When this condition is satisfied we can in Eq. (9) put approximately

$$\omega_k+\omega_{k_1}=\omega_{k_2}+\omega_{k+k_1-k_2}.$$

Changing to the one-dimensional case we write this formula in the form

$$\frac{d^d|E_k|^2}{dt}=\left(\frac{4\pi e^4}{km^3\partial\varepsilon/\partial\omega_k}\right)^2\frac{2}{\gamma}\int dv\,dv'\,\frac{1}{kv-\omega_k}\left(\frac{1}{kv'-\omega_k}\right)^*\frac{\partial}{\partial v}\frac{\partial}{\partial v'}$$
$$\times\frac{1}{2\pi}\int dk_1\frac{|E_{k_1}|^2}{(k+k_1)v-\omega_k-\omega_{k_1}}\left(\frac{1}{(k+k_1)v'-\omega_k-\omega_{k_1}}\right)^*\frac{\partial}{\partial v}\frac{\partial}{\partial v'}$$
$$\times\frac{1}{2\pi}\int dk_2|E_{k_2}|^2|E_{k+k_1-k_2}|^2\frac{1}{k_2v-\omega_{k_2}}\left(\frac{1}{k_2v'-\omega_{k_2}}\right)^*\frac{\partial f_0}{\partial v}\frac{\partial f_0}{\partial v'}. \quad (12)$$

An order of magnitude estimate of the integral on the right-hand side of (12) gives

$$\left(\frac{4\pi e^4}{m^3k\partial\varepsilon/\partial\omega_k}\right)^2\left(\frac{\partial f_0}{\partial v}\right)^2\frac{k^2|E_k|^6}{\gamma_k v^2\delta_k{}^6}\sim\gamma_L|E_k|^2\frac{e^4k^4|E_k|^4}{m^4v^2\delta_k{}^6}. \quad (13)$$

Substituting $\delta_k\sim(k^2D_k)^{1/3}$ we find that the non-linear correction is of the order of the quasi-linear growth rate. The authors of Ref. 7 restricted themselves to just such an estimate and they based upon that estimate the incorrect conclusion that it is necessary to take the non-linear corrections to the quasi-linear theory into account. In fact, the integrals on the right-hand side of Eq. (12) can be evaluated exactly without particular difficulties. Putting approximately $|E_{k_i}|^2\approx|E_k|^2$ and using the residue theorem to evaluate the integrals over k_1 and k_2 (the poles of the integrand are on both sides of the real k_1, k_2 axis) we get the following result:

$$\frac{d^d|E_k|^2}{dt}=\left(\frac{8\pi^2e^4}{m^3k\partial\varepsilon/\partial\omega_k}\right)^2\frac{1}{\gamma}\int dv\,dv'\,\frac{1}{kv-\omega_k-i\delta_k}$$
$$\times\frac{1}{\omega_k-i\delta_k-kv'}\frac{\partial f_0}{\partial v}\frac{\partial f_0}{\partial v'}\left(\frac{\partial}{\partial v}\frac{\partial}{\partial v'}\frac{1}{\omega_p(v'-v)+2i\delta_kv}\right)^2. \quad (14)$$

In each of the integrals over v and v' all singularities of the integrand are on one side of the real axis (in the upper half-plane in the integral over v and in the lower half-plane in the integral over v'). Replacing approximately the derivative $\partial f_0/\partial v$ by its value for $v=\omega_k/k$ one shows easily that the integrals over v and v' do in fact not contain the higher-order singularities which lead to the estimate (13). The non-linear corrections to the growth rate (12) are thus small compared to the quasi-linear value. This conclusion is obtained for four-plasmon decays but a completely analogous proof can be given also for higher-order decays.

When conditions (1) and (2) which were first formulated in Ref. 1 as the conditions for the applicability of the quasi-linear theory are satisfied the non-linear corrections to the quasi-linear equations turn out to be indeed small. In the present paper we showed that the non-linear corrections to the growth rate are small; however, it follows from the energy conservation law that in that case the non-linear corrections to the diffusion coefficient are also small.

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Dynamo of small-scale fields

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An equation for the dynamics of small-scale magnetic fields is derived for a non-Markov model by using the Lagrangian statistical characteristics of turbulence. It is shown that the behavior of the fields is closely connected with the correlation characteristics of the scalar admixture. In the case of extremely low magnetic viscosity $D\ll\nu$ (ν is the kinematic viscosity), the dynamics of the fields is described at large wave numbers by a universal equation. It is shown in this case that a dynamo solution, i.e., a solution that increases without limit, exists. The problem of the dynamo of a small scale field is thus solved for the case $D\ll\nu$.

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While the question of the dynamics of large-scale magnetic fields in a turbulent medium is now regarded as solved in the main outline, the problem of pulsation fields remains open despite its large urgency. Indeed, it is regarded as established that turbulence leads to the onset of turbulent diffusion and to generation (in the case of reflection non-invariance) of a large-scale field.[1,2] The question lies here only in the accuracy of

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the obtained diffusion and generation coefficients,[3,4] and the absence of a small parameter raises great difficulties in the quantitative calculations.

As for small-scale fields, no success was reached so far in strictly analyzing the principal question: does a turbulent medium produce a small-scale field dynamo or not? The first attempt was made by Batchelor,[5] and was the cause of much discussion and criticism (mainly on the part of Zel'dovich[6]). Batchelor used the analogy between a magnetic field and a vortex in a liquid. A solution of this problem was presented in the Kraichnan direct-interaction approximation,[7] in the Markov model,[8] and for acoustic turbulence.[9]

The difficulty of the problem is that turbulence leads to two competing effects: the transfer of the small-scale field energy into a region of ever smaller scales, and the amplification of the small-scale fields; the rates of these processes are of the same order of magnitude. Quantitative calculations therefore play the decisive role in the question of whether a small-scale-field dynamo can exist or not, in contrast to the question of the dynamics of large-scale fields, where the quantitative calculations determine only the velocity of the dynamo. The existing actual models[8,9] make use of the small parameter $\tau v/l$, where τ is the correlation times, l is the correlation length, and v is the mean squared velocity. The smallness of the correlation time corresponds to a white-noise process for the velocity field and to a Markov process for the magnetic field. In this case one uses in fact the perturbation-theory series in the form of an expansion in the parameter $\tau v/l$, and by the same token the rates of the aforementioned competing processes are determined by regular methods. Disregarding acoustic turbulence, which is not as frequently encountered in nature, the case $\tau v/l \ll 1$ is not encountered in applications. It is even difficult to imagine how to produce a turbulence with such exotic properties. This is precisely why the question of the small-scale-field dynamo has remained open to this day.

The situation changes radically after the publication of Kraichnan's paper.[10] He made use of a Lagrangian description of the turbulence. It turns out that in this approach there is no need for the small parameter $\tau v/l$, but at the same time, when considering the dynamics of a large-scale field (with scale L) Kraichnan uses a small parameter l/L. Of course, this small parameter does not appear in the dynamics of small-scale fields. Following the construction of an exact model in which there was no need for the small parameter l/L (Ref. 11)(a parameter, incidentally, which is realistic when it comes to large-scale fields), it became clear that the dynamics of small-scale fields can be explained by regular methods with the aid of a Lagrangian description. This in fact is the subject of the present article.

