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Kinetic equation for atoms interacting with laser radiation

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The motion of atoms in a resonant light wave that excites the atoms in a transition from the ground state to an excited state is considered. A kinetic equation of the Fokker-Planck type is obtained to describe the motion of the atoms due to the recoil of the induced and spontaneous transitions. The equation is used to analyze velocity monochromatization and focusing of an atomic beam in a laser beam.

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1. INTRODUCTION. FORMULATION OF PROBLEM

The recent calculations and the results of the first experiments (see the reviews[1-3]) have demonstrated convincingly the effectiveness of using laser-radiation pressure to act on the spatial motion of neutral particles. Thus, by using the pressure of laser light it becomes possible to deflect[4,5] focus,[6] and slow down[7] atom beams.

On the theoretical level, the investigations of light pressure were based so far on the study of the motion of atoms either in plane light waves or in light waves of constant bounded cross section. These approaches have revealed the role played by the main processes responsible for the existence of light pressure, and the character of the motion of the atoms in the simplest field configurations. At the same time, the use of these models is insufficient for the analysis of experimental situations in which an essential role is played by the laser-beam divergence. This, in one of the most promising applications, that of radiative cooling and dragging of atoms,[3] it is expedient to use light beams both with bounded cross section and with definite angle divergence.[8,9] The need for analyzing such problems calls for knowledge of the laws of motion of atoms in real laser beams.

The present paper presents a derivation of a kinetic equation that describes the evolution of the distribution function of atoms interacting with diverging or converging laser beams. The equation is derived for laser radiation of the fundamental TEM_{00q} mode and for atoms whose interaction with the laser field can be described by a two-level scheme. Particular attention in the analysis is paid to the conditions under which the equation is valid. Velocity monochromatization of an atomic beam in a plane light wave and the focusing of an atomic beam in a light wave with an inhomogeneous transverse distribution of the field are considered by way of examples of the derived equation.

2. INITIAL EQUATIONS

To obtain the equation of motion of an ensemble of atoms in a laser beam, we start from the equation

$$i\frac{\partial\hat\rho}{\partial t}=(\hat H'-\hat H''^{*})\hat\rho-i\hat\Gamma\hat\rho \tag{1}$$

For the density matrix $\hat\rho(\mathbf{r}', \mathbf{r}'', t)$ that describes the interaction of the atom with a classical light field E. In this equation, the Hamiltonian of the interaction consists of three terms:

$$\hat H=\hat H_0-(\hbar^2/2M)\nabla^2+\hat V. \tag{2}$$

The first determines the internal states of the atom, the second the translational state of the atom, and the third the dipole interaction of the atom with the field:

$$\hat V=-\hbar^{-1}\hat{\mathbf{d}}\mathbf{E}. \tag{3}$$

The relaxation operator $\hat\Gamma$ describes the change of the state of the atom on account of spontaneous decays.

We specify the laser radiation in the form of a fundamental TEM_{00q} mode (Fig. 1). The corresponding field E takes a cylindrical coordinate system with z axis along the beam axis the form[10]

$$\mathbf{E}(\mathbf{r},t)=\mathbf{e}E_0\frac{q_0}{q}\exp\left(-\frac{\rho^2}{2q^2}\right)\cos\left[\omega t-\left(k+\frac{\rho^2}{bq^2}\right)z\right], \tag{4}$$

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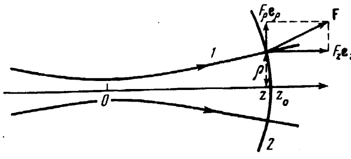

FIG. 1. Intersection of laser radiation of mode *TEM* by a plane passing through the *z* axis, and the components of the light-pressure force *F* acting on an atom with coordinates *z* and ρ: curve 1—line of constant intensity of the laser beam, curve 2—wave front of the beam.

where **e** is the unit vector of the wave polarization, the parameter q determines the scale of variation of the field over the beam cross section:

$$q=q_0(1+4z_0^2/b^2)^{1/2}, \tag{5}$$

$$q_0={}^1/_2(b\lambda/\pi)^{1/2}, \tag{6}$$

z_0 is the point of intersection of the wave front of the beam defined by the equation

$$z-z_0=-\frac{2z/b}{1+4z^2/b^2}\frac{\rho^2}{b}, \tag{7}$$

with the z axis. The parameter b, the so-called laser-radiation in variant, determines the beam divergence and the minimum dimension of the caustic. Its value depends on the geometry of the laser cavity and on the parameters of the focusing elements (mirrors, lenses, etc.).

We assume below that the variation of the field along the z axis, due to the laser beam divergence, takes place over a distance greatly exceeding the wavelength $\lambda=2\pi/k$. This condition corresponds formally to the relation

$$b\gg\lambda. \tag{8}$$

In addition, we consider only values of ρ that are of practical importance, and are not too large compared with the transverse scale of the field:

$$\rho\leqslant q. \tag{9}$$

Under conditions (8) and (9) we then have

$$\rho^2/bq^2\ll k. \tag{10}$$

Conditions (8)–(10) simplify greatly the final results. At the same time we cover the overwhelming majority of cases of practical interest.

