_{-\infty}^{0}\int\limits_{\tau}^{0}V_{yjy}[u(y)\tau, y, 0. \tau; u(y)\tau', y, 0, \tau'; x'-x, y', z'-z, 0]\,d\tau'\,d\tau\right\}_{\tau'=\tau}. \qquad (20)$$

The correlators of the velocity pulsations in (20) are determined by (10).

Expression (20) for the turbulent flux shows that even in the given region, where the connection between J_{turb} and $\nabla\bar{c}(y)$ does not depend on the coefficient of molecular diffusion D and has a "local" character, the value of J_{turb} is determined not only by $\nabla\bar{c}$, but also by the higher-order derivatives $d^n\bar{c}/dy^n$. However, estimates of the second term in (20) on the basis of the experimental data for T_{corr}, $L_{\mathrm{corr}}^{(x)}$, $L_{\mathrm{corr}}^{(z)}$, and the correlators V show that within the entire viscous sublayer it is many times smaller than the first term. If we confine ourselves to this term only, then at $y\gg(DT_{\mathrm{corr}})^{1/2}$ we obtain relation (1), where the turbulent-diffusion coefficient is

$$D_{\mathrm{turb}}(y)=\int\limits_{0}^{\infty}V_{yy}[u(y)t, y, 0, t; 0, y, 0, 0]\,dt. \qquad (21)$$

Thus, the behavior of D_{turb} is determined by the form of the paired correlator of the normal components of the pulsation velocity v_y' in different space-time points.

Further simplification can be obtained in the interior of the viscous sublayer, where the change of the correlator V_{yy} on account of the argument $u(y)t$ is small compared with the influence of the argument t [owing to the decrease of the average velocity $u(y)$]. Therefore, in this region

$$D_{\mathrm{turb}}(y)=\int\limits_{0}^{\infty}\langle v_y'(\mathbf{r}, t)\,v_y'(\mathbf{r}, 0)\rangle\,dt, \qquad (22)$$

in agreement with earlier results,[10-12] and also with qualitative estimates.[1,2]

We investigate now the connection between $J_{\mathrm{turb}}(y)$ and $d\bar{c}/dy$ inside the region $y\leqslant(DT_{\mathrm{corr}})^{1/2}$. At large Prandtl numbers, this region lies in the interior of the viscous sublayer. Estimates similar to those considered above make it possible to calculate the integrals with respect to the variables x_i and z_i in (13) and (16). The contribution of the higher correlators V turns out to be negligibly small. As a result

$$J_{\mathrm{turb}}(y)=-\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}G(y, y'; t)\,V_{yy}(0, y, 0, t; 0, y', 0, 0)\frac{d\bar{c}(y')}{dy'}\,dy'\,dt. \qquad (23)$$

$$G(y, y'; t)={}^1/_2(\pi Dt)^{-1/2}\{\exp[-(y-y')^2/4Dt]-\exp[-(y+y')^2/4Dt]\}.$$

This formula is valid both for $y\leqslant(DT_{\mathrm{corr}})^{1/2}$ and in a certain region $y\gg(DT_{\mathrm{corr}})^{1/2}$; in particular, it can be used to obtain again relation (22).

Using (22) to estimate $D_{\mathrm{turb}}(y)$ at $y\sim(DT_{\mathrm{corr}})^{1/2}$, we can show that in the entire region $y\leqslant(DT_{\mathrm{corr}})^{1/2}$ the coefficient $D\gg D_{\mathrm{turb}}(y)$, i.e., the molecular transport predominates. We can therefore replace $d\bar{c}(y')/dy'$ in (23) by $d\bar{c}(y)/dy$:

$$J_{\mathrm{turb}}(y)=-\frac{d\bar{c}(y)}{dy}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}G(y, y'; t)\,V_{yy}(0, y, 0, t; 0, y', 0, 0)\,dy'\,dt. \qquad (24)$$

For small y ($y\ll(DT_{\mathrm{corr}})^{1/2}$) the Green's function takes the form

$$G(y, y'; t)\approx\frac{yy'}{2\pi^{1/2}(Dt)^{3/2}}\exp\left(-\frac{y'^2}{4Dt}\right).$$

Thus, near a solid surface the turbulent-diffusion coefficient in (24) depends substantially on D.