§1. DESCRIPTION OF SMALL-SCALE FIELDS BY A LAGRANGIAN APPROACH

We note first that in the problem of the kinematic dynamo (to the analysis of which we confine ourselves) the velocity field is assumed given and it is required to determine the dynamics of the magnetic field. It is quite immaterial whether the velocity field is specified in the form of a characteristic functional, correlation tensors, or Lagrangian characteristics. Equations in Euler form admit of lucid interpretation but it is well known that to solve problems with large magnetic Reynolds numbers it becomes necessary to use a perturbation-theory series. This poses immediately the problem of the small parameter, which does not exist in the case of a real turbulence with $\tau v/l \approx 1$. We shall specify Lagrangian characteristics. Then, as made clear in Ref. 11, the dynamics of the large-scale fields are determined in exact form.

To describe the dynamics of small-scale fields, we use the exact solution for ideal magnetohydrodynamics of an incompressible liquid for a magnetic field $\mathbf{H}$:

$$H_i(\mathbf{x}, t) = \frac{\partial x_i}{\partial a_m} H_m(\mathbf{a}), \qquad (1)$$

$\mathbf{x}$ is the coordinate of a liquid particle leaving a point $\mathbf{a}$ at the instant $t=0$, i.e., $\mathbf{x}=\mathbf{a}$ at $t=0$, and $H_m(\mathbf{a})$ is the initial magnetic field. We replace $\partial x_i/\partial a_m$ by

$$\frac{\partial x_i}{\partial a_m} = \lim_{{}^2\mathbf{a}\to{}^1\mathbf{a}} \frac{{}^2x_i - {}^1x_i}{{}^2a_m - {}^1a_m},$$

${}^1\mathbf{x}$ and ${}^2\mathbf{x}$ are the coordinates of the particles ${}^1\mathbf{a}$ and ${}^2\mathbf{a}$. Then

$$H_i({}^1\mathbf{x}, t) H_j({}^3\mathbf{x}, t) = \lim_{\substack{{}^2\mathbf{a}\to{}^1\mathbf{a} \\ {}^4\mathbf{a}\to{}^3\mathbf{a}}} \frac{{}^2x_i - {}^1x_i}{{}^2a_m - {}^1a_m} \frac{{}^4x_j - {}^3x_j}{{}^4a_n - {}^3a_n} H_m({}^1\mathbf{a}) H_n({}^3\mathbf{a}). \qquad (2)$$

Here ${}^3\mathbf{x} = {}^3\mathbf{a}$ and ${}^4\mathbf{x} = {}^4\mathbf{a}$ at $t=0$.

We introduce a four-point distribution function (see Ref. 12, part I):

$$p({}^\alpha\mathbf{x} | {}^\beta\mathbf{a}, t) = p({}^1\mathbf{x}, {}^2\mathbf{x}, {}^3\mathbf{x}, {}^4\mathbf{x} | {}^1\mathbf{a}, {}^2\mathbf{a}, {}^3\mathbf{a}, {}^4\mathbf{a}, t)$$

is the probability density of finding the liquid particles at the points ${}^\alpha\mathbf{x}$ if at $t=0$ there were located at the points ${}^\alpha\mathbf{a}$ (the Greek superscripts run through the values 1, 2, 3, and 4, and the Latin superscripts through 1, 2, 3). We multiply expression (2) by $p({}^\alpha\mathbf{x} | {}^\beta\mathbf{a}, t)$ from the right and from the left, integrate with respect to $d^1\mathbf{a}d^3\mathbf{a}d^2\mathbf{x}d^4\mathbf{x}$, and average over the initial distribution of the magnetic field. We then obtain

$$\begin{aligned} \langle H_i({}^1\mathbf{x}, t) H_j({}^3\mathbf{x}, t)\rangle &= B_{ij}({}^3\mathbf{x} - {}^1\mathbf{x}, t) \\ &= \lim_{\substack{{}^2\mathbf{a}\to{}^1\mathbf{a} \\ {}^4\mathbf{a}\to{}^3\mathbf{a}}} \int d^1\mathbf{a}d^3\mathbf{a}d^2\mathbf{x}d^4\mathbf{x} \frac{{}^2x_i - {}^1x_i}{{}^2a_m - {}^1a_m} \frac{{}^4x_j - {}^3x_j}{{}^4a_n - {}^3a_n} \\ &\quad \times p({}^\alpha\mathbf{x} | {}^\beta\mathbf{a}, t) B_{mn}({}^3\mathbf{a} - {}^1\mathbf{a}, 0) \\ &= \int d^1\mathbf{a}d^3\mathbf{a} T^{ij}_{mn}({}^1\mathbf{x}, {}^3\mathbf{x}) | {}^1\mathbf{a}, {}^3\mathbf{a}, t) B_{mn}({}^3\mathbf{a} - {}^1\mathbf{a}, 0). \end{aligned} \qquad (3)$$

Equation (3) was derived assuming statistical homogeneity, therefore the correlation tensor B_{ij} depends only on the coordinate difference. Integration with respect to $d^1\mathbf{a}\, d^3\mathbf{a}$ corresponds to averaging over the initial position of the particles, and by integrating with respect to $d^2\mathbf{x}d^4\mathbf{x}$ we arrive at averaging of the tensor $(\partial^1x_i/\partial^1a_m)(\partial^3x_j/\partial^3a_n)$. Finally, expressing the magnetic fields in the right-hand side of (3) in the form of a correlation tensor corresponds to averaging over the initial distribution of the field.

In the right-hand side of (3), the hydrodynamic characteristics (the tensor $\hat{T}$) and the magnetic-field characteristics (the tensor $\hat{B}$) are separated and are represented in the form of a product. This corresponds to assuming statistical independence of the initial field and of the hydrodynamic characteristics in succeeding instants of time. Expression (3) must therefore be considered at $t > \tau$, where τ is the "memory time," when the system forgets the initial data and a definite correlation is established between the hydrodynamic characteristics and the magnetic field. Physically, τ is the correlation time of the velocity field. Homogeneity of the turbulence corresponds to invariance of the tensor $\hat{T}$ to the shift

$$\hat{T}({}^1\mathbf{x}, {}^3\mathbf{x} \mid {}^1\mathbf{a}, {}^3\mathbf{a}) = \hat{T}({}^1\mathbf{x}-\mathbf{a}, {}^3\mathbf{x}-\mathbf{a} \mid {}^1\mathbf{a}-\mathbf{a}, {}^3\mathbf{a}-\mathbf{a}).$$

From this follows, as can be easily seen, invariance of the right-hand side of (3), as a function of ${}^1\mathbf{x}$ and ${}^3\mathbf{x}$, to the shift. This is precisely why the left-hand side of (3) depends only on ${}^3\mathbf{x} - {}^1\mathbf{x}$, i.e., the homogeneity of the field at $t > 0$ follows automatically from the homogeneity of the initial distribution of the field.