We assume furthermore that the laser radiation excites atomic transitions only between two levels, the lower ground level $|g\rangle$ and the upper $|e\rangle$ that decays to the ground level with a total probability 2γ. It must be immediately emphasized that, strictly speaking, the two-level scheme chosen by us for the interaction of the atom with the field contradicts the chosen form of the field (4). In fact, diverging laser radiation is always elliptically polarized and excites, in the transition $nS-nP$ of interest to us, all three ($m=0$, ±1) sublevels of the upper state nP. Under conditions (8)–(10), however, the light field that intereacts with the atom does not differ greatly from a plane wave. In addition, practical interest usually attaches either to circular or to linear polarization of the laser radiation. Confining ourselves to these two cases and bearing the conditions (8)–(10) in mind, we can assume with sufficient degree of accuracy that in the case of linearly polarized radiation there is excited only a state with $m=0$, and in the case of circularly polarized radiation there is excited either a state with $m=1$, or with $m=-1$.

For the chosen scheme of the interaction of the atom with the field, and for the two indicated types of wave polarization, the relaxation operators $\hat{\Gamma}$ were determined in Refs. 11 and 12. Taking into account the results of these references, we can immediately deduce from (1) the equations for the elements of the density matrix $\rho_{ij}(\mathbf{r}', \mathbf{r}'', t)$. Without dwelling on the latter, we proceed to the equations for the Wigner density matrix $\hat{\rho}(\mathbf{r}, \mathbf{p}, t)$. To this end we carry out the following chain of transformations, which were previously used in Ref. 13. We replace the coordinates $\mathbf{r}'$ and $\mathbf{r}''$ by the coordinates $\mathbf{r}$ and $\mathbf{x}$:

$$\mathbf{r}'=\mathbf{r}-\mathbf{x}/2,\quad \mathbf{r}''=\mathbf{r}+\mathbf{x}/2, \tag{11}$$

We expand the field in plane waves

$$V_{ij}(\mathbf{r},t)=\int V_{ij}(\varkappa,t)e^{i\varkappa\mathbf{r}}d\varkappa \tag{12}$$

and introduce the density matrix in the Wigner representation by the relation

$$\rho_{ij}(\mathbf{r}-\mathbf{x}/2,\mathbf{r}+\mathbf{x}/2,t)=\hbar^{-3}\int\rho_{ij}(\mathbf{r},\mathbf{q},t)e^{-i\mathbf{q}\mathbf{x}/\hbar}d\mathbf{q}. \tag{13}$$

After these transformations, Eq. (1) reduces to a system of difference integro-differential equations for the elements of the Wigner density matrix

$$\begin{aligned}
i\frac{d}{dt}\rho_{ee}(\mathbf{r},\mathbf{p},t)&=\int V_{eg}(\varkappa,t)\rho_{ge}\left(\mathbf{r},\mathbf{p}-\frac{\hbar\varkappa}{2},t\right)e^{i\varkappa\mathbf{r}}d\varkappa\\
&-\int V_{ge}(\varkappa,t)\rho_{eg}\left(\mathbf{r},\mathbf{p}+\frac{\hbar\varkappa}{2},t\right)e^{i\varkappa\mathbf{r}}d\varkappa-2i\gamma\rho_{ee}(\mathbf{r},\mathbf{p},t),\\
i\frac{d}{dt}\rho_{eg}(\mathbf{r},\mathbf{p},t)&=\int V_{eg}(\varkappa,t)\left[\rho_{gg}\left(\mathbf{r},\mathbf{p}-\frac{\hbar\varkappa}{2},t\right)\right.\\
&\left.-\rho_{ee}\left(\mathbf{r},\mathbf{p}+\frac{\hbar\varkappa}{2},t\right)\right]e^{i\varkappa\mathbf{r}}d\varkappa-i\gamma\rho_{eg}(\mathbf{r},\mathbf{p},t),\\
i\frac{d}{dt}\rho_{gg}(\mathbf{r},\mathbf{p},t)&=-\int V_{eg}(\varkappa,t)\rho_{ge}\left(\mathbf{r},\mathbf{p}+\frac{\hbar\varkappa}{2},t\right)e^{i\varkappa\mathbf{r}}d\varkappa\\
&+\int V_{ge}(\varkappa,t)\rho_{eg}\left(\mathbf{r},\mathbf{p}-\frac{\hbar\varkappa}{2},t\right)e^{i\varkappa\mathbf{r}}d\varkappa+2i\gamma\int d\mathbf{n}\,\Phi(\mathbf{n})\rho_{ee}(\mathbf{r},\mathbf{p}+\mathbf{n}\hbar k_0,t),
\end{aligned} \tag{14}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial\mathbf{r}}, \tag{15}$$

$\mathbf{v}=\mathbf{p}/M$ is the velocity of the atom, $k_0=\omega_0/c$ is the wave vector of the photon emitted by the atom in spontaneous decay from the state $|e\rangle$ to the state $|g\rangle$. The function $\Phi(\mathbf{n})$ determines the relative probability of spontaneous emission of a photon in the direction of the unit vector $\mathbf{n}$. For linearly polarized radiation with a polarization vector along the x axis we have[11]

$$\Phi(\mathbf{n})=\frac{3}{16\pi}[1-(\mathbf{n}\mathbf{e}_x)^2], \tag{16}$$

and for circularly polarized radiation[12]

$$\Phi(\mathbf{n})=\frac{3}{8\pi}[1+(\mathbf{n}\mathbf{e}_z)^2]. \tag{17}$$

A similar system of equations for the case of a plane traveling light wave was derived earlier.[11,14]

