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Noise-induced phase transition and the percolation problem for fluctuating media with diffusion

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The diffusion problem is considered for a medium in which processes of disintegration and of reproduction of the diffusing substance are possible. The critical intensities of the fluctuations in the disintegration and reproduction rates, which lead to the occurrence of noise-induced explosive instability, are determined. The effects of suppression of such instability by nonlinear limitation mechanisms are analyzed. Fluctuation phenomena near the noise-induced phase-transition point are investigated.

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The action of external noise can lead to a qualitative reorganization of the behavior of a dynamical system.[1)] In those cases in which such a reorganization is observed in a distributed system and is accompanied by establishment of a new stationary mode, it is customary to speak of a noise-induced phase transition in such a system. The study of noise-induced transitions is one of the problems of the general theory of fluctuating and disordered media, in the construction of which there are today used a whole series of traditional methods of the of the theory of equilibrium phase transitions.

In the present paper, we consider the problem of diffusion through a fluctuating medium, in which processes of disintegration and reproduction of the diffusing substance are possible. In the paper, the threshold of noise-induced explosive instability in such a system is determined, and a possible mechanism for its suppression is considered. Fluctuation phenomena near the noise-induced phase-transition point are analyzed, and the spatially inhomogeneous problem is investigated.

§1. FORMULATION OF THE MODEL

Let $K_1(\mathbf{r},t)$ be the rate of disintegration of the diffusing substance at the instant t at the point $\mathbf{r}$ of the medium, and let $K_2(\mathbf{r},t)$ be its reproduction rate at this point, so that the diffusion equation has the form

$$\dot{n}=-(K_1-K_2)n+D\Delta n. \tag{1}$$

In general, the values of K_1 and K_2 depend on the local concentration $n(\mathbf{r},t)$ of the diffusing substance. Limiting ourselves to consideration of small concentrations, we shall suppose that this dependence is linear:

$$K_1-K_2=k_1-k_2+\beta n, \tag{2}$$

where the coefficient β is positive; that is, increase of the concentration n suppresses reproduction or increases the rate of disintegration of the diffusing component.

In a closed system, reproduction cannot continue without limit and must end upon exhaustion of the substrate required for this. We shall suppose, however, that the medium is open and that the mean rate of reproduction is maintained constant by influx of substrate from external sources.

We assume that the reaction rates k_1 and k_2 fluctuate in a prescribed manner in space and in time (fluctuations in the coefficient β may be disregarded, since the concentration n is small). Then we may distinguish in them the regular and the fluctuating components,

$$k_{1,2}(\mathbf{r},t)=\bar{k}_{1,2}+\delta k_{1,2}(\mathbf{r},t),\qquad \bar{k}_{1,2}\equiv\langle k_{1,2}(\mathbf{r},t)\rangle \tag{3}$$

and may finally rewrite the diffusion equation (1) in the form

$$\dot{n}=-\alpha n-\beta n^2+D\Delta n+f(\mathbf{r},t)n, \tag{4}$$

$$\alpha=\bar{k}_1-\bar{k}_2,\qquad f(\mathbf{r},t)=\delta k_2(\mathbf{r},t)-\delta k_1(\mathbf{r},t). \tag{5}$$

We shall hereafter assume that the coefficient α is positive; this corresponds to a situation in which the mean rate of disintegration exceeds the mean rate of reproduction.

The random field $f(\mathbf{r},t)$ is external with respect to the problem under consideration, since it does not depend on the concentration distribution $n(\mathbf{r},t)$. The mean value of this field is zero by definition, and we shall suppose that its pair correlators fall off exponentially in space and in time:

$$\langle f(\mathbf{r},t)f(\mathbf{r}',t')\rangle=S\exp\{-k_0|\mathbf{r}-\mathbf{r}'|-\gamma|t-t'|\}. \tag{6}$$

We shall also assume that the random field $f(\mathbf{r},t)$ is Gaussian.

With allowance for fluctuations of the rates of reproduction and disintegration, there is, in a medium described by equation (4), a randomly time-variable spatial structure of centers of reproduction, within which the condition $f(\mathbf{r},t) > \alpha$ is satisfied locally at a given instant of time. A characteristic spatial dimension of an individual center of reproduction is the quantity $r_0 = 1/k_0$, and a characteristic time of existence of one such a center is $\tau_0 = 1/\gamma$. In the limit $\tau_0 \to \infty$, in particular, equation (4) describes diffusion through a medium with a stationary random distribution of centers of reproduction.

On increase of the noise intensity S, the concentration of centers of reproduction in the medium increases; and at a certain critical noise intensity S_{cr}, the total growth of the density n within the centers of reproduction will begin to compensate the falling off of the density n outside such regions. This critical value determines the noise-induced phase-transition point.

If the coefficient β of nonlinear decay were zero, a stationary state of the system for $S > S_{cr}$ would be impossible; the observed effect would be an "explosion," typical of systems with chain reactions.[2] But if $\beta \neq 0$, then the explosive instability is suppressed, and there is established in the medium a stationary mean density $\langle n \rangle$ that increases with increase of S above S_{cr} and that vanishes at $S = S_{cr}$.

For investigation of the noise-induced phase transition, we shall use standard methods of the theory of phase transitions.

§2. THE GINZBURG-LANDAU EQUATION

Starting from the parameters α, D, k_0, and γ that occur in this problem, one can construct three characteristic combinations with the dimension length.

a) *The characteristic diffusion distance*

$$r_{diff} = (D/\alpha)^{1/2} \tag{7}$$

gives the mean depth of penetration from the boundary in the absence of fluctuations.

b) *The characteristic stationarity distance*

$$l = (D/\gamma)^{1/2} \tag{8}$$

is the mean distance over which a particle can pass by diffusion during the time $\tau_0 = \gamma^{-1}$ of existence of an individual center of reproduction.

c) *The characteristic dimension of a center of reproduction is*

$$r_0 = k_0^{-1}. \tag{9}$$

The three distances enumerated play the role of microscopic scales of our problem.