In a large-scale-field investigation,[11] a two-point distribution function was introduced, its explicit form was specified, and expressions of the type (3) were calculated in final form. In the present case it is practically impossible to proceed in this manner: it would be necessary to specify a four-point distribution function of utterly unknown form. It turns out that it is simpler to specify not the distribution function itself, but an equation for it. To introduce such an equation it is convenient, from the methodological point of view, to use the properties of a scalar admixture with density $\vartheta(\mathbf{x}, t)$ in the turbulent medium. It is known that, in Lagrangian coordinates, $\vartheta(\mathbf{x}, t) = \vartheta(\mathbf{a}, 0)$, so that in place of (3) we have

$$\begin{gathered}\Theta({}^\alpha\mathbf{x}, t) = \int p({}^\alpha\mathbf{x} \mid {}^\beta\mathbf{a}, t)\, \Theta({}^\gamma\mathbf{a}, 0)\, d^\delta\mathbf{a},\\ \Theta({}^\alpha\mathbf{x}, t) = \langle \vartheta({}^1\mathbf{x}, t)\, \vartheta({}^2\mathbf{x}, t)\, \vartheta({}^3\mathbf{x}, t)\, \vartheta({}^4\mathbf{x}, t) \rangle,\\ d^\delta\mathbf{a} = d^1\mathbf{a}\, d^2\mathbf{a}\, d^3\mathbf{a}\, d^4\mathbf{a}.\end{gathered} \tag{4}$$

Since the equation for ϑ is linear and the dynamics of ϑ is determined (in the case of homogeneous turbulence) only by the initial data, the equation for Θ takes the following general form:

$$\partial\Theta({}^\alpha\mathbf{x}, t)/\partial t = \int Q({}^\alpha\mathbf{x} \mid {}^\beta\mathbf{y})\, \Theta({}^\gamma\mathbf{y}, t)\, d^\delta\mathbf{y}, \tag{5}$$

and Q is independent of time by virtue of the stationary character of the turbulence. Taking the derivative of (4) with respect to time, we verify that Eq. (5) is obtained if the equation for p is of the form

$$\partial p({}^\alpha\mathbf{x} \mid {}^\beta\mathbf{a}, t)/\partial t = \int Q({}^\alpha\mathbf{x} \mid {}^\gamma\mathbf{y})\, p({}^\varepsilon\mathbf{y} \mid {}^\beta\mathbf{a}, t)\, d^\delta\mathbf{y}. \tag{6}$$

It is interesting to note that the Lagrangian approach is used here when the distribution function is introduced. Ultimately, however, the result must be obtained in Euler coordinates, since both Θ in (4), (5) and B_{ij} in (3) are ordinary correlation tensors. Such a mixed approach was developed by Kraichnan[10] for large-scale fields.

§2. CHOICE OF MODEL. DYNAMICS OF LARGE-SCALE FIELD IN THIS MODEL

We assume the turbulence to be isotropic, invariant under reflection, and homogeneous (the latter property was already taken into account in §1). We use the following version of the kinetic equation (6):

$$\begin{gathered}\partial p/\partial t = T_{ij}({}^\alpha\mathbf{x}, {}^\beta\mathbf{x})\, {}^\alpha\partial_i\, {}^\beta\partial_j\, p = {}^\alpha\partial_i\, {}^\beta\partial_j\, T_{ij}({}^\alpha\mathbf{x}, {}^\beta\mathbf{x})\, p,\\ T_{ij}({}^\alpha\mathbf{x}, {}^\beta\mathbf{x}) = (\delta_{ij}\Delta - \partial_i\partial_j)\, u(|{}^\alpha\mathbf{x} - {}^\beta\mathbf{x}|) = \langle u_i({}^\alpha\mathbf{x})\, u_j({}^\beta\mathbf{x}) \rangle\end{gathered} \tag{7}$$

(summation is implied over both Greek and Latin repeated indices). From the definition (7) of the tensor T_{ij} it is seen that T_{ij} has the properties of the correlation tensor of a certain homogeneous isotropic solenoidal field, $\operatorname{div}\mathbf{u} = 0$. The derivatives in the expression for T_{ij} are taken with respect to one of the variables in $\boldsymbol{u}$.

We assume that Eq. (7) describes correctly the hydrodynamic characteristics at least $t \gg \tau$. The kinetic equation (7) recalls the diffusion (Fokker-Planck) approximation and is actually obtained for a rapid process $\tau v/l \gg 1$. To verify this, it suffices to expand the distribution function in a Taylor series in the displacements

$$\Delta x_i = \int_0^t v_i(t_1, \mathbf{x})\, dt_1.$$

A nonzero contribution is obtained only in second order (i.e., from the second derivatives $\partial_i\partial_j$). The higher orders are insignificant in the $\tau v/l \to 0$ approximation. The principal difference between (7) and the Fokker-Planck equation is that (7) corresponds to a process that is stationary in the statistical sense, whereas the diffusion approximation describes a relaxation of a distribution function. The evolution of the probability density p in (7) describes, roughly speaking, the dispersal of liquid particles that were initially close to one another—a process which by itself acts continuously.

A kinetic equation in the form (7) contains no information on the parameter $\tau v/l$: the tensor T_{ij} has the dimensionality of the diffusion coefficient, we have also the correlation length of the function u, and these two parameters are insufficient to determine the parameter $\tau v/l$. In the particular case $\tau v/l \ll 1$, the tensor T_{ij} is defined as

$$T_{ij}({}^\alpha\mathbf{x}, {}^\beta\mathbf{x}) = {}^1/{}_2 \int_{-\infty}^{+\infty} \langle v_i({}^\alpha\mathbf{x}, t)\, v_j({}^\beta\mathbf{x}, t+s) \rangle\, ds;$$

here $\mathbf{v}$ is the Euler velocity. In the general case, however, the parameter $\tau v/l$ need not be small, and then T_{ij} cannot be expressed in simple fashion in terms of the velocity.

The distribution function p is subject to rather stringent requirements. In particular, when two points coincide, p must vanish; e.g., if ${}^1\mathbf{x} = {}^2\mathbf{x}$, then $p = 0$, with the exception of the degenerate case ${}^1\mathbf{a} = {}^2\mathbf{a}$. Indeed, two liquid particles cannot land in the same point, unless they coincided at the initial instant (i.e., they were in fact not two liquid particles but one). Equation (7) does not contradict this property. This can be verified in the following manner. If the function p has this property at the instant $t = t_1$, then the right-hand side of (7) vanishes at ${}^1\mathbf{x} = {}^2\mathbf{x}$, meaning that $p = 0$ at the next instant. Indeed, if $p({}^1\mathbf{x}={}^2\mathbf{x}, {}^3\mathbf{x}, {}^4\mathbf{x})=0$ then in the vicinity of the point ${}^1\mathbf{x}={}^2\mathbf{x}$ the function p is approximated by the positive-definite form $p\sim a_{ij}({}^1x_i-{}^2x_i)({}^1x_j-{}^2x_j)$; a_{ij} depends, generally speaking, on all the coordinates. It is then immediately clear that all the terms of the right-hand side of (7), which contain the derivatives ${}^3\partial_i$ and ${}^4\partial_j$, vanish at ${}^1\mathbf{x}={}^2\mathbf{x}$. The remaining terms also vanish; this can be easily verified by using the fact that in this case

$$T_{ij}({}^1\mathbf{x}, {}^2\mathbf{x})=T_{ij}({}^2\mathbf{x}, {}^1\mathbf{x})=T_{ij}({}^1\mathbf{x}, {}^1\mathbf{x})=T_{ij}({}^2\mathbf{x}, {}^2\mathbf{x}).$$

The most important property of the distribution function p is that it is positive-definite. Equation (7) should not lead to the appearance of negative sections of the function p. That this requirement is indeed satisfied can be verified in the following manner. Assume that $p\geqslant 0$ at $t=t_1$. In order to become negative, this function must go through zero. If p vanishes at the point ${}^\alpha\mathbf{x}^{(0)}$ at $t=t_1$, then p is approximated in the vicinity of this point by the positive-definite form

$${}^{\alpha\beta}b_{ij}({}^\alpha x_i-{}^\alpha x_i^{(0)})({}^\beta x_j-{}^\beta x_j^{(0)}).$$

We represent the tensor $\hat{b}$ in the form ${}^{\alpha\beta}b_{ij}={}^\alpha b_i{}^\beta b_j^*$, and then the right-hand side of (7) is written in the positive definite form

$$2{}^\alpha b_i{}^\beta b_j^*\langle u_i({}^\alpha\mathbf{x}^{(0)})u_j({}^\beta\mathbf{x}^{(0)})\rangle.$$

Consequently, at the point ${}^\alpha\mathbf{x}={}^\alpha\mathbf{x}^{(0)}$ the derivative $\partial p/\partial t=0$ and the distribution function cannot become negative.