In (14), the order of magnitude of the wave vector $\varkappa$ depends on its direction. Along the propagation direction of the wave (4) the order of $\varkappa$ is determined by the magnitude of the wave vector $k \approx k_0$ of the light wave, in the transverse direction the order of $\varkappa$ is determined by the reciprocal of the transverse scale of the field q, with $l/q \ll k$ by virtue of the condition (8). We assume that the width Δp of the momentum distribution of the atoms along any direction exceeds the photon momentum

$$\mu = \hbar k/\Delta p \ll 1. \tag{18}$$

Under this condition, and with (8) taken into account, the ratio of the transverse momentum of the field $\hbar/q$ to the width Δp of the transverse momentum distribution will always be less than the ratio (18):

$$\eta = \frac{\hbar/q}{\Delta p_\rho} \ll \mu \ll 1. \tag{19}$$

Using the conditions (18) and (19), we can expand the elements of the density matrix $\rho_{ij}(\mathbf{r}, \mathbf{p}+\mathbf{p}', t)$ in Eqs. (14) in powers of μ and η near the point $\mathbf{p}$. Carrying out this procedure and writing down explicitly the terms of zeroth and first order in μ and η and only those second-order terms that are important for the subsequent analysis, we obtain for $\rho_{ij}(\mathbf{r}, \mathbf{p}, t) = \rho_{ij}$ the equations

$$\begin{aligned}
&i\frac{d}{dt}\rho_{ee} = V_{eg}\rho_{ge} - V_{ge}\rho_{eg} + \frac{1}{2}i\hbar\left(\frac{\partial V_{eg}}{\partial \mathbf{r}}\frac{\partial \rho_{ge}}{\partial \mathbf{p}} + \frac{\partial V_{ge}}{\partial \mathbf{r}}\frac{\partial \rho_{eg}}{\partial \mathbf{p}}\right) - 2i\gamma\rho_{ee} + \ldots,\\
&i\frac{d}{dt}\rho_{eg} = V_{eg}(\rho_{gg} - \rho_{ee}) + \frac{1}{2}i\hbar\frac{\partial V_{eg}}{\partial \mathbf{r}}\left(\frac{\partial \rho_{gg}}{\partial \mathbf{p}} + \frac{\partial \rho_{ee}}{\partial \mathbf{p}}\right) - i\gamma\rho_{eg} + \ldots,\\
&i\frac{d}{dt}\rho_{gg} = -V_{eg}\rho_{ge} + V_{ge}\rho_{eg} + \frac{1}{2}i\hbar\left(\frac{\partial V_{eg}}{\partial \mathbf{r}}\frac{\partial \rho_{ge}}{\partial \mathbf{p}} + \frac{\partial V_{ge}}{\partial \mathbf{r}}\frac{\partial \rho_{eg}}{\partial \mathbf{p}}\right)\\
&\qquad + 2i\gamma\left[\rho_{ee} + \frac{1}{2}\hbar^2 k^2 \sum_{i,j}\alpha_{ij}\frac{\partial^2 \rho_{ee}}{\partial p_i \partial p_j}\right] + \ldots,
\end{aligned} \tag{20}$$

in which the tensor α_{ij} determines the angular anisotropy of the spontaneous emission:

$$\alpha_{ij} = \int d\mathbf{n}\, \Phi(\mathbf{n})\, n_i n_j. \tag{21}$$

We substitute now the field (4) in (20), use the standard rotating-wave approximation, and replace the off-diagonal elements of the density matrix:

$$\rho_{eg} = \rho_{eg}' \exp[-i\Omega t + i(k + \rho^2/bq^2)z], \tag{22}$$

where $\Omega = \omega - \omega_0$. After these transformations, Eqs. (20) go over into equations that do not contain rapidly oscillating factors.

From the latter it is convenient to change over to equations for the Bloch variables

$$w = \rho_{gg} + \rho_{ee},\ u = \rho_{gg} - \rho_{ee},\ c = \rho_{ge}' + \rho_{eg}',\ -is = \rho_{ge}' - \rho_{eg}'. \tag{23}$$

In terms of these variables, the equations of interest to us take the form

$$\frac{dw}{dt} = -\hbar k V \frac{\partial s}{\partial p_z} + \hbar V \frac{\rho}{q^2}\frac{\partial}{\partial p_\rho}\left(c - \frac{z}{b}s\right) + \frac{1}{2}\hbar^2 k^2 \gamma \sum_{i,j}\alpha_{ij}\frac{\partial^2 (w-u)}{\partial p_i \partial p_j} + \ldots, \tag{24.1}$$

$$\frac{du}{dt} = 2\gamma(w-u) - 2Vs + \ldots, \tag{24.2}$$

$$\frac{dc}{dt} = \left(kv_z + \frac{2\rho z v_\rho}{bq^2} - \Omega\right)s - \gamma c + \hbar V \frac{\rho}{q^2}\frac{\partial w}{\partial p_\rho} + \ldots, \tag{24.3}$$

$$\frac{ds}{dt} = 2Vu - \gamma s - \hbar k V \frac{\partial w}{\partial p_z} - \left(kv_z + \frac{2\rho z v_\rho}{bq^2} - \Omega\right)c - 2\hbar V \frac{\rho}{q^2}\frac{z}{b}\frac{\partial w}{\partial p_\rho} + \ldots, \tag{24.4}$$

where $v_z = p_z/M$ and $v_\rho = p_\rho/M$.