Inside the viscous sublayer we can use for qualitative estimates an expansion of the correlator V_{yy} in powers of y and y':

$$V_{yy}(0,y,0,t;0,y',0,0)\approx\left(\frac{y}{L_b}\right)^n\left(\frac{y'}{L_b}\right)^m\tilde{V}_{yy}(t),$$
$$\tilde{V}_{yy}(t)=L_b^{n+m}\left(\frac{\partial^{n+m}}{\partial y^n\,\partial y'^m}\right)V_{yy}(0,y,0,t;0,y',0,0),\quad y=y'=0, \tag{25}$$

$$V_{yy}(\mathbf{r},t;\mathbf{r},0)\approx\left(\frac{y}{L_b}\right)^N\tilde{V}_{yy}(t),\quad N=n+m, \tag{26}$$

$$L_b=5(\nu/A)^{1/2},\quad A=du(0)/dy, \tag{27}$$

where L_b is the thickness of the viscous sublayer, $n\geqslant 2$, $m\geqslant 2$, and $N\geqslant 4$.

Using similar expansions for the velocity-pulsation correlators V, we can show that inside the viscous sublayer

$$D_{\rm turb}(y)\propto y^N\left[1+a_1\frac{y}{L_b}+a_2\left(\frac{y}{L_b}\right)^2+\ldots\right]\quad\text{if}\quad y\gg(DT_{\rm corr})^{1/2}, \tag{28}$$

$$D_{\rm turb}(y)\propto y^{n+1}\left[1+\frac{b_1}{{\rm Pr}^{1/2}}+\frac{b_2}{\rm Pr}+\ldots\right]\quad\text{if}\quad y\ll(DT_{\rm corr})^{1/2}. \tag{29}$$

It follows from (28) that owing to the damping of the turbulent pulsations inside the viscous sublayer, the series (15) converges rapidly at $y\ll L_b$. Therefore at $\Pr\gg 1$ we can definitely confine ourselves to the first term of this expansion [see formulas (21)–(24)]. We note that at small y the ratio of the terms of this series no longer depends on y, this being due to the nonlocality of the connection of $D_{\rm turb}$ and V_{yy} in (24). Estimates of the numerical values of the coefficients in (28) and (29) show that $a_1\ll 1$ and $b_1\ll 1$ [mainly because of the smallness of the normal component of the pulsations of the velocity v'_y compared with $u(y)$ at $y\leqslant L_b$]. Therefore the conclusion that we can confine ourselves to the contribution [formulas (21)–(24)] of only the first term of the series (15) remains valid also at $\Pr\sim 1$, when the thickness δ_D of the diffusion boundary layer is close to L_b.

The use of expansions (25) and (26) yields simple expressions for the coefficient of turbulent diffusion at various distances from the surface:

$$D_{\rm turb}(y)\approx\alpha\left(\frac{y}{L_\Gamma}\right)^{n+1}\quad\text{if}\quad y\ll(DT_{\rm corr})^{1/2},$$
$$\alpha=\frac{2\Gamma(1+{}^1/_2m)}{\pi^{1/2}}\left(\frac{4D}{L_b}\right)^{(m-1)/2}\int_0^\infty\tilde{V}_{yy}(t)\,t^{(m-1)/2}\,dt, \tag{30}$$

$$D_{\rm turb}(y)\approx\beta\left(\frac{y}{L_b}\right)^N\quad\text{if}\quad y\gg(DT_{\rm corr})^{1/2},$$
$$\beta=\int_0^\infty\tilde{V}_{yy}(t)\,dt. \tag{31}$$

With increasing distance from the solid boundary, the region of validity of formula (31) is restricted by the condition that the expansion (26) be valid, as well as by the condition $L^{(x)}_{\rm corr}\gg u(y)T_{\rm corr}$ [according to (21)].