We have illustrated here the fact that Eq. (7) does not contradict the most important properties of a distribution function. In principle, it is necessary to verify the satisfaction of all the properties. This is, however, difficult: we have previously[11] listed eight requirements imposed on a two-point distribution function. The number of requirements will be much larger for a four-point function. At the same time it is clear that there is no need to verify the satisfaction of all the conditions. The point is that, as mentioned above, Eq. (7) can be obtained for a real random process with the property $\tau v/l\ll 1$, yet there is no information on the parameter $\tau v/l$ in (7). Consequently, Eq. (7) satisfies the necessary requirements automatically.

To clarify the dynamics of large-scale fields, let us dwell on the two-point distribution function obtained when (7) is integrated with respect to $d^3\mathbf{x}d^4\mathbf{x}$ over all of space:

$$\begin{gathered}\partial p/\partial t=T({}^1\Delta+{}^2\Delta)p+2T_{ij}({}^1\mathbf{x}, {}^2\mathbf{x}){}^1\partial_i{}^2\partial_j p,\\ T_{ij}({}^1\mathbf{x}, {}^1\mathbf{x})=T_{ij}({}^2\mathbf{x}, {}^2\mathbf{x})=\delta_{ij}T.\end{gathered}\tag{8}$$

It can be verified that the function p defined by Eq. (8) satisfies the eight requirements cited in the preceding paper.[11]

In addition, in analogy with (4)–(6), we have

$$\begin{gathered}\partial f/\partial t=2T\Delta f-2T_{ij}({}^1\mathbf{x}, {}^2\mathbf{x})\partial_i\partial_j f,\\ f=\langle\vartheta({}^1\mathbf{x}, t)\vartheta({}^2\mathbf{x}, t)\rangle;\end{gathered}\tag{9}$$

where the derivatives are taken with respect to the variable $\mathbf{r}={}^2\mathbf{x}-{}^1\mathbf{x}$. This is the equation for the correlation function $f(r)$ of the pulsations of the scalar admixture. That $f(r)$ possesses the required properties can be verified by changing to the Fourier transform of (9), i.e., to an equation for the spectral function. The latter turns out to be positive-definite. The vanishing of the right-hand side at ${}^1\mathbf{x}={}^2\mathbf{x}$ corresponds to conservation of the quantity $\langle\vartheta^2\rangle$—a property obtainable from the Euler equation for the scalar admixture.

The model of the velocity field (8) does not agree, generally speaking, with the previously considered model,[11] so that it is sensible to obtain an equation for the large-scale component $\langle\mathbf{H}\rangle$ of the field. In place of (3) we have

$$\langle H_i({}^1\mathbf{x}, t)\rangle=\lim_{{}^2\mathbf{a}\to{}^1\mathbf{a}}\int d^1\mathbf{a}d^2\mathbf{x}\frac{{}^2x_i-{}^1x_i}{{}^2a_m-{}^1a_m}p({}^1\mathbf{x}, {}^2\mathbf{x}|{}^1\mathbf{a}, {}^2\mathbf{a}, t)\langle H_m({}^1\mathbf{a}, 0)\rangle.\tag{10}$$

We shall use below the property

$$\lim_{{}^2\mathbf{a}\to{}^1\mathbf{a}}\int d^2\mathbf{x}\frac{({}^2x_j-{}^1x_j)({}^2x_i-{}^1x_i)}{{}^2a_m-{}^1a_m}p({}^1\mathbf{x}, {}^2\mathbf{x}|{}^1\mathbf{a}, {}^2\mathbf{a}, t)=0,\tag{11}$$

which follows from the fact that p behaves like $p_1({}^1\mathbf{x}, {}^2\mathbf{x})\delta({}^1\mathbf{x}-{}^2\mathbf{x})$ as ${}^2\mathbf{a}\to{}^1\mathbf{a}$, and p_1 is a slowly varying function without singularities. Replacing the δ function by any of its approximations that include the parameter ${}^2\mathbf{a}-{}^1\mathbf{a}$, e.g.,

$$\delta=[\pi({}^2\mathbf{a}-{}^1\mathbf{a})^2]^{-3/2}\exp[-({}^2\mathbf{x}-{}^1\mathbf{x})^2/({}^2\mathbf{a}-{}^1\mathbf{a})^2],$$

we verify that (11) is correct.

To derive an equation for $\langle\mathbf{H}\rangle$ we take the time derivative of (10). We replace $\partial p/\partial t$ in the right-hand side of (10) with the aid of (8); using the convergence of the integrals, we integrate with respect to $d^2\mathbf{x}$ (in those cases when it is necessary to integrate an expression with the operator ${}^2\partial_j$). We next reduce this equation to a form in which the operator ${}^1\partial_i$ acts on the entire expression in the right-hand side of (10). This gives rise, in particular, to terms of the type

$${}^1\partial_i\int d^2\mathbf{x}pT_{ji}({}^1\mathbf{x}, {}^2\mathbf{x})/({}^2a_m-{}^1a_m),$$

which can be calculated by using the proximity of ${}^2\mathbf{x}$ to ${}^1\mathbf{x}$ as ${}^2\mathbf{a}\to{}^1\mathbf{a}$, i.e., expand T^{ji} in terms of ${}^2x_f-{}^1x_f$. Then, taking (11) into account, all the terms cancel out, with the exception of the diffusion terms

$$\partial\langle\mathbf{H}\rangle/\partial t=T\Delta\langle\mathbf{H}\rangle.\tag{12}$$

We have used here again Eq. (10).

If we now forgo the invariance with respect to reflections, then it is necessary to add to the definition (7) of T_{ij} the term

$$C(|{}^\alpha\mathbf{x}-{}^\beta\mathbf{x}|)\varepsilon_{ijf}({}^\alpha x_f-{}^\beta x_f),$$

where C is a pseudoscalar. In the expansion of $T_{ij}({}^1\mathbf{x}, {}^2\mathbf{x})$ with respect to ${}^2x_f-{}^1x_f$, the term with C makes then its contribution

$$\partial\langle\mathbf{H}\rangle/\partial t=T\Delta\langle\mathbf{H}\rangle-2C(0)\,\mathrm{rot}\,\langle\mathbf{H}\rangle.\tag{13}$$

This well known equation for the generation of a large-scale field was initially obtained using a small parameter (a small magnetic Reynolds number Rm or $\tau v/l\ll 1$), by Kraichnan[10] using the real small parameter l/L, and in our preceding paper[11] in an exact model [but not the same as the model (8)].