3. KINETIC EQUATION

A distinguishing feature of the system (24) is the presence of a characteristic time

$$\tau_{int} \approx \gamma^{-1}, \tag{25}$$

that determines the change of the internal state of the atom. This time, as seen from (24), characterizes directly the variation of the functions u, c, and s. For the reasoning that follows we assume that there exists one more time $\tau_{tr} \gg \tau_{int}$, which characterizes the variation of the function $w = w(\mathbf{r}, \mathbf{p}, t)$. We shall call this the time of variation of the translational state of the atom. Under this assumption, we can distinguish between two time intervals within which the solutions (24) evolve in qualitatively different ways.

At $t \lesssim \tau_{int}$ only the change of the internal state of the atom is important. Accordingly, in this time interval the function w varies little, and the functions u, c, and c undergo rapid oscillations with a characteristic time τ_{int}, and the values of these functions depend strongly on the initial conditions. At $t \gg \tau_{int}$, the translational state of the atom changes considerably. Therefore, over long times, the functions u, c, and s should undergo, besides rapid oscillations with characteristic time τ_{int}, also changes compatible with the variation of the distribution function w.

Being interested in times $t \gg \tau_{int}$, we assume that the functions $h = u$, c, s are in the case of long times functionals of the distribution function[1)] w:

$$h(\mathbf{r}, \mathbf{p}, t) = h(\mathbf{p}; w(\mathbf{r}, \mathbf{p}, t)). \tag{26}$$

Then relation (26) enables us to write down the time derivatives (15) in the left-handed sides of (24.2)–(24.4) in the form

$$\frac{dh}{dt} = \frac{\partial h}{\partial w}\frac{dw}{dt}; \quad h = u, c, s. \tag{27}$$

We consider now Eq. (2.4) and Eqs. (24.2)–(24.4) with left-hand sides in the form (27) in different orders in the small parameters μ and η, with an aim at obtaining in closed form an equation for the distribution function $w(\mathbf{r}, \mathbf{p}, t)$ in second order in μ and η. In the zeroth approximation in μ and η, the equation for the distribution function [Eq. (24.1)] takes the form

$$dw/dt = 0. \tag{28}$$

Equations (24.2)–(24.4) in the zeroth approximation should then be written in the form

$$\begin{aligned}
&\gamma(w-u) - Vs = 0, \quad -\gamma c + (kv_z + 2\rho z v_\rho/bq^2 - \Omega)s = 0,\\
&2Vu - \gamma s - (kv_z + 2\rho z v_\rho/bq^2 - \Omega)c = 0.
\end{aligned} \tag{29}$$

Solving the latter, we can easily determine the dependences of u, c, and s on the zeroth-approximation function w.

To obtain an equation for the function w in first-order in μ and η, we note that (24.1) already contains the functions s and c in first order in μ and η. It suffices therefore to use expressions in zero order for these functions in (24.1). Obtaining s and c from (29) and substituting them in (24.1), we get an equation for w in first order in μ and η:

$$\frac{dw}{dt} = -\frac{\partial}{\partial p_z}(F_z^{(1)} w) - \frac{\partial}{\partial p_\rho}(F_\rho^{(1)} w), \tag{30}$$

where $F_z^{(1)}$ and $F_\rho^{(1)}$ are the components of the light-pressure force along the axis z and ρ, determined in first order in μ and η:

$$F_z^{(1)} = \hbar k\gamma L, \tag{31}$$

$$F_\rho^{(1)} = \hbar \frac{\rho}{q^2}\left(\Omega - kv_z - \frac{2\rho z v_\rho}{bq^2} + \frac{\gamma z}{b}\right) L. \tag{32}$$

We have introduced above the notation

$$G = 2V^2/\gamma^2, \tag{33}$$

$$L = G\left[1 + G + \left(\Omega - kv_z - \frac{2\rho z v_\rho}{bq^2}\right)^2 \Big/ \gamma^2\right]^{-1}. \tag{34}$$

From Eq. (30) we can determine the functions u, c, and s in first order in μ and η. The latter, according to (24.2)–(24.4), (27), and (30) satisfy the equations

$$\begin{aligned} \left(\frac{\partial u}{\partial w}\right)^{(0)} \left(\frac{dw}{dt}\right)^{(1)} &= 2\gamma(w-u) - 2Vs, \\ \left(\frac{\partial c}{\partial w}\right)^{(0)} \left(\frac{\partial w}{\partial t}\right)^{(1)} &= -\gamma c + \left(kv_z + \frac{2\rho z v_\rho}{bq^2} - \Omega\right) s + \hbar V \frac{\rho}{q^2}\frac{\partial w}{\partial p_\rho}, \\ \left(\frac{\partial s}{\partial w}\right)^{(0)} \left(\frac{\partial w}{\partial t}\right)^{(1)} &= -\gamma s + 2Vu - \left(kv_z + \frac{2\rho z v_\rho}{bq^2} - \Omega\right) c \\ &\quad - \hbar kV \frac{\partial w}{\partial p_z} - 2\hbar V \frac{\rho}{q^2}\frac{z}{b}\frac{\partial w}{\partial p_\rho}, \end{aligned} \tag{35}$$

where $(dw/dt)^{(1)}$ is determined by relation (30), and the derivatives $(\partial h/\partial w)^{(0)}$, where $h = u$, c, or s, should be determined from Eqs. (29).