The distribution of the average density $\bar{c}(y)$ is specified by the equation (32) which follows from (6):

$$J_0=-D\frac{d\bar{c}(y)}{dy}+J_{\rm turb}(y), \tag{32}$$

where J_0 is the total diffusion flux. Inasmuch as at all y we have

$$J_{\rm turb}(y)\approx-D_{\rm turb}(y)\frac{d\bar{c}}{dy},$$

it follows that

$$\bar{c}(y)=\bar{c}(0)-J_0\int_0^y\frac{dy}{D+D_{\rm turb}(y)};$$

$$J_0=\frac{D[\bar{c}(0)-\bar{c}(\infty)]}{\delta_D},\quad\delta_D=\int_0^\infty\left[1+\frac{1}{D}D_{\rm turb}(y)\right]^{-1}dy,$$

where $D_{\rm turb}(y)$ is determined by (22) and (24). The main contribution to δ_D (the thickness of the diffusion layer) is made by the region where $D_{\rm turb}(y)\sim D$.

Using for estimates formulas (30) and (31), it can be shown[10, 11] that

$$(DT_{\rm corr})^{1/2}\ll\delta_D\ll L_b.$$

Therefore, for qualitative estimates of δ_D we shall use formula (31). As a result

$$\delta_D\approx L_b\left(\frac{D}{\beta}\right)^{1/N}\frac{\pi}{N\sin(\pi/N)},\quad N\geqslant 4.$$

This relation agrees with the one used in semi-empirical theories of turbulent heat and mass transport.[1-7] The obtained formulas make it possible to find the parameter β, which is determined in accordance with (31) and (26) by the form of the correlator $\langle v'_y(\mathbf{r},t)v'_y(\mathbf{r},0)\rangle$ on the boundary of the viscous sublayer:

$$\beta\approx\int_0^\infty\langle v_y'(\mathbf{r},t)v_y'(\mathbf{r},0)\rangle|_{y=L_\Gamma}dt\approx T_{\rm corr}u^2(L_\Gamma)\varepsilon_{yy}^2,$$

$$\varepsilon_{yy}=\frac{\langle v_y'^2\rangle^{1/2}}{u(y)}\quad\text{at}\quad y=L_b=5\nu^{1/2}\left(\frac{du(0)}{dy}\right)^{-1/2}.$$

The experimental data[13] yield for ε_{yy} a value of the order of 0.03–0.05. There is much less known experimental information on $T_{\rm corr}$; it appears that at the present time it is possible to establish on the basis of direct hydrodynamic measurements only that this quantity varies over rather a wide range:

$$0{,}5\left(\frac{du(0)}{dy}\right)^{-1}\leqslant T_{\rm corr}\leqslant 50\left(\frac{du(0)}{dy}\right)^{-1}.$$

This does not permit at present the calculation of the coefficient γ in the formula

$$\delta_D\approx\gamma L_b({\rm Pr})^{-1/N},\quad\gamma\approx\left(25\varepsilon_{yy}^2T_{\rm corr}\frac{du(0)}{dy}\right)^{-1/N}. \tag{33}$$

On the other hand, using the experimental data for the $\delta_D(\Pr)$ dependence, we can determine N. We note that on the basis of the theory developed above, the exponent N is not less than four. The experimental results[1-9] yield for this quantity a value in the range from 3 to 4. This shows that N apparently is equal to its minimum possible value, $N=4$. On the basis of these same experimental data we can estimate $T_{\rm corr}$:

$$3\leqslant T_{\rm corr}\frac{du(0)}{dy}\leqslant 8.$$

In conclusion, we discuss now the applicability of the Reynold's analogy between the momentum, heat, and matter transport by turbulent pulsations, as well as the premises of the "mixing path" theory based on the analogy with the molecular-kinetic theory of gases. According to these approaches, the turbulent Prandtl number

$$\mathrm{Pr}_{\mathrm{turb}}(y)=\frac{\nu_{\mathrm{turb}}(y)}{D_{\mathrm{turb}}(y)},$$

$$\nu_{\mathrm{turb}}(y)=\left(\frac{du(y)}{dy}\right)^{-1}\langle v_x'(\mathbf{r},t)v_y'(\mathbf{r},t)\rangle$$

is constant in the interior of the entire viscous sublayer and is close to unity in order of magnitude. The relations obtained above for D_{turb} allow us to express $\mathrm{Pr}_{\mathrm{turb}}$ in terms of purely hydrodynamic characteristics and of the molecular-diffusion coefficient. In the most important region $(DT_{\mathrm{corr}})^{1/2} \ll y < L_b$ we have

$$\mathrm{Pr}_{\mathrm{turb}}\approx\frac{\langle v_x'v_y'\rangle}{\langle v_y'v_y'\rangle}\left(T_{\mathrm{corr}}\frac{du(0)}{dy}\right)^{-1}.$$