§3. DYNAMICS OF SMALL-SCALE FIELDS

To derive an equation for the correlation tensor B_{ij} of the small-scale fields, we take the time derivative of (3). The procedure is here quite analogous to that used in the derivation of (12). It is necessary only to use expression (7) [in place of (8) in the derivation of (12)]. All the tensors T_{ij} must be expanded in the vicinity of the points $^1\mathbf{x}$ and $^3\mathbf{x}$. After cumbersome but straightforward calculations we obtain

$$\tfrac{1}{2}\partial B_{ij}/\partial t=\partial_m\partial_n(\delta_{mn}TB_{ij}+T_{in}B_{mj}+T_{mj}B_{in}-T_{mn}B_{ij}-T_{ij}B_{mn}). \tag{14}$$

All the tensors depend here on $\mathbf{r}={}^3\mathbf{x}-{}^1\mathbf{x}$, the derivatives in the right-hand side are taken with respect to the same variable, B_{ij} depends additionally on t, and T is constant [see (8)].

We can verify directly that (14) satisfies the most important requirement, that the magnetic field be solenoidal ($\partial_i B_{ij}=0$), i.e., application of the operator ∂_i should cause the right-hand side of (14) to vanish. It is obvious that

$$\int_0^\infty B_{ij}d\mathbf{r}=\Pi_{ij}=\text{const},$$
$$\frac{1}{2}\frac{d}{dt}B_{ii}(0,t)=-\frac{1}{3}B_{ii}(0,t)(\Delta T_{jj})_{r=0}>0, \tag{15}$$

and the analog of the Loitsyanskiĭ invariant Π_{ij} is apparently equal to zero. It follows from the second property of (15) that B_{ij} increases exponentially. However, as noted above, enhancement of the field is accompanied by a decrease of the scale, and ultimately finite ohmic damping sets in, and Eqs. (14) and (15) are no longer valid.

By reducing (14) to a Schrödinger equation, we verify that the eigenfunction decreases too slowly outside the "potential well." Formally this is due to the vanishing of the coefficient of the highest-order derivative, i.e., to a singularity in the potential well. Physically all this is explained by the fact that the scale decreases just the same: if Eq. (14) were to have a "good" eigenfunction, this would mean that there exists a solution without a decrease of scale. It can thus be stated that (14) contains no eigenfunctions having correlation-function properties (this circumstance manifests itself more clearly in Fourier space, see below).

Since the decrease of the scale is so important in the field dynamics, we turn to equation for quite small-scale fields, i.e., such that $l_H\ll l$, where l_H is the correlation length of the tensor B_{ij}. It suffices then to retain the following terms of the expansion of the tensor T_{ij}:

$$T_{ij}(\mathbf{r})=T\delta_{ij}-T_2(r^2\delta_{ij}-\tfrac{1}{2}r_ir_j),\quad T,T_2>0.$$

The equation for the fluctuations with $l_H\ll l$ will take the form

$$\frac{1}{2}\frac{\partial}{\partial t}B_{ij}=T_2\left(r^2\Delta B_{ii}-\frac{1}{2}r_mr_n\partial_m\partial_nB_{ii}+r_m\partial_mB_{ii}+5B_{ii}\right), \tag{16}$$

or, in the Fourier representation

$$\partial B(k)/\partial t=T_2(k^2B''(k)+2kB'(k)+4B(k)); \tag{17}$$

the primes correspond to derivative with respect to k, and $B(k)$ is the Fourier transform of the correlation function B_{ii}.

The problem of the eigenfunctions of Eq. (17) will be formulated in the form $B(k)=B_n(k)\exp(-E_nt)$. It is clear that an exponentially increasing solution with $E_n<0$ [corresponding to the presence of a deep well in the potential of Eq. (17)] remains in the asymptotic regime as $t\to\infty$. The largest contribution is made as $t\to\infty$ by the lowest harmonic with a maximum increment $|E_0|$. Of course, the eigenfunction must be positive here, since this is a spectral function. However, the eigenfunctions of Eq. (17)

$$B_n\sim k^\alpha,\quad \alpha=\tfrac{1}{2}\{-1\pm[1-4(4+E_n/T_2)]^{1/2}\}$$

with $E_n<0$ have a singularity either at zero or at infinity. This demonstrates clearly the validity of the statement made above that a solution of the general equation (14) as an eigenvalue problem is incorrect. The reason is the decrease of the scale, i.e., the increase of k, and follows directly from (17). It is easily seen that

$$\frac{d}{dt}\int_0^\infty Bdk=4T_2\int_0^\infty Bdk,\quad \frac{d}{dt}\int_0^\infty Bkdk=6T_2\int_0^\infty Bkdk,$$

i.e., the first moment increases more rapidly than the zeroth moment ($6>4$), which illustrates in fact the growth of the effective k.

The situation changes substantially if ohmic dissipation is introduced. It is this dissipation which determines the characteristic scale of the eigenfunction. If, e.g., we add to the right-hand side of (14) diffusion terms $D\Delta B_{ij}$, and correspondingly in the right-hand side of (17) $2k^2DB(k)$, $D=c^2/4\pi\sigma$, σ is the electric conductivity, then the singularity at the zero potential of (14) disappears: the highest derivative at $r=0$ will have a coefficient D. The actual reason is that (in Euler coordinates) the equation for $\mathbf{H}$ is of the form

$$\partial\mathbf{H}/\partial t=\text{rot}\,[\mathbf{v}\times\mathbf{H}]+D\Delta\mathbf{H} \tag{18}$$

and at $D=0$ the order of the system changes, the equation becomes a partial differential equation for which the eigenvalue problem is meaningless. In the general case when $D\neq0$ and v is a stationary field, the eigenvalue problem is correct, and the same can be stated concerning the equation for B_{ij} in the case of a turbulence that is stationary in the statistical sense.

It must be borne in mind, however, that the general form of the equation for B_{ij}, with account taken of finite D, is not known. Addition to (14) of a diffusion term is possible only for the Markov model, i.e., when $\tau v/l\ll1$ (Ref. 8). We, however, started from Eq. (1), which was obtained with diffusion neglected. Therefore (14) is valid when

$$l,\ l_H\gg D/v,\quad k_0,\ k_H\ll v/D,\quad k_0=2\pi/l,\quad k_H=2\pi/l_H, \tag{19}$$

and (17) is valid when

$$k_0\ll k\ll(T_2/D)^{1/2}=k_1. \tag{20}$$

The general eigenvalue problem can therefore be posed for the equation

$$\partial B(k)/\partial t=\int A(\mathbf{k},\mathbf{q})B(q)d\mathbf{q}, \tag{21}$$

which goes over at $k \ll k_1$ into (14), in the interval (20) into (17), at $k \gg k_1$ into $\partial B/\partial t = -2Dk^2 B$, and into an unknown form at $k \approx k_1$.

We shall seek the solution of (21) in the form $B(k) = B_n(k)\exp(-E_n t)$. Then for the eigenfunctions of Eq. (21) and for its conjugate we have the equations

$$
\begin{aligned}
-E_n B_n(k) &= \int A(\mathbf{k},\mathbf{q}) B_n(q)\, d\mathbf{q},\\
-E_m^* \tilde{B}_m(k) &= \int A^*(\mathbf{q},\mathbf{k}) \tilde{B}_m(q)\, d\mathbf{q}.
\end{aligned}
\tag{22}
$$