We solve now Eqs. (35) and substitute in (24.1) the functions s and c in first order in μ and η, and the function u in zeroth order, since the latter is already contained in (24.1) in second order in μ and η. As a result we obtain for the distribution function $w(\mathbf{r}, \mathbf{p}, t)$ an equation of the Fokker-Planck type, which is exact to second order in μ and η.

Without writing down the resultant equation, we note immediately that by virtue of conditions (8) and (19) it suffices to retain in this equation terms of second order only in the parameter μ. Then, neglecting the terms proportional to $\mu\eta$ and η^2, we write down finally a Fokker-Planck equation that is accurate in second order in μ and first order in η:

$$\frac{\partial w}{\partial t} + \mathbf{v}\frac{\partial w}{\partial \mathbf{r}} = -\frac{\partial}{\partial p_z}(F_z^{(2)} w) - \frac{\partial}{\partial p_\rho}(F_\rho^{(1)} w) + \sum_{i,j} \frac{\partial^2}{\partial p_i \partial p_j}(D_{ij} w), \tag{36}$$

where the longitudinal component of the light-pressure force in second order in μ and in first order in η is

$$\begin{aligned} F_z^{(2)} = \hbar k\gamma L\Big\{ 1 + \frac{2\varepsilon L}{\gamma}\left(\Omega - kv_z - \frac{2\rho z v_\rho}{bq^2}\right) \\ \times\left[1 + \frac{1}{G}L^2\left[\frac{1}{\gamma^2}\left(\Omega - kv_z - \frac{2\rho z v_\rho}{bq^2}\right)^2 - 7 - G\right]\right]\Big\}, \end{aligned} \tag{37}$$

The transverse component of the force $F_\rho^{(1)}$ is determined by relation (32), while the components D_{ij} determine the diffusion tensor

$$D_{ij} = \tfrac{1}{2}\hbar^2 k^2 \gamma \chi_{ij} L, \tag{38}$$

$$\chi_{ij} = \alpha_{ij} + \delta_{3i}\delta_{3j}(1+d), \tag{39}$$

$$d = \left[\frac{1}{\gamma^2}\left(\Omega - kv_z - \frac{2\rho z v_\rho}{bq^2}\right)^2 - 3\right]\frac{1}{G}L^2. \tag{40}$$

We have introduced above the symbol

$$\varepsilon = kv_r/2\gamma = R/\hbar\gamma, \tag{41}$$

where $v_r = \hbar k/M$ is the recoil velocity and $R = \hbar^2 k^2/2M$ is the recoil energy.

For linearly (π) and circularly (σ) polarized radiation, only the diagonal elements of the tensor $\hat{\chi}$ differ from zero. The values of the diagonal elements χ_{xx}, χ_{yy}, and χ_{zz}, calculated from (39) and (21), are as follows:

σ:	$3/10$	$3/10$	$7/5+d$
π:	$1/5$	$2/5$	$7/5+d$

4. CONFIRMATION OF ASSUMPTIONS AND DISCUSSION OF THE EQUATION

The foregoing derivation of the kinetic equation makes it possible to determine immediately the conditions of its applicability. We stipulate that the second-approximation forces be close to those of the first approximation. Then, comparing (37) with (31), we find that the necessary condition for the proximity of the first and second approximation is

$$\varepsilon = R/\hbar\gamma \ll 1. \tag{42}$$

Thus, Eq. (36) is valid for atomic transitions whose natural line width greatly exceeds the recoil energy.

We recall now that the main assumption made in the derivation of (36) is that there exists a time $\tau_{tr} \gg \gamma^{-1}$ that characterizes the kinetic stage of the evolution of the distribution function. Taking the condition (42) into account, we easily find from (36) that this time actually exists and its order of magnitude is

$$\tau_{tr} \approx \hbar/R \gg \gamma^{-1}. \tag{43}$$

Thus, the assumption of condition (42) automatically justifies the procedure used in Sec. 3 to derive Eq. (36).

We point out finally that in the derivation of (36) we have assumed that the width of the momentum distribution exceeds the photon momentum [see (18)]. Interest attaches therefore to the question of the minimum width of the momentum distribution. This problem was solved in Ref. 16, where it was shown that the narrowing of the velocity distribution is limited in principle to the width

$$\Delta v \approx (2\hbar\gamma/M)^{1/2} = v_r \varepsilon^{-1/2}.$$

Therefore the maximum value of the parameter under the condition (42) is limited to $\mu_{max} \approx \varepsilon^{1/2} \ll 1$, so that the assumption (18) is likewise always justified.

We conclude finally that, subject to the only condition (42), all the assumptions made in the derivation of (36) are satisfied, and the coefficients in (36) should be those given by expressions (31), (32), and (38)–(40), i.e., the components of the light-pressure force should be taken in first orders in μ and η, while the diffusion tensors should be taken in second order in μ.