We introduce a smoothed concentration $\eta(\mathbf{r},t)$, defining it by a spatial average over a small volume ΔV with linear dimensions $(\Delta V)^{1/d}$ large in comparison with the microscopic scales (7)–(9):

$$\eta(\mathbf{r},t) = (\Delta V)^{-1} \int_{\Delta V} n(\mathbf{r}+\boldsymbol{\rho},t)\,d\boldsymbol{\rho}. \tag{10}$$

To obtain an equation for η, we carry out a spatial smoothing in the original equation (4):

$$\dot\eta = -\alpha\eta - \beta\eta^2 + D\Delta\eta + \bar f(\mathbf{r},t)\eta - \beta\langle \tilde n^2\rangle_{\Delta V} + \langle \tilde f \tilde n\rangle_{\Delta V}. \tag{11}$$

Here the symbol $\tilde n$ has been introduced for the concentration component that fluctuates rapidly in space ($\tilde n = n - \eta$); the random field $\bar f(\mathbf{r},t)$ is determined by smoothing $\bar f(\mathbf{r},t)$ over the volume ΔV, and $\tilde f(\mathbf{r},t) = f(\mathbf{r},t) - \bar f(\mathbf{r},t)$ is the rapidly fluctuating component of the field $f(\mathbf{r},t)$.

Subtraction of (11) from (4) gives the equation for $\tilde n$:

$$\dot{\tilde n} = -\alpha\tilde n - \beta[2\eta\tilde n - (\tilde n^2 - \langle \tilde n^2\rangle_{\Delta V})] + D\Delta\tilde n + \bar f\tilde n + \eta\tilde f + (\tilde f\tilde n - \langle \tilde f\tilde n\rangle_{\Delta V}). \tag{12}$$

In order to obtain a closed equation for η, we must determine from (12) the correlators of the rapid quantities, $\langle \tilde f\tilde n\rangle_{\Delta V}$ and $\langle \tilde n^2\rangle_{\Delta V}$. In the calculation of these correlators, we may take into account that the averaging volume ΔV is large on a microscopic scale, and therefore averaging over this volume may be replaced by taking of the statistical mean at the prescribed local values of the smoothed quantities η and $\bar f$:

$$\langle \tilde n^2\rangle_{\Delta V} = \langle \tilde n^2\rangle_\eta, \quad \langle \tilde n\tilde f\rangle_{\Delta V} = \langle \tilde n\tilde f\rangle_\eta. \tag{13}$$

We may also note that under the condition

$$(\langle \tilde n^2\rangle_\eta)^{1/2} \ll \eta \tag{14}$$

linearization with respect to $\tilde n$ is justified in equation (12); as a result, it takes the form[2)]

$$\dot{\tilde n} = -(\alpha + 2\beta\eta - \bar f)\tilde n + D\Delta\tilde n + \eta\tilde f. \tag{15}$$

By starting from the linearized equation (15), one can easily determine the required correlators in the usual manner. They are given by the integrals

$$\langle \tilde n^2\rangle_\eta = \eta^2 \int \frac{S(q)\,d^{d+1}q}{\omega^2 + [\alpha_{eff} + Dk^2]^2}, \tag{16}$$

$$\langle \tilde n\tilde f\rangle_\eta = \eta \int \frac{S(q)\,d^{d+1}q}{-i\omega + [\alpha_{eff} + Dk^2]}. \tag{17}$$

Here $S(q)$ is the Fourier spectrum ($q = \mathbf{k}, \omega$) of the random field $f(\mathbf{r},t)$. For various dimensionalities d of the medium, the expressions for $S(q)$ are:

$$S(q) = S\gamma k_0\pi^{-2}(k^2+k_0^2)^{-1}(\omega^2+\gamma^2)^{-1}, \quad d=1, \tag{18a}$$

$$S(q) = {}^1/_2 S\gamma k_0\pi^{-2}(k^2+k_0^2)^{-3/2}(\omega^2+\gamma^2)^{-1}, \quad d=2, \tag{18b}$$

$$S(q) = S\gamma k_0\pi^{-3}(k^2+k_0^2)^{-2}(\omega^2+\gamma^2)^{-1}, \quad d=3. \tag{18c}$$

The notation $\alpha_{eff} = \alpha + 2\beta\eta - \bar f$ has also been used.

The integration in (16) and (17) should be limited to wave vectors $|\mathbf{k}| > 2\pi/(\Delta V)^{1/d}$. It can be shown, however, that the main contribution to these integrals comes from values of the wave vectors corresponding to microscopic scales; and therefore, neglecting small terms of the order $L_{micro}/(\Delta V)^{1/d}$, where L_{micro} are the characteristic microscopic scales (7)–(9), we may extend the integration to the region $|\mathbf{k}| < 2\pi/(\Delta V)^{1/d}$ as well.

The functions of η and $\bar f$ defined by the expressions (16) and (17) must then be substituted in equation (11) for η. Here it may be noted that if the condition (14) is satisfied, then the term $\beta\langle \tilde n^2\rangle_\eta$ in (11) is always small in comparison with the term $\beta\eta^2$ in the same equation and may be neglected. Furthermore, since we

η and small intensities of the smoothed fluctuational field $\bar{f}$, we may expand $\langle\tilde{n}\tilde{f}\rangle_\eta$ in powers of η and $\bar{f}$, keeping only terms through the second order in η and terms linear in $\bar{f}$.

Taking into account that, as is evident from the expression (17), the correlator $\langle\tilde{n}\tilde{f}\rangle_\eta$ is proportional to the noise intensity S, we may then write the resulting equation in the form

$$\dot{\eta}=-\alpha[1-(S/S_{cr})]\eta-\beta(1+C_1)\eta^2+D\Delta\eta+\bar{f}\eta(1+C_2); \tag{19}$$

$$\frac{S}{S_{cr}}=\frac{1}{\alpha}\left(\frac{1}{\eta}\langle\tilde{n}\tilde{f}\rangle_\eta\right)\bigg|_{\eta=\bar{f}=0},$$
$$C_1=\frac{1}{2\beta}\left(\frac{\partial^2}{\partial\eta^2}\langle\tilde{n}\tilde{f}\rangle_\eta\right)\bigg|_{\eta=\bar{f}=0}, \tag{20}$$
$$C_2=\left(\frac{\partial^2}{\partial\eta\,\partial\bar{f}}\langle\tilde{n}\tilde{f}\rangle_\eta\right)\bigg|_{\eta=\bar{f}=0}.$$

In the derivation of equation (19), the assumption (14) was used. Since we have now obtained the explicit expression (16) for the correlator $\langle\tilde{n}^2\rangle_\eta$, we can test the validity of this assumption. It must first of all be noted that according to (16), the root-mean-square of the rapid component $\tilde{n}$ is proportional to the local value of η. Therefore, in contrast to the situation near an equilibrium second-order phase-transition point, satisfaction of the criterion (14) does not impose limitations on the proximity to the noise-induced phase-transition point, and the calculation of the integral in (16) can be carried out directly at $S=S_{cr}$, $\eta=0$.