Using for estimates the power-law approximation (26) and $\langle v'_x v'_y\rangle \approx \tilde{V}_{xy}(y/L_b)^{N-1}$ we obtain for $\mathrm{Pr}_{\mathrm{turb}}$

$$\mathrm{Pr}_{\mathrm{turb}}(y)\approx\frac{L_b}{y}\left(T_{\mathrm{corr}}\frac{du(0)}{dy}\right)^{-1}\frac{\tilde{V}_{xy}}{\tilde{V}_{yy}}\quad \text{at}\quad (DT_{\mathrm{corr}})^{1/2}\ll y<L_r. \tag{34}$$

Formula (34) and the experimental data for T_{corr}, $\tilde{V}_{xy}$, and $\tilde{V}_{yy}$ make it possible to estimate $\mathrm{Pr}_{\mathrm{turb}}$ on the outer boundary of the viscous sublayer:

$$\mathrm{Pr}_{\mathrm{turb}}(L_b)\lesssim 1.$$

It follows from (34) that within the viscous sublayer $[(DT_{corr})^{1/2} \ll y < L_b]$ the turbulent Prandtl number $\mathrm{Pr}_{\mathrm{turb}}$ is inversely proportional to y, so that ν_{turb} can exceed D_{turb} in this region by dozens of times. In particular, in the region of the diffusion boundary layer, ν_{turb} can be larger by an order of magnitude than D_{turb}.

Furthermore, on going through the region $y \sim (DT_{\mathrm{corr}})^{1/2}$, the functional form of the turbulent-diffusion coefficient changes, and this coefficient becomes strongly dependent on the molecular-diffusion coefficient. At the same time, the functional form of ν_{turb} is the same in the entire viscous sublayer. Thus, the premises of the molecular-kinetic theory, and particularly the Reynolds analog, cannot be used to describe the processes of turbulent heat and mass transport through a viscous sublayer.

5. TURBULENT MASS TRANSPORT IN THE INLET SECTION

In this section we consider the development of a diffusion boundary layer along a surface in a turbulent shear flow $\mathbf{u}=u(y)\mathbf{i}$ of an incompressible liquid. In the oncoming stream (at $x<0$) we have $c(x,y)\equiv c_1$, and at $x>0$ the surface density $c(x,0)$ is equal to c_2. Just as in the preceding section, we can confine ourselves in the general expressions (13)–(16) for the turbulent mass flow $\mathbf{J}_{\mathrm{turb}}(x,y)$ to the first term of the series:

$$\mathbf{J}_{\mathrm{turb}}(x,y)=-\int_0^\infty d\tilde{y}\int_{-\infty}^\infty d\tilde{x}\left[\Delta_x(\tilde{x},y,\tilde{y})\frac{\partial\bar{c}(x+\tilde{x},\tilde{y})}{\partial\tilde{x}}+\Delta_y(\tilde{x},y,\tilde{y})\frac{\partial\bar{c}(x+\tilde{x},\tilde{y})}{\partial\tilde{y}}\right], \tag{15'}$$

$$\Delta_{x,y}(\tilde{x},y,\tilde{y})=\iint_{-\infty}^{\infty}d\tau\,d\tilde{z}\,\mathbf{V}_{x,y}(\tau,\tilde{x},\tilde{z},\tilde{y},y)G(-\tau,-\tilde{x},-\tilde{z},y,\tilde{y}); \tag{16'}$$

$$\mathbf{V}_{x,y}(\tau,\tilde{x},\tilde{z},y',y)\equiv\langle v_{x,y}'(\mathbf{r},t)\mathbf{v}'(\mathbf{r}',t')\rangle;\quad \tau=t'-t,\quad \tilde{x}=x'-x,\quad \tilde{z}=z'-z,$$

where the Green's function G is defined in (13′).