We discuss now the form of the diffusion tensor. One part of the diffusion tensor, as is well known, is connected with the change of the velocity of the atom on account of the recoil of the spontaneous decays. As a result of these processes the velocity of the atom varies in an ellipsoid with axes α_{ii}: $v'_i = v_{0i} + \alpha_{ii} n_i v_r$. This continuous diffusion has already been considered in Refs. 11 and 12. In addition to the continuous variation of the atom velocity, the scattering of the photons from a directional light beam causes a discrete change in the atom velocity by $\pm v_r$ along the z axis. This process is responsible for the discrete diffusion, the existence of which was indicated in Ref. 14. On the whole, the previously employed approaches[11,12,14] made it possible to determine the first two terms of (39), i.e., the continuous diffusion (α_{ij}) and the first part ($\delta_{3i}\delta_{3j}$) of the discrete diffusion.

A rigorous derivation of the equation shows that the exact expression (39) for the diffusion tensor contains one more term $\delta_{3i}\delta_{3j}d$. In contrast to the first two, the value of this term depends substantially on the intensity of the light wave, on the deviation of the light frequency from the frequency of the atomic transition, and on the velocity of the atom. The maximum value of d is reached under the condition

$$|\Omega-kv_z-2\rho z v_\rho/bq^2|=\gamma(G+7)^{1/2} \quad (44)$$

and is equal to

$$d_{max}=G/4(4+G)<1/4. \quad (45)$$

The minimum value is assumed by d under the condition

$$\Omega-kv_z-2\rho z v_\rho/bq^2=0. \quad (46)$$

It is equal to

$$d_{min}=-3G/(1+G)^2\geqslant-3/4. \quad (47)$$

On the whole, the minimum value of the discrete diffusion $(1+d)$ is limited to 0.25 [this value is reached under condition (46) and $G=1$], while the maximum of $(1+d)$ is limited to 1.25, which is reached under condition (44) and at $G=\infty$.

We indicate finally that the description of the motion of one atom or of an ensemble of atoms by a distribution function $w(r, p, t)$ is subject to a number of quantum-mechanical limitations. They can be obtained by starting from the fact that the function $w(\mathbf{r}, \mathbf{p}, t)$ is not sensitive to a change of time by an amount γ^{-1} and to a change of momentum by an amount $\hbar k$. The uncertainty in the translational energy lies in the region

$$R\ll\Delta E\ll\hbar\gamma. \quad (48)$$

The uncertainty in the momentum satisfies the relation

$$\hbar k\ll\Delta p\ll\hbar k(\hbar\gamma/R)^{1/2}. \quad (49)$$

The limit of the spatial localization of the atoms, described by Eq. (36), is Δz:

$$\lambda\!\!\!^{-}(R/\hbar\gamma)^{1/2}\ll\Delta z\ll\lambda\!\!\!^{-}. \quad (50)$$

In practice these uncertainties never limit the solutions of (36), since they are always less than the minimum possible energy $\hbar\gamma$ and the minimum momentum $\hbar k(\hbar\gamma/R)^{1/2}$.[16]

5. EVOLUTION OF ATOMIC BEAM IN A LASER FIELD

We shall use Eq. (36) to analyze the influence of the light-pressure force and of velocity diffusion on an atomic beam propagating along a laser beam. The most interesting physical features of such a problem are the monochromatization of the longitudinal velocities and the focusing (defocusing) of the beam.

Velocity monochromatization. When considering the evolution of the longitudinal velocity distribution of the atoms, we shall assume the wave to be plane and disregard the insignificant variation of the distribution function along the z axis and the diffusion of the transverse

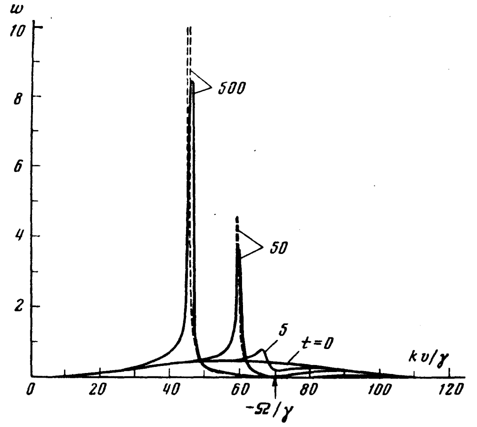

FIG. 2. Deceleration of beam of Ca^{40} atoms and narrowing of its velocity distribution by an opposing light wave resonant to the transition $4S-4P$. The dimensionless detuning is $\delta=\Omega/\gamma=-70$, and the saturation parameter is $G=10$. The solid curves represent the solutions of Eq. (51), and the dashed curves the solutions of (51) without the diffusion term.

velocities. Under these simplifications, Eq. (36) reduces to the one-dimensional diffusion equation

$$\frac{\partial w}{\partial t}=-\frac{\partial}{\partial v}(Lw)+\varepsilon(1+\alpha_{zz}+d)\frac{\partial^2}{\partial v^2}(Lw). \quad (51)$$

We write down the latter in terms of dimensionless variables, choosing the units of time and velocity to be $(kv_r)^{-1}$ and γ/k.