The results of a calculation of the critical noise intensities S_{cr} and a test of the fulfillment of the criterion (14), for various dimensionalities of the medium under study ($d=1,2,3$) and for various relations between the characteristic distances (7)–(9), are given in the Appendix. It is evident that the criterion (14) is far from being fulfilled in all cases. For its validity, it is first of all necessary that the characteristic dimension r_0 of a center of reproduction be much smaller than the characteristic diffusion distance $r_{diff}=(D/\alpha)^{1/2}$. Also important is the relation between the stationarity distance $l=(D/\gamma)^{1/2}$ and the value of r_{diff}. When the lifetime $\tau_0=\gamma^{-1}$ of an individual center of reproduction is large, i.e., $l\gg r_{diff}\gg r_0$, the criterion (14) is satisfied only in the three-dimensional case.

In the present paper, we shall restrict ourselves to consideration solely of those situations in which the criterion (14) is satisfied. If this is not so, the field $n(\mathbf{r},t)$ is a strongly fluctuating one; and in the analysis, complexities occur that are typical of the fluctuation region near an equilibrium second-order phase transition. Calculations also show that in all the cases of interest to us, the corrections C_1 and C_2 [see (20)] are small, and therefore the Ginzburg-Landau equation (19) for the smoothed concentration η, which plays the role of order parameter of the problem, can be rewritten in a simpler form:

$$\dot{\eta}=\alpha[(S/S_{cr})-1]\eta-\beta\eta^2+D\Delta\eta+\bar{f}(\mathbf{r},t)\eta. \tag{21}$$

We recall that the random field $\bar{f}$ is obtained by spatial smoothing

$$\bar{f}(\mathbf{r},t)=\frac{1}{\Delta V}\int_{\Delta V} f(\mathbf{r}+\boldsymbol{\rho},t)\,d\boldsymbol{\rho}$$

over a volume ΔV with linear dimensions much greater than the microscopic scales (7)–(9); and that since $\langle f\rangle=0$, the intensity of the field $\bar{f}$ approaches zero with increase of ΔV.

§3. NOISE-INDUCED PHASE TRANSITION

For an infinite medium, the averaging volume ΔV may become infinite, so that the smoothed concentration η will be the mean value $\langle n\rangle$ at the instant of time under consideration, and the value of $\langle n\rangle$ will be governed by the equation

$$\frac{d}{dt}\langle n\rangle=\alpha\left(\frac{S}{S_{cr}}-1\right)\langle n\rangle-\beta\langle n\rangle^2. \tag{22}$$

We assume that at the initial instant, there was in the medium some uniform concentration distribution $\langle n(0)\rangle=n_0$. If the intensity of the fluctuations of the rates of disintegration and reproduction is small ($S<S_{cr}$), then with passage of time $\langle n(t)\rangle\to 0$; the characteristic time of extinction is

$$\tau=(\alpha[1-S/S_{cr}])^{-1}. \tag{23}$$

But if $S>S_{cr}$, then there is established in the medium a self-sustaining stationary distribution with mean concentration $\langle n\rangle$ independent of the initial value n_0. Thus for the steady-state, stationary mode:

$$\langle n\rangle=0,\quad S<S_{cr}$$
$$\langle n\rangle=\frac{\alpha}{\beta}\left(\frac{S}{S_{cr}}-1\right),\quad S\geqslant S_{cr}. \tag{24}$$

We interpret the occupation of the medium when $S>S_{cr}$ as a noise-induced phase transition.

In order to treat fluctuational phenomena near such a phase-transition point, we note first of all that for an infinite medium, when $\Delta V\to\infty$, the criterion that we used in deriving equation (21) can be put into the form

$$(\langle n^2\rangle)^{1/2}\ll\langle n\rangle, \tag{25}$$

so that it takes the form of the condition for applicability of the self-consistent field approximation (the Ginzburg criterion) for the nonequilibrium phase transition under study. As follows from the expression (16) (see also the Appendix), the specific character of a noise-induced phase transition manifests itself in the fact that the root-mean-square fluctuation of the concentration n turns out to be proportional to the mean concentration $\langle n\rangle$, and therefore the Ginzburg criterion (25) either is satisfied all the way to the transition point itself, or is not satisfied at all, depending on the dimensionality d of the medium and on the relation between the characteristic lengths (7)–(9). Such an unusual fluctuational behavior is due to the fact that the order parameter in this problem is a positive definite quantity and that for $S\to S_{cr}$, the root-mean-square fluctuation of the order parameter disappears along with the order parameter itself, which vanishes at $S=S_{cr}$. Calculation of the single-time pair correlator $\langle n(\mathbf{r})n(\mathbf{r}')\rangle$ under the conditions for validity of the criterion (25) shows that the correlation radius determined by this correlator remains finite at the transition point and is determined by the microscopic scales (7)–(9).

Consideration of the percolation problem is also of interest. Let a medium described by equation (4) occupy the half-space $\{x,y,z;x>0\}$, and let the concentration n on the boundary $x=0$ be maintained constant by an external source. Although the mean rate of disintegration of the substance n on the average exceeds the mean rate of reproduction of it, as a result of fluctuations of these rates there arises in the medium a random structure of centers of reproduction, within which the diffusion current from the boundary $x=0$ is capable of obtaining reinforcement. Because of this, the mean depth of diffusion penetration increases as compared with the characteristic diffusion distance $r_{diff}=(D/\alpha)^{1/2}$ in the absence of fluctuations. For noise intensities S sufficiently close to S_{cr}, this characteristic distance can be determined from equation (21):

$$\delta L=r_{diff}(1-S/S_{cr})^{-1/2}. \tag{26}$$

Thus δL becomes infinite when $S=S_{cr}$, so that for $S>S_{cr}$ the medium becomes transparent for the diffusing substance n.