Outside the "diffusion tip," i.e., at $x \gg L_D=(D/A)^{1/2}$ [see (27)], a diffusion boundary layer is produced, with a thickness $\delta_D(x)$ that increases at $L_D \ll x \ll x_{\mathrm{in}}$; at $x \gg x_{\mathrm{in}}$, the stabilized layer considered above is produced. The quantity x_{in} is defined below. The entry section can be subdivided into two characteristic regions.

At $x \gg x_*$, owing to the abrupt change of $\bar{c}(x,y)$, the connection between $\mathbf{J}_{\mathrm{turb}}$ and $\nabla\bar{c}$ is essentially nonlocal at all values of y, so that it is impossible to go over to the local relation (17) even approximately. The length x_* is equal to $(DA^2T_{xx}^3)^{1/2}$, where T_{xx} is the characteristic correlation time of the longitudinal components of the velocity pulsations $v_x'(\mathbf{r},t)$ inside the viscous sublayer.

At $x \gg x_*$, in the region of greatest importance for mass transport $y \sim \delta_D(x)$, we can go over approximately to the local relations

$$J_i(x,y)=-D_{ij}(y)\nabla_j\bar{c}(x,y),$$

where the components of the turbulent-diffusion tensor are equal to

$$D_{ij}(y)=\int_{-\infty}^{t'}\langle v_i'(\mathbf{r},t)v_j'(\mathbf{r},t')\rangle\,dt.$$

We proceed now to finding the distributions of the average density $\bar{c}(x,y)$ and of the density of the diffusion flux through the surface $J(x)=-D\partial\bar{c}(x,0)/\partial y$. At $x \ll x_{\mathrm{in}}$, owing to the small thickness of the diffusion boundary layer, these characteristics correspond in the zeroth approximation to the $c^0(x,y)$ and $J^0(x)$ distributions for laminar flow with the same average-velocity profile $u(y)$. We shall therefore analyze the deviation $J^1(x)=J(x)-J^0(x)$, whose measurement can yield information on the characteristics of the hydrodynamic turbulent pulsations.

In the nonlocal region $(L_D \ll x \ll x_*)^{1)}$

$$\frac{J^1(x)}{J^0(x)}\propto\varepsilon_{xx}^2+\varepsilon_{xy}^2\frac{x}{L_b}+\varepsilon_{yy}^2\frac{x^2}{L_b^2},\qquad \varepsilon_{ij}^2=\left.\frac{\langle v_i'(r,t)v_j'(r,t)\rangle}{u^2(y)}\right|_{y=L_b}. \tag{35}$$

The largest is the first term, which is connected with the longitudinal turbulent transport under the influence of the longitudinal density gradient. We note that this region the ratio (35) does not decrease when x decreases, despite the weakening of the velocity pulsations; this is due to the nonlocality of the connection of (15′) and (16′).

In the local region of the inlet section $(x_* \ll x \ll x_{\mathrm{in}})$

$$\frac{J^1(x)}{J^0(x)}\propto\varepsilon_{xx}^2AT_{xx}\left(\frac{L_D}{x}\right)^{2/3}+\varepsilon_{xy}^2AT_{xy}\frac{L_D^{2/3}x^{1/3}}{L_b}+\varepsilon_{yy}^2AT_{yy}\frac{L_D^{2/3}x^{4/3}}{L_b^2}. \tag{36}$$

Here T_{ij} is the correlation time of the velocity-pulsation components $v_i'(\mathbf{r},t)$ and $v_j'(\mathbf{r},t')$ in the viscous sublayer. At $x_* \ll x \ll x_{\mathrm{min}}$, where

$$x_{min}=\left(\frac{T_{xx}}{T_{yy}}\right)^{1/2}\frac{\varepsilon_{xx}}{\varepsilon_{yy}}L_b\quad x_*\ll x_{min}\ll x_{\mathrm{in}},$$

the first term, the xx transport, predominates just as in the nonlocal region $x \lesssim x_*$. The relative magnitude of $J^1(x)$ decreases with increasing x. At $x \gg x_{\mathrm{min}}$, normal turbulent transport due to the normal concentration gradient predominates (yy transport). In this region, $J^1(x)$ increases with increasing x. Thus, $J^1(x)$ passes through a minimum at $x \sim x_{\min}$ without the inlet section.