The solution (51) has different qualitative behavior at short and long times. During the initial stage of the evolution, besides the change of its average rate, the distribution becomes narrower,[16] owing to the monotonic decrease of the force when the atomic velocity deviates from the resonant velocity $v_r=\delta$ ($\delta=\Omega/\gamma$). This feature is clearly seen in Fig. 2, which shows the time evolution of w for an atomic beam irradiated by an opposing light wave. During this stage of the evolution, the diffusion exerts a weak influence on the width of the distribution (see the solid and dashed curves in Fig. 2).

Over long times, allowance for diffusion is essential, and the question of fundamental interest is that of the asymptotic form of the distribution $w(v)$. To obtain the answer, we recognize that in this case one should take a long time to mean one in which the average velocity v_0 of the distribution is far from the resonant one $(v-\delta)\gg(1+G)^{1/2}$. Then, assuming the distribution to be narrow enough, we expand $L(v)$ in powers of $u=v-v_0$ about the mean velocity v_0. Confining ourselves further in (51) to those expansion terms that take into account the change of the average velocity and of the distribution width, we write down an asymptotic form of (51):

$$\frac{\partial w}{\partial t}+L_0\frac{\partial w}{\partial u}=\varepsilon(1+\alpha_{zz})L_0\left(\frac{\partial^2 w}{\partial u^2}+\frac{1}{T_0}\frac{\partial(uw)}{\partial u}\right) \quad (52)$$

and the equations for the first two moments of the distribution ($v_0=\langle v\rangle$, $T=\langle u^2\rangle$):

$$\frac{\partial v_0}{\partial t}=L_0, \quad (53)$$

$$\frac{\partial T}{\partial t}=2\varepsilon(1+\alpha_{zz})L_0(1-T/T_0), \quad (54)$$

where

$$L_0=G/(\delta-v_0)^2,\quad T_0={}^1/_2\varepsilon(1+\alpha_{zz})(v_0-\delta).$$

Recognizing now that the time dependence of w is functional:

$$w(v,t)=w(u;v_0(t)),\tag{55}$$

we calculate the derivative with respect to time in (52)

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial v_0}\frac{\partial v_0}{\partial t}=L_0\frac{\partial w}{\partial v_0}\tag{56}$$

and proceed to solve (52) by successive approximations. The latter allows us to represent a solution in the form of a series

$$w=w^{(0)}+w^{(1)}+\ldots,\tag{57}$$

whose terms take successively into account the change of the average velocity v_0 with time.

To obtain the zeroth-approximation function, we set the left-hand side of (52) equal to zero and, solving (52), obtain

$$w^{(0)}=(2\pi T_0)^{-1/2}\exp(-u^2/2T_0).\tag{58}$$

We now substitute (58) in the left-hand side of (52) and retain the normalization of the function w relative to the function $w^{(0)}$. Then, solving (52), we determine $w^{(1)}$:

$$w^{(1)}={}^1/_8w^{(0)}(1-u^2/T_0).\tag{59}$$

Continuing the calculations, we can successively determine the next terms of the expansion (57).

Thus, the exact asymptotic distribution $w(v)$ is determined by the series (57). We indicate in connection with this conclusion that in Ref. 16 the function $w^{(0)}$ was suggested as the asymptotic form of $w(v)$. It is easily seen, however, that cutting off the series (57) after the first term leads to the equality $T=T_0$ and contradicts (54). To determine the true asymptotic temperature it is necessary to recognize that, according to (57), T is proportional to T_0. We then find from (54) that in the case of long times we have $T=(4/5)T_0$. The last result makes it also possible to indicate that formally (57) is a series expansion in powers of the parameter $|T-T_0|/T_0=1/5$.

Focusing of the beam. To analyze the evolution of the transverse distribution of the atomic beam, we shall disregard the variation of the width of its longitudinal velocity distribution, assuming the latter to be a δ function. In addition, by virtue of the cylindrical symmetry of the problem, we consider only the one-dimensional transverse motion of the atoms, e.g., along the x axis. The corresponding equation for w is obtained from (36). In dimensionless variables, it takes the form

$$\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial z}+u\frac{\partial w}{\partial k}+L\frac{\partial w}{\partial v}-2\varepsilon\frac{(v_0-\delta)L}{s^2}x\frac{\partial w}{\partial u}=\varepsilon(1+\alpha_{zz})L\frac{\partial^2 w}{\partial u^2},\tag{60}$$

where u denotes the transverse velocity along the x axis, and $s=2qk$ is the dimensionless radius of the light beam.

We note immediately that in (60) the oscillations of the transverse velocity (and coordinate) distribution are due only to the action of the harmonic force, and are smoothed out by diffusion in the course of time. In this connection, from the point of view of focusing of a beam, principal interest attaches to short times, for which the width of the velocity distribution Δu exceeds the diffusion broadening. We therefore drop the right-hand side of (60) and write down the characteristics of the obtained linear equation

$$dt=\frac{dz}{v}=\frac{dx}{u}=\frac{dv}{L}=-\frac{s^2}{2\varepsilon(v_0-\delta)L}\frac{du}{x}.\tag{61}$$

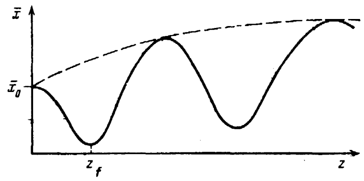

FIG. 3. Oscillations of transverse dimensions of atomic beam in the field of laser radiation of fundamental mode TEM_{00q}.