§4. THE ROLE OF RARE FLUCTUATIONS

The Ginzburg criterion (25) requires that on the average, the random variations of the density n must be small in comparison with its mean value $\langle n\rangle$. Thus fulfillment of this criterion does not prohibit the occurrence of strong fluctuations, for which δn is larger than or of the order of $\langle n\rangle$, but guarantees only their exponential rareness. The preceding treatment, based on development of a perturbation theory in the variations of the density n, did not take into account possible effects caused by exponentially rare strong fluctuations. The present section of the paper is devoted to a discussion of such effects.

Although the strong fluctuations are exponentially rare, the growth of the concentration n at them may reach extremely large values. As a result, against a macroscopically homogeneous background, due to typical weak fluctuations, there arises a structure of rare oases, each of which possesses its own lifetime τ.

The greatest interest attaches to two questions: first, whether there does not occur a self-sustaining population of rare oases at lower noise intensities S, as compared with the above-found critical value S_{cr} at which population of the medium begins to occur because of typical, weak fluctuations; and second, what the contribution of the rare, populated oases is to the mean steady-state concentration.

We note that in solution of the first of these problems, it is sufficient to consider an equation linear in the concentration n, by disregarding the term βn^2 in Eq. (4). In fact, we shall suppose that at the initial instant of time, there is prescribed a uniform concentration distribution $n(\mathbf{r},t=0)=n_0$, and that the value of n_0 is small. Then even if population occurs and the concentration n begins to grow with time, in the first stage (which can be made as prolonged as one wishes by choosing a sufficiently small value of n_0) the term $-\beta n^2$, which causes the limiting of the growth of the concentration, can be disregarded. Furthermore, it is natural first to carry out a renormalization of the coefficients of equation (4) because of typical weak fluctuations, obtaining

$$\dot{n}=-\alpha_{eff}n+D\Delta n+f(\mathbf{r},\ t)n, \tag{27}$$

where $\alpha_{eff}=\alpha(1-S/S_{cr})$, and where $f(\mathbf{r},t)$ contains only the exponentially rare fluctuations.

The substitution

$$n(\mathbf{r},\ t)=n'(\mathbf{r},\ t)\exp(-\alpha_{eff}t)$$

reduces equation (27) to the form

$$\dot{n}'=f(\mathbf{r},\ t)n'+D\Delta n'. \tag{28}$$

Below, we shall omit the primes in the notation for the new concentration.

Because the strong fluctuations are exponentially rare, they may be treated independently. Let the volume ΔV within the time interval Δt contain one such fluctuation, so that the increase of the quantity of reproducing substance in this fluctuation is

$$\Delta N=\int_{\Delta V}\int_{\Delta t}n(\mathbf{r},t)\,d\mathbf{r}\,dt. \tag{29}$$

We consider formally the eigenvalue problem

$$\hat{L}(t)\nu=\lambda(t)\nu, \tag{30}$$

where the linear operator $\hat{L}(t)$ has the form

$$\hat{L}(t)=D\Delta+f(\mathbf{r},\ t). \tag{31}$$

For each instant of time t equation (30) is equivalent to a stationary Schrödinger equation, reduction to which is accomplished by the substitution

$$D\to(\hbar^2/2m),\quad f(\mathbf{r},\ t)\to-U(\mathbf{r},\ t),\quad \lambda\to-E. \tag{32}$$

If the field f did not change with time, the general solution of equation (28) for a single strong fluctuation would be given by the expansion

$$n(\mathbf{r},t)=\sum_l C_l\exp(\lambda_l t)\nu_l(\mathbf{r}), \tag{33}$$

where $\{\lambda_l\}$ is the spectrum of the operator $\hat{L}$. Then the fastest growing term in (33) would be that for which $\lambda=\max\lambda_l$. In the language of the Schrödinger equation, it would correspond to the deepest level in the well $U(\mathbf{r})=-f(\mathbf{r})$. Actually, the field f changes with time, and therefore $\lambda_l=\lambda_l(t)$ and $\lambda_l=v_l(\mathbf{r};t)$. We also define

$$\lambda(t)\equiv\max_l\lambda_l(t).$$

In order to characterize each fluctuation roughly, it is possible to indicate for it the mean (over time) value of λ and the lifetime of the fluctuation τ. If $\lambda\tau\ll1$, then this fluctuation is weak, since growth of concentration n in it is small and it can be taken into account by perturbation theory. Thus strong fluctuations satisfy the condition $\lambda\tau\gtrsim1$. For such fluctuations, the potential $f(\mathbf{r},t)$ in equation (28) is a slowly varying function of time; and in analogy with the nonstationary Schrödinger equation, we may use the adiabatic approximation,[3] obtaining for the total increase of the quantity of substance the expression

$$\Delta N\sim n(0)\exp\left(\int_0^{\Delta t}\lambda(\tau)\,d\tau\right). \tag{34}$$

By introducing a characteristic parameter

$$s=\int_{-\infty}^{\infty}\lambda(\tau)d\tau,$$

describing each individual strong fluctuation ($s \gtrsim 1$), we can put the expression (34) into the form

$$\Delta N(s)\sim n(0)e^{s}, \qquad (35)$$

where $n(0)$ is the mean value of the concentration in the medium at the instant immediately preceding the occurrence of the given fluctuation.

Let $p=p(s)$ be the probability that unit volume within unit time contains a strong fluctuation with parameter s. Then the mean increase of concentration at strong fluctuations per unit time is

$$\overline{\Delta n}=\int \Delta N(s)p(s)ds. \qquad (36)$$

Since strong fluctuations are exponentially rare, $p(s) \sim\exp(-\Phi(s))$, where $\Phi(s)\gg 1$. Therefore by use of (35) and (36) we get

$$\overline{\Delta n}\sim\bar{n}\int_{1}^{\infty}\exp(s-\Phi(s))ds=Q\bar{n}. \qquad (37)$$

Thus the growth of the volume-average concentration $\bar{n}$ because of strong fluctuations occurs according to the law $\dot{\bar{n}}=Q\bar{n}$; or, after return to the old concentration n by the substitution $\bar{n}=\bar{n}'\exp(-\alpha_{eff}t)$,

$$\dot{\bar{n}}=(Q-\alpha_{eff})\bar{n}. \qquad (38)$$

Thus we must estimate the value of

$$Q\sim\int_{1}^{\infty}\exp(s-\Phi(s))ds\equiv\int_{1}^{\infty}\exp(F(s))ds. \qquad (39)$$

The order of magnitude of this quantity is determined by competition of two factors: the exponential rareness of strong fluctuations, and the exponential growth of concentration at each of them. If $F(s)<0$ for all s, then the value of Q is exponentially small; but if $F(s)>0$ within some interval of values of s, then the coefficient Q will be exponentially large.