The system of functions $\tilde{B}_n$ is conjugate (dual) to the system of eigenfunctions B_n, the asterisk denotes complex conjugation. It is important in the following that Eq. (21) can be reduced to a self-adjoint form, i.e., the corresponding eigenfunction problem (22) reduces to two identical equations. Indeed, the equation conjugate to (18)

$$\partial\tilde{\mathbf{H}}/\partial t = -[\mathbf{v}\times \mathrm{rot}\,\tilde{\mathbf{H}}] + D\Delta\tilde{\mathbf{H}},$$

coincides with the equation for the vector potential of the magnetic field $\mathbf{H}$ following the substitutions $\mathbf{x}\to -\mathbf{x}$ and $\mathbf{v}(\mathbf{x})\to \mathbf{v}(-\mathbf{x})$. The transformation $\mathbf{x}\to-\mathbf{x}$ is the reflection transformation, to which the statistical properties of the turbulence are assumed to be invariant. This means, in particular, that the equation for the spectral function of the field $\tilde{\mathbf{H}}$ is connected with Eq. (21) for the spectral function of the field $\mathbf{H}$ in the same way as the spectral equation for the vector potential is connected with the corresponding equation for $\mathbf{H}$, i.e., one of the equations in (22) should follow from the other upon the substitution $\tilde{B}_n(k)k^2 \to B_n(k)$:

$$k^{-2}A^*(\mathbf{k},\mathbf{q})q^2 = A(\mathbf{q},\mathbf{k}).$$

From this we get that for a function $z = B(k)/k$ the operator of the right-hand side of (21) in self-adjoint, $\mathrm{Im}E_m = 0$, and the equations for the eigenfunctions (22) for z coincide. In fact, as can be easily verified, the Fourier transform of Eq. (14) and its particular form, Eq. (17), have the indicated property, namely, they can be reduced to a self-adjoint form for the function z.

To obtain the eigenvalues we use a variational principle that is valid for self-adjoint operators:

$$
\begin{aligned}
E &= -\int z^*(k) k^{-1} A(\mathbf{k},\mathbf{q})\, q z(q)\, d\mathbf{q}\, d\mathbf{k} \Big/ \int z^*(k) z(k)\, d\mathbf{k},\\
\delta E &= 0, \quad \int z^*(k) z(k)\, d\mathbf{k} = \mathrm{const}.
\end{aligned}
\tag{23}
$$

It is clear that if one can find a trial function z that can make the functional (23) negative, then the lower eigenfunction (with eigenvalue $E_0 = \min E_n$) will make this functional more negative in absolute value, i.e., $E_0 < 0$ and an exponentially increasing solution (dynamo solution) with an increment $|E_0|$ does exist (see Ref. 9).

We choose trial functions in the form

$$z = \frac{k^m}{((2m+2)!)^{1/2}} \exp\left(-\frac{k}{k_2}\right)\left(\frac{2}{k_2}\right)^{(2m+3)/2}, \quad m = 0, 1, 2, \ldots, \tag{24}$$

k_2 lies in the interval (20). Then the contribution of the region with $q > k_1$ to the integral with respect to $\mathbf{q}$ in (23) is exponentially small. In fact, the inverse statement, that a substantial contribution is made by the integral at $q \gtrsim k_1$, would mean that according to (21) [in which $B(q)$ is replaced by zq, which also attenuates exponentially at $q \approx k_1$], the dynamics of a field with scale $l_H = 2\pi/k_2$ is determined by ohmic diffusion. This contradicts the well known fact that for a field with scale $l_H = 2\pi/k_2 \gg 2\pi/k_1$ the "freezing-in" condition is satisfied, i.e., the ohmic diffusion does not play any role in (18). It is clear at the same time that in the integration with respect to $\mathbf{q}$ the contribution of the region $q \lesssim k_0$ is quite small, therefore we can use (17) to calculate (23). All the calculations then become simple and it is easily seen that trial functions with $m = 0, 1, 2, 3, 4, 5, 6$ make the functional (23) negative. Consequently, a dynamo solution exists.

§4. CONNECTION WITH REAL TURBULENCE

The problem of generation of small-scale fields was solved in the non-Markov model (7). The extent to which this model is universal and describes the real turbulence is unknown. At the same time, as shown in §3, to prove the existence of a small-scale field dynamo we have actually used not the general equation (14), which follows from the model (7), but only its asymptotic form (17), since the trial functions (24) make a negligibly small contribution to the functional (23) in the region where it is necessary to use the general equation (14). For this approach to be valid, it is necessary above all that the interval (20) exist. For turbulence with a power-law spectrum of the Kolmogorov type the transition from (14) to (16) and (17) is possible if the scale l is taken to mean the viscous length $l = (\nu^3/\bar{\varepsilon})^{1/4}$, ν is the kinematic viscosity, and $\bar{\varepsilon}$ the energy flux over the spectrum. To estimate T_2 it is then necessary to take the characteristic frequency of the velocity field variation for the scale l, $T_2 \approx (\bar{\varepsilon}/\nu)^{1/2}$. From this it follows directly that for the existence of the interval (20) it is necessary that

$$\nu \gg D, \tag{25}$$

i.e., the magnetic Reynolds number calculated for the external turbulence scale l_2 must strongly exceed the ordinary Reynolds number.

Thus, for real turbulence, the solution of the problem of the small-scale field dynamo is determined when (25) is satisfied by the extent to which Eq. (17) represents correctly the real dynamics of the field. As will be shown later on, Eq. (17) is equally universal for the region $k \gg k_0$, as the Kraichnan approximation[10] is for $k \ll 2\pi/l_1$ and $k \ll k_0$. First of all, the interval (20) corresponds to a small-scale magnetic-field structure for which the velocity field can be represented by the expansion $v_i = v_i^0 + a_{ij}x_j$, i.e., the basic equation (18) can be represented in the form

$$\partial H_i/\partial t + a_{jl}x_l\partial_j H_i = H_j a_{ij}. \tag{26}$$

We have not written out the term $(\mathbf{v}^0\cdot\nabla)\mathbf{H}$ which leads to a simple field shift, since $\mathbf{v}^0$ does not depend on the coordinates, and therefore plays no role in the dynamics of the field in the presence of homogeneous turbulence. The diffusion term is likewise insignificant because of the second inequality of (20).

It is seen from (26) that the statistical characteristics of the field are connected only with the tensor a_{ij}, which does not depend on the coordinates and is a random process in time. Therefore all the hydrodynamic characteristics in the equation for the correlation function and the spectral function must not contain a length parameter. Then the only possible representation of the operator (21) is in the form of a differential operator with coefficients connected with the hydrodynamic characteristics, but independent of k and q [T_2 in (16) and (17) is precisely such a coefficient]. The coefficients are likewise independent of time in view of the stationary character of the turbulence.

The same conclusion can be reached by using the representation of the distribution function with the aid of similarity considerations (see Ref. 12, part II). For the small particle displacements considered by us, p should depend on

$$({}^2\mathbf{x}-{}^1\mathbf{a})^2/l^2, \quad ({}^1x_i-{}^1a_i)({}^2x_i-{}^2a_i)/l^2,$$

etc., and on tT_2. At various small displacements (much smaller than l), all the spatial dependences are replaced in first-order approximation by zeros. We are left with the time dependence, and consequently the operator (21) can be only differential with coefficients characterizing the hydrodynamic turbulence. This circumstance simplifies very greatly the general form of the kinematic equation for p. One can expect that by satisfying a large number of requirements imposed on the kinetic equation, we shall be left with only a small number of coefficients. And indeed, as shown in the Appendix, only one coefficient is left, and the equation obtained for p leads to Eq. (17) for the region (20). Consequently, the conclusion that a dynamo solution for small-scale fields exists is generally in character if (25) is satisfied.