The solution of the nonlinear system (61) entails a number of serious difficulties. We therefore simplify the problem, but bear nevertheless in mind the situation of practical importance. We assume that $|x|\ll q$, and then L becomes a function of the longitudinal velocity only. We assume the velocity v to be limited by the conditions $(1+G)^{1/2}\ll v-\delta\ll\delta$. After these simplifcations, Eqs. (61) can be solved by the method of slowly varying amplitudes. Under the initial conditions

$$z(0)=0,\quad v(0)=v_0,\quad x(0)=x_0,\quad u(0)=0,$$

which correspond to an unfocused beam, the dependence of the transverse coordinate x on the longitudinal z is represented in parametric form

$$x=x_0\left(\frac{v-\delta}{v_0-\delta}\right)^{1/4}\cos\left(\frac{2\varepsilon G}{v-\delta}\right)^{1/2}\frac{z}{s\delta},\tag{62}$$

$$z=\frac{\delta}{3G}[(v-\delta)^3-(v_0-\delta)^3].\tag{63}$$

The obtained solution shows that in the course of propagation the transverse dimension of the atomic beam oscillates with decreasing frequency, and the envelope of the amplitude of the oscillations increases nonlinearly (Fig. 3). Under the condition of practical importance

$$(v_0-\delta)^{3/2}\gg{}^1/_2\pi(G/2\varepsilon)^{1/2}s,$$

which means that during the focusing time the longitudinal velocity of the beam changes little $(v-v_0\ll v_0-\delta)$, we can obtain from (62) the position of the first focus

$$z_f=\frac{\pi}{2}\delta s\left(\frac{v_0-\delta}{2\varepsilon G}\right)^{1/2}.\tag{64}$$

For the beam of Ca^{40} atoms considered above, at a transverse light-beam dimension $q=10^{-2}$ cm, an initial velocity $v_0=7\times10^4$ cm/sec, and a detuning $\delta=50$, the focusing length is $z_f=30$ cm.

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[1)] To become aquainted with the details of the method see, e.g., the monograph of Akhiezer and Peletminskiĭ.[15]

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Propagation of an electromagnetic field in noninertially moving optically active media

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The propagation of electromagnetic waves in an optically active medium at rest in a rotating coordinate frame is investigated on the basis of the method of local Lorentz transformations. The results are used to determine the splitting of the natural frequencies of a ring resonator rotating together with the medium. The theoretical-analysis data agree well with the results of the performed experiment.

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1. To determine the character of propagation of electromagnetic waves in an arbitrary medium that is at rest in a noninertial reference frame, it is necessary to solve Maxwell's equations, which in covariant form are given by[1]

$$\nabla_\alpha H^{\mu\alpha}=0; \quad \nabla_{[\lambda}F_{\mu\nu]}=0, \tag{1}$$

where $H^{\mu\alpha}$ and $F_{\mu\alpha}$ are bivectors of the electromagnetic field, ∇_α is the covariant derivative, and a correspondence is established between the components of the tensors $H^{\mu\alpha}$, $F_{\mu\alpha}$ and the three-dimensional vectors in accordance with the rule

$$H^{\mu\nu}=\{H^{0i};\ H^{mn}\}=\{(-g)^{1/2}\mathbf{D};\ (-g)^{1/2}\mathbf{H}\},$$
$$F_{\mu\nu}=\{F_{i0};\ F_{mn}\}=\{\mathbf{E};\ \mathbf{B}\}, \tag{2}$$

which transforms (1) to the usual three-dimensional form.[1]

The system (1) must be supplemented by the equations that connect the bivectors $H^{\mu\alpha}$ and $F_{\mu\alpha}$—the material equations. To establish the covariant form of this connection in noninertially moving media one used customarily generalizations of various schemes for the construction of the material equations of inertially moving media.[2] However, even in the first few theoretical papers the use of different versions of generalizations of the electrodynamics of moving isotropic media, within the framework of special relativity theory, to include the case of general relativity theory, has led to discrepancies in the main results.[3-5] It has been shown[6-7] that these discrepancies are due to the erroneous assumptions[5] made in the construction of the material tensor $\varepsilon^{\mu\nu\alpha\beta}$ that determines the coupling equations. This was confirmed in an experimental determination[8] of the influence of the refractive index n of an isotropic medium on the splitting $\Delta\nu$ of two opposing waves of a rotating laser with a ring resonator (RR). Volkov and Kiselev[9], within the framework of a method developed by them,[5] determined the splitting $\Delta\nu$ for the case when, in the proper system of the rotating resonator, the permittivity and the permeability of the medium are specified by gyrotensors $\hat{\varepsilon}$ and $\hat{\mu}$. The result in Ref. 9 differs from an analogous result obtained in their own paper[10] with qualitative account taken of the gyrotropy of the medium.

Thus, the available theoretical data do not make it possible to determine rigorously the character of the propagation of the electromagnetic field in rotating optically active media, and the solution of this problem calls for a detailed theoretical and experimental study.

In the present paper we construct the material equations and solve subsequently the propagation problem on the basis of the method of local Lorentz transforma-

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