We note that $Q=Q(S)$, where S is the noise intensity. It is important to know what the value of $Q_c=Q(S_{cr})$ is. If Q_c is exponentially small, then strong fluctuations cause only an exponentially small shift of the threshold for population calculated earlier ($\alpha_{eff}=0$); while if the value of Q_c is exponentially large, then population at strong fluctuations is sustained even at much lower noise intensities than $S=S_{cr}$.

A typical fluctuation contains a deepest level $\bar{\lambda}$ (if there are any levels at all in it; see the three-dimensional case), whose lifetime is of order τ_0, where τ_0 is the correlation time for Gaussian noise $f(\mathbf{r},t)$; for such a typical fluctuation, the condition $\lambda\tau_0\ll 1$ is satisfied if the Ginzburg criterion is fulfilled. The appearance of strong fluctuations with $s\gtrsim 1$ is an exponentially rare event and may be achieved for two reasons. First, it may occur because of a fluctuation with a level λ of very great depth ($\lambda\gg\bar{\lambda}$). Since the occurrence of such a fluctuation is already an exponentially rare event, one must expect that its lifetime will be typical ($\tau\sim\tau_0$). Second, some one of the fluctuations with a typical depth $\lambda\sim\bar{\lambda}$ may turn out to be exceptionally long-lived, so that $\tau\gg\tau_0$ and $\bar{\lambda}\tau\gtrsim 1$.

We shall begin such a treatment with the three-dimensional case ($d=3$) under the condition r_{diff}, $l\gg r_0$, when, as is evident from the Appendix, the Ginzburg criterion is satisfied. Then the threshold noise intensity S_{cr} is

$$S_{cr}^{1/2}=\alpha\left(\frac{r_{diff}}{r_0}\right)=\frac{D}{r_0^2}\frac{r_0}{r_{diff}}\ll\frac{D}{r_0^2}. \qquad (40)$$

By considering the equivalent eigenvalue problem (30) of a stationary Schrödinger equation with the substitution $D\to\hbar^2/2m$, $f\to -U$, $\lambda\to -E$, we see that near the threshold for population at weak fluctuations, a typical well, having depth $U_0\sim S_{cr}$ and width $a\sim r_0$, contains no levels at all. Consequently, the very appearance of a discrete level is already an exponentially rare event; and if it occurs, the level will exist for a time of about τ_0. Therefore in making estimates, we may set

$$Q\sim\int_{1/\tau_0}^{\infty}\exp(\lambda\tau_0-\Phi(\lambda))d\lambda, \qquad (41)$$

where $w(\lambda)\sim\exp(-\Phi(\lambda))$ is the probability of occurrence of the level λ.

The function $\Phi(\lambda)$ must be calculated on the basis of equation (30), which at each instant of time reduces, by the substitution (32), to a stationary Schrödinger equation. Therefore we may use the known calculations[4] of the density of states in the fluctuation region for the Schrödinger equation in the case of a Gaussian random potential with intensity S and correlation radius r_0. Under the condition $S\ll D^2/r_0^4$, the expressions presented for the three-dimensional case in the review of Liftshitz, Gredeskul, and Pastur[4] given

$$\Phi(\lambda)=\lambda^2/2S \quad \text{if} \quad \lambda r_0^2/D\gg 1, \qquad (42a)$$

$$\Phi(\lambda)\approx(D^2/Sr_0^4)[1+\xi(\lambda r_0^2/D)^{1/2}] \quad \text{if} \quad \lambda r_0^2/D\ll 1, \qquad (42b)$$

where the coefficient ξ is of order unity.

It is then easy to demonstrate that the maximum of the function $F(\lambda)=\lambda\tau_0-\Phi(\lambda)$ is attained in the range $\lambda\sim D/r_0^2$; and because of the condition $l\gg r_0$, we have $1/\tau_0\ll D/r_0^2$. Since

$$\max F(\lambda)\leqslant D^2/Sr_0^4-D\tau_0/r_0^2, \qquad (43)$$

the condition $\max F(\lambda)<0$, which insures an exponentially small coefficient Q, is satisfied at noise intensities $S<S_c$, where

$$S_c\sim D/\tau_0 r_0^2. \qquad (44)$$

Thus self-sustaining population of oases—rate strong fluctuations—occurs in this case at noise intensities $S>S_c$. On comparing the value of S_c with the value of the threshold S_{cr} for population of the medium because of typical weak fluctuations, not containing levels within them, we find that $S_c\ll S_{cr}$ when $l\gg r_{diff}$ (that is, $\tau_0\gg 1/\alpha$), and that $S\gg S_{cr}$ when $l\ll r_{diff}$ (that is, $\tau_0\ll 1/\alpha_0$).

We shall consider further a situation likewise in the three-dimensional case, but with a different relation of the characteristic lengths, $r_{diff}\gg r_0\gg l$, when, as follows from the Appendix, the Ginzburg criterion is, as before, fulfilled. In this case, the expression for

S_{cr} may be written in the form

$$S_{cr}^{1/2}=(\alpha/\tau_0)^{1/2}=(r_0^2/r_{diff}l)(D/r_0^2), \quad (45)$$

and therefore if $r_0 \gtrsim (r_{diff} l)^{1/2}$, we have $S_{cr}^{1/2} \gtrsim D/r_0^2$, and a typical well contains levels within it.

The condition $r_0 \gg l$ means $1/\tau_0 \gg D/r_0^2$; and therefore when $\lambda > 1/\tau_0$, the expression for $\Phi(\lambda)$ is given by formula (42a). For $S=S_{cr}$, we have

$$F(\lambda)=\lambda\tau_0-(\tau_0\lambda^2/2\alpha), \quad (46)$$

and since the relation $F(\lambda)\approx -1/2\alpha\tau_0 \ll -1$ is already valid at $\lambda=1/\tau_0$, the condition $F(\lambda)<0$ is certainly satisfied for all $\lambda>\tau_0^{-1}$, and therefore the contribution to the coefficient Q from deep levels $(\lambda>\tau_0^{-1})$ is exponentially small.