§5. DISCUSSION

1. It was made clear first that if Lagrangian statistical characteristics are specified for the turbulence, then the dynamics of the magnetic field can be explained for the non-Markov model $\tau \approx l/v$. To determine the dynamics of the large-scale field it is necessary to specify a distribution function for two liquid particles, or, equivalently, a two-point correlation for a scalar admixture [see Eqs. (8) and (9) and the result (12) and (13)]. To determine the dynamics of small-scale fields, i.e., to calculate the two-point correlation function of the magnetic fluctuations, it is necessary to use a four-point distribution function. For the (non-Markov) turbulence model (7), Eq. (14) is obtained for the dynamics of small-scale fields. A shortcoming of this approach is that this method cannot explain the behavior of small-scale fields in the region $k \approx k_1$, where the dissipation begins to come into play [see the definition (20)]. If $k \ll k_1$, then the freezing-in approximation is satisfied and the solution (1), meaning also (14), is valid; in the region $k \gg k_1$ we have simply $\partial B/\partial t = -2Dk^2B$. For the Markov approximation it is possible to obtain an equation for B in the entire region.[8]

2. The dynamics of a large-scale field is described by the general-form equation (13). In the present article this result is obtained in an exact model, but the results of Kraichnan[10] is more general, Eq. (13) is obtained by expansion in a small parameter l/L for any model. The dynamics of the small-scale fields for $kl \gg 1$ is described by the asymptotic equation (17), which is obtained here both for the exact model (7) and as an expansion in the parameter $(kl)^{-1}$ [Eq. (A.II) a small-scale field dynamo can be deduced only on the basis of Eq. (17)]. It can therefore be stated that the dynamo solution exists if only the condition (25) is satisfied.

3. It is clear from the foregoing how greatly important for the elucidation of the dynamics of magnetic fields is the experimental determination of the distribution function, or, equivalently, the determination of the dynamics of the two-point and four-point correlation functions of the scalar admixture. This makes it possible to determine exactly the coefficients of the turbulent diffusion and generation in Eq. (13). In addition, this method can explain the behavior of the small-scale fields in the entire region $k \ll k_1$, i.e., to verify that the model (7) is general. This is particularly necessary if the condition (25) is not satisfied: in this case there is no region (20) and the question of the small-scale-field dynamo remains open. On the other hand if (25) is satisfied, then a study of the scalar admixture makes it possible to determine the coefficient T_2 in (17), which agrees in order of magnitude with the growth rate of the dynamo instability. We note in this connection also the importance of measuring the Lagrangian characteristics in astrophysical investigations. We recall that the problem of turbulent dynamo arose precisely in astrophysics. For example, it becomes possible in principle to determine the distribution function from the motions of distinct details in the solar photosphere.

APPENDIX

The right-hand side of (17) can contain terms of the type a_0B, $a_0^{(1)}kB$, $a_0^{(2)}k^2B$, a_1kB', $a_1^{(1)}k^2B'$, a_2k^2B'', $a_2^{(1)}k^3B''$, a_3k^3B''', etc. At the same time the presence of terms with derivatives of order higher than the second contradicts the requirement that the spectral function be positive-definite. In fact, whereas at the initial instant the spectral function in the vicinity of $k = k_\alpha$ is represented in the form

$$B=(k-k_\alpha)^2a+(k-k_\alpha)^3b, \quad a>0,$$

then at the point $k = k_\alpha$ we have

$$B=0, \quad B'=0, \quad B''=2a>0,$$

[therefore the coefficient of B'' should be positive, as in (17), $T_2 > 0$] and $B''' = 6b$. However, since both the magnitude and the sign of b are generally speaking arbitrary, the term with the third derivative can give a negative contribution such that $\partial B/\partial t$ becomes less than zero and B becomes negative at the point $k = k_\alpha$, which is inadmissible. It can be similarly proved that there are no terms with fourth and higher derivatives.

We analyze now the terms proportional to B. It is clear that the right-hand side of (17) cannot contain terms of the type $a_0^{(1)}kB$, $a_0^{(2)}k^2B$, $a_0^{(3)}k^3B$, $a_0^{(4)}k^4B$, etc. The point is that $a_0^{(2)}$ and $a_0^{(3)}$ contain the dimension of length, whereas in (26) the components a_{ij} have the dimension of frequency and depend only on the time. A similar selection for the terms $\propto B'$ and $\propto B''$ reveals the following general form of the operator:

$$a_0 B + a_1 k B' + a_2 k^2 B''$$

(all the coefficients have the dimension of frequency). This means that the equation for the four-point distribution function, if the points are close enough to one another, contains derivatives of order not higher than the second, and the coefficients have the dimension of frequency.

We now write down this equation after integrating it with respect to two points. In other words, we write down the equation for the two-point distribution function:

$$\partial p/\partial t = {}^{\alpha\beta\gamma}b^{\alpha}\partial_i{}^{\beta\gamma}X_i p + {}^{\alpha\beta\gamma\delta\varepsilon\eta}b_{ijfm}{}^{\varepsilon}\partial_i{}^{\eta}\partial_j{}^{\alpha\beta}X_f{}^{\gamma\delta}X_m p. \qquad {}^{\alpha\beta}X_i = {}^{\alpha}x_i - {}^{\beta}x_i. \tag{A.I}$$

The Greek letters run here through the values 1 and 2. The equation is written in divergent form, i.e., so that the integral with respect to p over space is preserved (this accounts for the term without the derivative). Setting all possible combinations for the isotropic reflection-invariant tensor ${}^{\alpha\beta\gamma}b$, we see that they inevitably contain a certain vector: $\delta_{\alpha\beta}{}^{\gamma}b$, ${}^{\alpha}b^{\beta}b^{\gamma}b$, etc. In turbulence, however, there is no selected direction and ${}^{\alpha}b = 0$. In other words, all the coefficients ${}^{\alpha\beta\gamma}b$ vanish.

To determine the coefficients of the term with the second derivative, we recall that Eq. (A.I) coincides according to (5), (6), (8), and (9) with the equation for the correlation function of the scalar admixture, $f = f(|{}^2\mathbf{x} - {}^1\mathbf{x}|)$. Then, recognizing that ${}^1\partial_i = -{}^2\partial_i$, we have in the right-hand side of (A.I) only a term of the form

$$b_{ijfm}{}^1\partial_i{}^1\partial_j{}^{21}X_f{}^{21}X_m p.$$

The requirement of isotropy and obvious symmetry with respect to the subscripts m and f leads to the following form of the tensor

$$b_{ijfm} = A\delta_{ij}\delta_{fm} + B(\delta_{if}\delta_{jm} + \delta_{im}\delta_{jf}).$$

Finally, we recall that the equation for the scalar admixture

$$\partial\vartheta/\partial t = v_i\partial_i\vartheta + \chi\Delta\vartheta$$

(χ is the molecular diffusion coefficient) and its conjugate

$$\partial\tilde{\vartheta}/\partial t = -v_i\partial_i\tilde{\vartheta} + \chi\Delta\tilde{\vartheta}$$

go over into each other with the aid of the transformations $\mathbf{x} \to -\mathbf{x}$ and $\mathbf{v}(\mathbf{x}) \to \mathbf{v}(-\mathbf{x})$. Consequently, just as in §3, the operator acting in the right-hand side of (A.I) (and in the equation for f) should be self-adjoint. From this follows the connection between the coefficients A and B, namely $B = -A/4$.