At $x \sim x_{\text{in}}$, relations (22) and (24) are valid, so that

$$Ay\frac{\partial \bar{c}}{\partial x}=\frac{\partial}{\partial y}\left\{[D+D_{yy}(y)]\frac{\partial \bar{c}}{\partial y}\right\}.$$

The use of the expansion (26) enables us to estimate the length of the inlet section:

$$x_{\text{in}} \sim [L_{\text{b}}^{3N-6} D^{3-N} \varepsilon_{yy}{}^{-6} T_{yy}{}^{-3} A^{N-6}]^{1/N}.$$

6. CONCLUSIONS

We have proposed in this paper a method that made it possible to find the relation (13) between a turbulent diffusion (thermal) flux $\mathbf{J}_{\text{turb}}$ and the gradient of the average density (temperature) $\nabla\bar{c}$. The connection between these quantities turned out to be nonlocal both in space and in time. The kernel of this tensor integral relation Δ_{ij} (the density of the turbulent-diffusion tensor) is determined both by purely hydrodynamic characteristics:

$$\mathbf{u}(\mathbf{r}, t)\equiv\langle\mathbf{v}(\mathbf{r}, t)\rangle, \quad V_{ij}\equiv\langle v_i'(\mathbf{r}, t)v_j'(\mathbf{r}', t')\rangle \quad \text{etc.},$$

and by the coefficient of molecular diffusion D. A closed integro-differential equation was obtained for $\bar{c}(\mathbf{r}, t)$.

The obtained general relations were used to investigate the mass transport near a flat solid boundary. It is shown that in the initial section of the diffusion boundary layer $x < x_*$ the connection between $\mathbf{J}_{\text{turb}}$ and $\nabla\bar{c}$ is essentially nonlocal. In this region, the contribution to the diffusion flux onto the surface from the turbulent hydrodynamic pulsations increases with decreasing x:

$$J^1(x)\propto x^{-1/3}.$$

At $x \gg x_*$, including in the section where the diffusion-layer thickness $\delta_D(x)$ is stabilized, and in the case of transport through a boundary layer of constant thickness, the local relation between J_{turb} and $\nabla\bar{c}$ can be approximately introduced, but the turbulent-diffusion tensor $D_{ij}(y)$ has different functional forms at different distances from the surface [see, e.g., (21), (22), (24), (30), and (31)]. At $x \ll x_{\min}$, the decisive role is played by the longitudinal (xx) transport [formulas (35) and (36)], while at $x \gg x_{\min}$ the normal (yy) transport predominates. The spatial correlation of the longitudinal pulsations $v_x'(\mathbf{r}, t)$ the temporal correlation of $v_x'(\mathbf{r}, t)$ and the temporal correlation of $v_y'(\mathbf{r}, t)$ predominate at $x \ll x_*$, $x^* \ll x \ll x_{\min}$, and $x \gg x_{\min}$, respectively.

With increasing x, the correction $J^1(x)$ to the flux density through the surface, due to turbulent pulsations, begins to decrease: $J^1(x)\propto x^{-1/3}$ at $L_D \ll x \ll x_*$ and $J^1(x)\propto x^{-1}$ at $x_* \ll x \ll x_{\min}$. $J_1(x)$ goes through a minimum at $x = x_{\min}$, after which it increases linearly (at $x_{\min} \ll x \ll x_{\text{in}}$). In the region $x \gg x_{\text{in}}$, where the diffusion boundary-layer thickness δ_D becomes stabilized, the approximate expansions of D_{turb} in powers of y take different froms at small y and in the region of the diffusion layer $y \sim \delta_D$; in particular, the exponents differ. At $y \sim \delta_D$, the exponent in the expansion in powers of y is not less than four.

On the basis of experimental data on mass transport, we determine the behavior of the pulsation correlator V_{yy} within the viscous sublayer; estimates were obtained also for the correlation time T_{corr} of the normal pulsations $v_y'(\mathbf{r}, t)$ in this region. It is shown that the Reynolds analogy is not suitable for the description of turbulent convective diffusion through a viscous sublayer.

The authors thank I. M. Lifshitz, Kh. A. Rakhmatullin, S. I. Anisimov, and P. I. Geshev for helpful discussions of the work.

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Translated by J. G. Adashko