But now we must take into account the possibility that one of the levels with a typical depth $\lambda\sim\bar{\lambda}$ turns out to be long-lived. We shall first estimate the probability that a given level with a typical value $\bar{\lambda}$ will survive a time $\tau \gg \tau_0$. The probability functional of Gaussian noise $f(\mathbf{r},t)$ is

$$P[f(\mathbf{r},t)]\sim \exp\left\{-{}^1/_2\iint g(\mathbf{r}_1-\mathbf{r}_2,t_1-t_2)f(\mathbf{r}_1,t_1)f(\mathbf{r}_2,t_2)d\mathbf{r}_1\,d\mathbf{r}_2\,dt_1\,dt_2\right\}, \quad (47)$$

where the function $g(\mathbf{r},t)$ is determined by the condition

$$\int g(\mathbf{r}-\mathbf{r}',t-t')\sigma(\mathbf{r}'-\mathbf{r}_1,t'-t_1)d\mathbf{r}'\,dt'=\delta(\mathbf{r}-\mathbf{r}_1)\delta(t-t_1)$$

and according to the expression (6)

$$\sigma(\mathbf{r},t)=S\exp(-|\mathbf{r}|/r_0)\exp(-|t|/\tau_0).$$

If the given fluctuation is long-lived $(\tau\gg\tau_0)$, then its potential $f(\mathbf{r},t)$ varies smoothly in time as compared with the correlation time τ_0, and therefore at such a fluctuation the functional (47) takes the same value as does the corresponding functional for temporally white noise:

$$P[f(\mathbf{r},t)]\sim\exp\left\{-{}^1/_4\tau_0^{-1}\iint\bar{g}(\mathbf{r}_1-\mathbf{r}_2)f(\mathbf{r}_1,t)f(\mathbf{r}_2,t)d\mathbf{r}_1\,d\mathbf{r}_2\,dt\right\} \quad (48)$$

$$\int\bar{g}(\mathbf{r}-\mathbf{r}')S\exp(-r_0^{-1}|\mathbf{r}-\mathbf{r}_1|)d\mathbf{r}'=\delta(\mathbf{r}-\mathbf{r}_1).$$

We take for an estimate the function

$$f(\mathbf{r},t)=\chi(\mathbf{r}) \quad \text{at} \quad 0<t<\tau,$$
$$f(\mathbf{r},t)=0 \quad \text{at} \quad t<0 \quad \text{or} \quad t>\tau. \quad (49)$$

The functional P takes on it the value

$$P\sim\exp\left\{(-\tau/4\tau_0)\iint\bar{g}(\mathbf{r}_1-\mathbf{r}_2)\chi(\mathbf{r}_1)\chi(\mathbf{r}_2)d\mathbf{r}_1\,d\mathbf{r}_2\right\}. \quad (50)$$

If the fluctuation (apart from its "longevity") is typical, this means that

$$\iint\bar{g}(\mathbf{r}_1-\mathbf{r}_2)\chi(\mathbf{r}_1)\chi(\mathbf{r}_2)d\mathbf{r}_1\,d\mathbf{r}_2\lesssim 1. \quad (51)$$

Therefore the probability $p=p(\tau)$ of the appearance of levels typical as regards depth, but long-lived, is

$$p(\tau)\sim\exp(-\varphi(\tau)),\quad \varphi(\tau)\sim\tau/\tau_0 \quad \text{at} \quad \tau\gg\tau_0. \quad (52)$$

The contribution to the coefficient Q from such fluctuations may then be estimated as

$$Q\sim\int_{\tau_0}^{\infty}\exp(\bar{\lambda}\tau-\varphi(\tau))d\tau. \quad (53)$$

It is evident that, since under the Ginzburg condition $\bar{\lambda}\ll 1/\tau_0$, the contribution (53) is exponentially small.

Thus when $d=3$, for relations between the characteristic distances r_{diff}, l, and r_0 that insure satisfaction of the Ginzburg criterion, population at rare fluctuations can occur below the population threshold S_{cr} found in Sec. 3 only for sufficiently large correlation times $\tau_0 \gg 1/\alpha$ (and, in particular, for stationary random media).

We shall further consider the situation in the two-dimensional case $(d=2)$, when the Ginzburg criterion, as is evident from the Appendix, is satisfied for $r_{diff}\gg r_0, l$. Since an arbitrarily shallow two-dimensional well contains levels within it, we must take into account the contribution to the coefficient Q both from levels that are deep $(\lambda>\tau_0^{-1})$ but of average lifetime $(\tau\sim\tau_0)$ and from levels that are typical in depth $(\lambda\sim\bar{\lambda})$ but long-lived. The latter contribution can be estimated by a method completely analogous to that used above for the three-dimensional case, and it is found to be exponentially small. The contribution from the deep levels is

$$Q\sim\int_{1/\tau_0}^{\infty}\exp(\lambda\tau_c-\Phi(\lambda))d\lambda. \quad (54)$$

We first analyze the case in which $r_{diff}\gg l\gg r_0$. In this case

$$S_{cr}=\frac{\alpha^2}{\ln(2l/r_0)}\frac{r_{diff}^2}{r_0^2}=\frac{\alpha D}{r_0^2\ln(2l/r_0)}, \quad (55)$$

and therefore near the threshold $S=S_{cr}$, values $\lambda>\tau_0^{-1}$ belong to the fluctuation region $(\lambda\gg Sr_0^2/D)$. After appropriate changes of notation, the expression for $\Phi(\lambda)$ given in Ref. 4 takes the form

$$\Phi(\lambda)=\lambda^2/2S \quad \text{at} \quad \lambda r_0^2/D\gg 1, \quad (56a)$$
$$\Phi(\lambda)\sim(D^2/Sr_0^4)(\lambda r_0^2/D) \quad \text{at} \quad Sr_0^4/D^2\ll\lambda r_0^2/D\ll 1. \quad (56b)$$

Since for $S\sim S_{cr}$ the relation

$$\tau_0\ll D/Sr_0^2\sim\alpha^{-1}\ln(2l/r_0), \quad (57)$$

holds, and since we are considering the case $r_{diff}\gg l\gg r_0$ (that is, $\tau_0\ll\alpha^{-1}$), the value of $F(\lambda)$ is negative for all $\lambda>\tau_0^{-1}$, and consequently the coefficient Q for $S\gtrsim S_{cr}$ is exponentially small.