We turn now to the four-point distribution function. It likewise is written in the form (A.I), and the terms with the first derivative vanish for the same reason as in Eq. (A.I) itself. As for the coefficient of the second derivative, isotopic combinations (of Kronecker symbols) are set up here. Taking into account the antisymmetry with respect to the indices $\alpha \to \beta$ and $\gamma \to \delta$ and the symmetry with respect to $\varepsilon \leftrightarrow \eta$, and taking also into account the requirement that p vanish at ${}^{\alpha}\mathbf{x} = {}^{\beta}\mathbf{x}$ (see §2), the only terms left are of the type

$$b_{ijfm}{}^1\partial_i{}^2\partial_j({}^1x_f - {}^2x_f)({}^1x_m - {}^2x_m),$$

and similar ones for the point combinations 1, 3; 1,4; 2, 3; and 3, 4. Then, only one term remains in the integration of the equation for p with respect to any two points, and the form of the tensor b_{ijfm} can be explained in exactly the same way as in the analysis of Eq. (A.I). Consequently, the general form of the equation for close points (distance much less than l) is

$$\partial p/\partial t = -T_2 \sum_{\alpha,\beta} {}^{\alpha}\partial_i{}^{\beta}\partial_j[({}^{\alpha}x - {}^{\beta}x)^2\delta_{ij} - {}^1/{}_2({}^{\alpha}x_i - {}^{\beta}x_i)({}^{\alpha}x_j - {}^{\beta}x_j)]p. \tag{A.II}$$

A single coefficient T_2 is used for different combinations of points, as follows from the obvious symmetry between the points. From (A.II) follow directly (16) and (17), with $T_2 > 0$ in order to satisfy the condition that the spectral function be positive-definite.

Note added in proof (31 October 1980). Recently (F. Krause, in "Dynamo Theory," Report of International Workshop, KARG, Alšcovice, 1979, p.20) a statement was made that the Markov model used by Kazantsev[8] is not a dynamo model. The reasoning is the following. Since τ is regarded in this model as a small quantity, we have the condition $\tau \ll 1/Dk^2$. Since k is arbitrary, it follows therefore that $D \to 0$. But at $D = 0$ the amplification of the field is not a dynamo, and the amplification is always temporary when $D \neq 0$ is turned on. Actually, there is no such condition in the model. In fact, from the Kazantsev equation, which differs from (17) in that a term Dk^2B is added to the left-hand side, it follows that the growth rate $\gamma = T_2 = (v/l)^2\tau = Dk_1^2$, where k_1 is a wave vector that cuts off the spectral function. Thus, τ is small but not zero! At the same time the use of δ-correlation in time is fully justified: τ is small compared with the quantities l/v, $1/\gamma$, l^2/D, and $1/Dk_1^2$. The last circumstance makes it possible to regard all the components of the type $\exp[-Dk^2(t_1 - t_2)]$, which arise when diagrams are shown, to be regarded as equal to unity, which is equivalent to multiplying these quantities by $\delta(t_1 - t_2)$. Thus, the Markov model by itself is naturally self-consistent. The only question is whether this model applies to real turbulence.

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Possibility of mutual enhancement of electromagnetic waves in semiconductors with narrow conduction band

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A quantum kinetic equation for the conduction electrons with arbitrary dispersion law, in the presence of two homogeneous alternating electric fields, is obtained in the approximation of weak electron-phonon coupling. In the case of electromagnetic fields with quantum energy exceeding the width of the conduction band, expressions for the coefficients of the intraband absorption of each of the fields are obtained on the basis of a solution of the kinetic equation. In the considered situation, one of the coefficients turns out to be negative, thus indicating the possibility of mutual enhancement of electromagnetic waves in semiconductors with narrow conduction band.

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Malevich and Épshtein[1] have investigated, on the basis of a quantum kinetic equation, the behavior of the electrons of a semiconductor in the presence of two strong homogeneous alternating electric fields: $\mathbf{E}(t)=\mathbf{E}_1 \sin\omega_1 t + \mathbf{E}_2 \sin\omega_2 t$. They used a quadratic isotropic electron dispersion law, therefore the results obtained by them are no longer valid at high frequencies, where the energy of the electromagnetic-field quanta is comparable with or larger than the width Δ of the conduction band. In a semiconductor with a narrow conduction band, Δ can be quite small ($\lesssim 10^{-2}$ eV in semiconductors with superlattice[2] and in organic[3-5] or polaron semiconductors), i.e., the results of Malevich and Épshtein cease to be valid already at $\omega_{1,2} \gtrsim 10^{13}$ sec^{-1}.

No assumptions whatever are made in this paper concerning the electron dispersion law in the conduction band. This makes it possible to investigate the situation[1)] $\omega_{1,2} \gtrsim \Delta$ (a system of units with $\hbar=1$ is used). The frequencies $\omega_{1,2}$ are bounded from above only by the assumption that there are no interband transitions.

The Hamiltonian that describes the interaction of the electrons with the phonons takes in the presence of an electromagnetic field the form

$$\hat{H}=\sum_{\mathbf{p}} \varepsilon\left(\mathbf{p}-\frac{e}{c}\mathbf{A}(t)\right) a_{\mathbf{p}}^{+}a_{\mathbf{p}}+\sum_{\mathbf{q}} \omega_{\mathbf{q}} b_{\mathbf{q}}^{+} b_{\mathbf{q}}$$

$$+\sum_{\mathbf{p},\mathbf{q}} C_{\mathbf{q}} a_{\mathbf{p}-\mathbf{q}}^{+} a_{\mathbf{p}} (b_{\mathbf{q}}+b_{-\mathbf{q}}^{+}).$$

Here $\varepsilon(\mathbf{p})$ is the electron dispersion law in the conduction band, $\mathbf{A}(t)$ is a certain potential connected with the electromagnetic-wave field

$$\mathbf{E}(t)=\mathbf{E}_1 \sin\omega_1 t+\mathbf{E}_2 \sin\omega_2 t$$

by the relation

$$\mathbf{E}(t)=-c^{-1}d\mathbf{A}(t)/dt,$$

a_p^+ and a_p (b_q^+, b_q) are the operators of creation and annihilation of an electron (phonon), ω_q is the phonon frequency, and C_q is the electron-phonon coupling constant.

In the derivation of the quantum kinetic equation for the description of the processes in the high-frequency fields ($\omega_{1,2} \gg \tau^{-1}$, where τ is the relaxation time), we follow the procedure developed by Épshtein.[7] Assuming the phonons to be in equilibrium and the electron gas to be nondegenerate, we obtain in the lowest order in the coupling constant C_q an equation for the electron distribution function

$$\partial\varphi(\mathbf{p},t)/\partial t=\left\{\sum_{\mathbf{q}} f(\mathbf{q}) \operatorname{Re}\int_{-\infty}^{t} dt' \exp\left[-i\int_{t'}^{t} d\tau\left(\varepsilon\left[\mathbf{p}+e\int^{\tau}\mathbf{E}(\tau')d\tau'-\mathbf{q}\right]\right.\right.\right.$$

$$\left.\left.\left.-\varepsilon\left[\mathbf{p}+e\int^{\tau}\mathbf{E}(\tau')d\tau'\right]\right)\right]\left[\varphi(\mathbf{p}-\mathbf{q},t')-\varphi(\mathbf{p},t')\right]\right\}-\{\mathbf{p}\to\mathbf{p}+\mathbf{q}\}, \quad (1)$$

where $\{\mathbf{p}\to\mathbf{p}+\mathbf{q}\}$ stands for the expression written out explicitly in the curly brackets, with the corresponding change in the arguments

and T is the temperature in energy units. In the derivation of (1), the phonon energy ω_q was assumed to be small compared with the characteristic electron energy $\bar{\varepsilon}$ (quasielastic scattering).

We consider now in greater detail the terms in the argument of the exponential in (1). We have

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