The case $d=2$ and $r_{diff}\gg r_0\gg l$, when $S_{cr}=\alpha/\tau_0$, must be treated separately. In this case $\tau_0^{-1}\gg D/r_0^2$, and $\Phi(\lambda)$ is given by the expression (56a). Then the condition $F(\lambda)<0$ for $\lambda>\tau_0^{-1}$ is satisfied if

$$\tau_0\ll(D^2/Sr_0^4)(r_0^2/D). \quad (58)$$

With allowance for the expression given above for S_{cr} in this case, this means $\alpha\tau_0\ll 1$, and therefore the requirement (58) is satisfied.

Thus when the dimensionality of the medium is two, in all the cases examined in which the Ginzburg criterion is satisfied, population at rare fluctuations for $S<S_{cr}$ does not occur.

We finally consider the one-dimensional case $(d=1)$, when the Ginzburg criterion is satisfied if $r_{diff}\gg r_0, l$. An arbitrarily shallow one-dimensional well contains levels within it; but the contribution to the coefficient Q from levels that are typical in depth but long-lived is again small, for the same reasons as in the three-dimensional case, when $r_{diff}\gg r_0\gg l$. The value $\lambda=\tau_0^{-1}$

again falls in the fluctuation region, where now (see Ref. 4)

$$\Phi(\lambda)=\lambda^2/2S \quad \text{at} \quad \lambda r_0^2/D\gg 1, \tag{59a}$$

$$\Phi(\lambda)\sim(D^2/Sr_0^4)(\lambda r_0^2/D)^{3/2} \quad \text{at} \quad Sr_0^4/D^2\ll\lambda r_0^2/D\ll 1. \tag{59b}$$

It is easy to show that when $S\sim S_{cr}$, the value of $F(\lambda)=\lambda\tau_0-\Phi(\lambda)$ is negative for all $\lambda>\tau_0^{-1}$, and therefore the coefficient Q for $S\sim S_{cr}$ is exponentially small; this means absence of population of oases for $S<S_{cr}$.

Thus population because of rare fluctuations is sustained at noise intensities S less than S_{cr} only in a three-dimensional medium, under the conditions $l\gg r_{diff}\gg r_0$.

We shall now discuss the contribution to the steady-state concentration $\langle n\rangle$ from population of oases—exponentially rare strong fluctuations. For this purpose (in contrast to the problem of the population threshold) we must take account, in the original equation (4), of the nonlinear term βn^2, which limits the growth of concentration at an individual oasis. Appropriate estimates may be made as follows.

With allowance for the nonlinear limiting, the total quantity of reproducing substance at an individual oasis—a rare fluctuation with level λ—is approximately

$$\Delta N(\lambda)\sim(\lambda-\alpha_{eff})V(\lambda)/\beta, \tag{60}$$

where $V(\lambda)\sim\lambda^{-d/2}$ is the characteristic volume over which the effect of a strong fluctuation spreads. The contribution of strong fluctuations to the mean concentration is then

$$\bar{n}\sim\int_{1/\tau_0}^{\infty}\Delta N(\lambda)\exp[-\Phi(\lambda)]d\lambda. \tag{61}$$

Since strong fluctuations are exponentially rare, a leading role in the integral (61) is played by the factor $\exp[-\Phi(\lambda)]$, in which $\Phi(\lambda)\gg 1$; $\Phi(\lambda)$ increases with increase of λ. Therefore the contribution $\bar{n}$ under consideration is exponentially small; and apart from a preexponential factor, we may write that

$$\bar{n}\sim\exp[-\Phi(\tau_0^{-1})]. \tag{62}$$

In view of the exponential smallness of $\bar{n}$, this contribution can certainly be neglected for $S>S_{cr}$, when the principal contribution (24) to the mean concentration is that due to population by typical weak fluctuations. Only in one case—for a three-dimensional medium with $l\gg r_{diff}\gg r_0$, when population at rare fluctuations is sustained far below the value $S=S_{cr}$—is allowance for this warranted. In this case, on the mean concentration (24) calculated above is superposed an exponentially small tail, $\bar{n}\sim\exp(-D^2/Sr_0^4)$, self-sustaining below S_{cr} down to noise intensity $S\sim S_c$ [see (44)].

§5. CONCLUSION

The basic role in the occurrence of the noise-induced phase transition considered in this paper is played by diffusion. As is evident from the Appendix, the criterion for weakness of typical fluctuations (the Ginzburg criterion) is violated when the characteristic diffusion distance $r_{diff}=(D/\alpha)^{1/2}$ is much smaller than the typical dimension r_0 of a center of reproduction. It is easy to show that in this limiting case ($r_0\gg r_{diff}$), noise-induced explosive instability does not occur, and therefore the nonequilibrium phase transition that we are studying is possible.

In fact, if the diffusion coefficient approaches zero, then different regions of space are not connected with each other, and for $\beta=0$ we actually have the equation

$$\dot{n}=-\alpha n+f(t)n. \tag{63}$$

Simple integration gives the solution of this equation:

$$n(t)=n_0\exp\left\{-\left[\alpha-t^{-1}\int_0^t f(\tau)d\tau\right]t\right\}. \tag{64}$$

Since the mean value of the random function $f(t)$ is zero, the relation

$$t^{-1}\int_0^t f(\tau)d\tau\to\langle f\rangle=0, \quad t\to\infty \tag{65}$$

is valid; and consequently, independently of the noise intensity $f(t)$, the concentration n approaches zero as time goes on, and therefore there is no explosive instability.

The same result is valid also if the spatial correlation radius r_0 of the noise becomes infinite, since then the fluctuations are homogeneous in space and diffusion is unimportant.

It is evident from the Appendix that the Ginzburg criterion is violated for two- and one-dimensional media in the case $l\gg r_{diff}\gg r_0$, when the correlation time τ_0 of the noise is the largest time scale of the problem. In these cases typical fluctuations are strong, and therefore the treatment presented in this paper is inapplicable.

It must also be noted that in the immediate vicinity of the nonequilibrium phase-transition point, where $\langle n\rangle\to 0$, the very description in terms of the diffusion equation (1) becomes inapplicable, and it is necessary to take account of the discrete (atomistic) character of the diffusion. Furthermore, in the paper no attention was paid to characteristic (for example hydrodynamic) fluctuations in the concentration of the reproducing substance, but only fluctuations in its rates of disintegration and reproduction were taken into account. Investigations of these questions will be the subject of separate communications.

The problem studied was formulated by us as a problem on diffusion of some reproducing substance through a medium where processes for its disintegration are also possible. It must be emphasized, however, that the original equation can also describe other situations in various physical, biological, and ecological systems, and therefore the results obtained have a broader range of applicability. A number of analogous effects for biophysical systems were discussed earlier.[5]

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APPENDIX

The value of the critical noise intensity $S_{cr} = \langle f^2 \rangle_{cr}$ and values of the parameter $G = [\langle \tilde{n} \rangle_\eta^2 / \eta^2]_{S = S_{cr}}$, which characterizes satisfaction of the criterion for weak fluctuations, for media of various dimensionalities ($d = 1, 2, 3$) for various relations between r_{diff}, l, and r_0:

I. $d=1$. a) $r_{diff} \gg r_0 \gg l$ $(\alpha \ll Dk_0^2 \ll \gamma)$,

$$S_{cr}^{1/2} = \alpha (r_{diff}/l), \quad G = r_0/r_{diff} \ll 1.$$

b) $r_{diff} \gg l \gg r_0$ $(\alpha \ll \gamma \ll Dk_0^2)$,

$$S_{cr}^{1/2} = \alpha (r_{diff}/r_0^{1/2} l^{1/2}), \quad G = l/r_{diff} \ll 1.$$

c) $r_0 \gg r_{diff}, l$ $(Dk_0^2 \ll \alpha, \gamma)$, $G \gtrsim 1$.

d) $l \gg r_{diff}, r_0$ $(\gamma \ll \alpha, Dk_0^2)$, $G \gtrsim 1$.

II. $d=2$. a) $r_{diff} \gg r_0 \gg l$ $(\alpha \ll Dk_0^2 \ll \gamma)$,

$$S_{cr}^{1/2} = \alpha r_{diff}/l, \quad G = (r_0/r_{diff})^2 \ln (2 r_{diff}/r_0) \ll 1.$$

b) $r_{diff} \gg l \gg r_0$ $(\alpha \ll \gamma \ll Dk_0^2)$,

$$S_{cr}^{1/2} = \frac{\alpha r_{diff}}{r_0} \left(\ln \frac{2l}{r_0} \right)^{-1/2}, \quad G = \left(\frac{l}{r_{diff}} \right)^2 \ln \frac{r_{diff}}{l} \Big/ \ln \frac{2l}{r_0} \ll 1.$$

c) $r_0 \gg r_{diff}, l$ $(Dk_0^2 \ll \gamma, \alpha)$, $G \gtrsim 1$.

d) $l \gg r_{diff}, r_0$ $(\gamma \ll \alpha, Dk_0^2)$, $G \gtrsim 1$.

III. $d=3$. a) $r_{diff} \gg r_0 \gg l$ $(\alpha \ll Dk_0^2 \ll \gamma)$,

$$S_{cr}^{1/2} = \alpha r_{diff}/l, \quad G = (r_0/r_{diff})^2 \ll 1.$$

b) $r_{diff} \gg l \gg r_0$ $(\alpha \ll \gamma \ll Dk_0^2)$

$$S_{cr}^{1/2} = \alpha r_{diff}/r_0, \quad G = 2 r_0 l / r_{diff}^2 \ll 1.$$

c) $r_0 \gg r_{diff}, l$ $(Dk_0^2 \ll \alpha, \gamma)$, $G \gtrsim 1$.

d) $l \gg r_0 \gg r_{diff}$ $(\gamma \ll Dk_0^2 \ll \alpha)$, $G \gtrsim 1$.

e) $l \gg r_{diff} \gg r_0$ $(\gamma \ll \alpha \ll Dk_0^2)$,

$$S_{cr}^{1/2} = \alpha r_{diff}/r_0, \quad G = r_0/r_{diff} \ll 1.$$

[1)] The best known example of such an effect is parametric excitation of a classical oscillator (see Ref. 1).

[2)] This procedure is exactly equivalent to the self-consistent field approximation in the theory of equilibrium phase transition.

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Transmission of sound across a boundary between liquid helium and a metal

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A description of the experimental procedure is followed by a report of the results of investigations of the angular dependence of the transmission of a plane monochromatic acoustic wave incident from liquid ^{4}He on the surfaces of tungsten and gold. The energy of 10–30 MHz sound was deduced from measurements of the Kapitza temperature jump at the liquid helium–metal boundary at temperatures 0.1–0.4 °K. A resonance energy transmission peak was observed experimentally outside the critical cone when sound was incident at the Rayleigh angle. This effect was considered using the generalized acoustic theory and the theory of Andreev. The contribution of the energy associated with the Rayleigh peak was approximately equal to the energy in the subcritical angle. A comparison was made of the attenuation of surface waves on tungsten with the known theoretical and experimental coefficients representing the bulk absorption of sound due to an electron mechanism.

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INTRODUCTION

The present author reported earlier[1] the observations of resonance absorption of sound by the surface of a metal. The effect was predicted theoretically by Andreev[2] and it was due to dissipation of the energy of a Rayleigh wave excited in the metal by the incident sound. The present paper describes an investigation of the angular and frequency dependences of the transmission coefficient $w(\theta, \omega)$ of the acoustic energy crossing the boundary between liquid ^{4}He and a metal.

The idea of measuring $w(\theta, \omega)$ has been put forward in several laboratories some years ago after numerous attempts to explain fully the Kapitza temperature jump[3] which appears at an interface when heat is transferred from a solid body to liquid helium. Kapitza showed that this temperature jump ΔT is proportional to the heat flux density $\dot{Q}/S$ and to the thermal resistance of the boundary (contact resistance) R_c, which varies with temperature as T^{-3}.

A theoretical explanation of the Kapitza jump was giv-

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