One-dimensional disordered system in an electric field

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A study is made of the influence of an electric field on the character of the electronic states in a one-dimensional system with randomly distributed centers. The existence of an electric field threshold is demonstrated: $E_c = 2\varepsilon_0/el(\varepsilon_0) = 2mcV(2p_0)e\hbar^2$ (c is the impurity density and $V(q)$ is the Fourier component of the potential of an individual impurity). At $E < E_c$ the electronic states are localized, just as in the absence of a field, but the character of the localization is changed. A particle emitted from a point x' at an instant of time $t = 0$ undergoes an average displacement along the field, and is distributed at $t \to \infty$ with a density that decreases as $|x| \to \infty$ not exponentially but in power-law fashion, with an exponent that depends on the field. If $E \to E_c$, then $K \to 1$. At $E > E_c$ the electronic states are expected to be delocalized.

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1. INTRODUCTION

It is known that in a one-dimensional disordered system all the electronic states are localized. This assertion is due to Mott and Twose[1] and was confirmed by a number of calculations.[1-4] The asymptotic form of the electron density for the localized state as $|x| \to \infty$ is in the main exponential.[2] Its explicit form was determined by Gogolin.[5]

Starting from the assertion of Mott and Twose, one can expect the static conductivity to vanish at zero temperature. This result was proved by direct calculation by Berezinskiĭ.[2] The same result was later obtained by a somewhat different method by Abrikosov and Ryzhkin.[4] In the last two references, however, the conductivity was calculated from the Kubo formula, in which it is assumed that the applied electric field is infinitely weak. It can therefore turn out actually that in the case of a weak but finite field the conductivity differs nonetheless from zero. The result of Refs. 2 and 4 would in this case mean a strong nonlinearity of the conductivity as a function of the field, for example $\sigma = \sigma_0 \exp(-E_0/E)$.

In the three-dimensional case, the electric field mixes the localized and delocalized states, and it is this which causes a finite conductivity in an electric field. This takes place, of course, if there exists a mechanism that ensures energy relaxation in the system. Otherwise the conductivity is infinite. In the one-dimensional case the localization is complete—all states are localized regardless of energy. Nonetheless in the one-dimensional case, where the localization, unlike in the three-dimensional case, takes place, at any degree of disorder, and is the result of strong interference effects in impurity scattering, the electric field causes a strong rearrangement of the electronic states, inasmuch as the very character of the electron motion changes in an external field. The present paper deals with precisely the question of the influence of an electric field on the character of the electronic states in a one-dimensional disordered system. The method proposed by Berezinskiĭ[2] is followed in the main.

We consider an electron situated in the field of randomly distributed centers and a constant electric field. To investigate the character of the electronic states it is necessary to calculate the density correlator.[6] For an electron with a given energy ε it can be defined as

$$X_\varepsilon(x,x'|\omega)=\int_0^\infty e^{i\omega t}\frac{\{\langle\varepsilon|\rho(x,t)\rho(x')|\varepsilon\rangle\}_c}{\{\langle\varepsilon|\rho(x')|\varepsilon\rangle\}_c}dt, \tag{1}$$

where $\rho(x)=\delta(\hat{x}-x)$ is the density operator and $|\varepsilon\rangle$ is the eigenfunction of the particle in an external field with energy ε. The curly brackets $\{\ldots\}$ in (1) denote averaging over all the realizations of the random potential. The normalization factor in (1) is chosen such that

$$\int X_\varepsilon(x,x'|\omega)\,dx=1. \tag{2}$$

There exists for Eq. (1) the following spectral representation:

$$X_\varepsilon(x,x'|\omega)=\int\frac{d\omega'}{2\pi i}\frac{S_\varepsilon(x,x'|\omega')}{\omega-\omega'+i0}, \tag{3}$$

$$S_\varepsilon(x,x'|\omega)=\frac{\{\psi_\varepsilon(x)\psi^*_{\varepsilon+\omega}(x)\psi_{\varepsilon+\omega}(x')\psi^*_\varepsilon(x')\}_c}{\{\psi_\varepsilon(x')\psi^*_\varepsilon(x')\}_c}. \tag{4}$$

If we include in (4) unity in the form

$$\int_{-\infty}^{\infty}d\varepsilon'\delta(\varepsilon'-\varepsilon)=\int_{-\infty}^{\infty}\frac{d\varepsilon'}{2\pi i}\left\{\frac{1}{\varepsilon-\varepsilon'-i0}-\frac{1}{\varepsilon-\varepsilon'+i0}\right\} \tag{5}$$

and then make in (3) the change of variable $\varepsilon''=\varepsilon+\omega'$, then we obtain for (1) the expression

$$X_\varepsilon(x,x'|\omega)=X_\varepsilon^-(x,x'|\omega)-X_\varepsilon^+(x,x'|\omega), \tag{6}$$

$$X_\varepsilon^\pm(x,x'|\omega)=\{G^\pm(x,x'|\varepsilon)G^+(x,x'|\varepsilon+\omega)\}_c/\{\operatorname{Im}G^-(x',x'|\varepsilon)\}_c, \tag{7}$$

$$G^\pm(x\,x'|\varepsilon)=\int\frac{d\omega}{2\pi i}\frac{\psi_\varepsilon(x)\psi^*_\varepsilon(x')}{\varepsilon-\omega\pm i0}. \tag{8}$$

In the subsequent averaging with respect to energy in (6), only the first term makes a contribution. We shall therefore take hereafter X_ε to mean just $X\bar{\varepsilon}$. The function

$$P(x,x'|t)=\int_{-\infty}^{\infty}d\varepsilon X_\varepsilon(x,x'|t) \tag{9}$$

has the meaning of the conditional probability function. It is used to describe the spreading-out of a particle placed at the point x' at the instant $t=+0$. The function

$$p_\infty(x|x')=X_\varepsilon(x,x'|t\to\infty) \tag{10}$$

specifies the distribution of the electron density of a particle with energy ε. From the behavior of $p_\infty(x|x')$, (6), and (9) we can assess the character of the electronic states.[6]

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2. ELECTRON GREEN'S FUNCTION IN AN EXTERNAL FIELD

The initial object in the calculation of (7) are the Green's functions. We obtain them in the presence of an external field. Assume the presence of a uniform field $U(x)=-Fx$. The force $F=eE$ acting on the electron is directed along the x axis. In such a field, the electron wave function corresponding to the energy ε takes in the p-representation the form[7]

$$\Psi_\varepsilon(p)=\frac{1}{(2\pi F)^{1/2}}\exp\left[\frac{i}{F}\left(\varepsilon p-\frac{p^3}{6m}\right)\right] \tag{11}$$

with normalization to a δ function of the energy ($\hbar=1$). Substituting this equation in (8) and integrating with respect to ω, we obtain for the quantum-mechanical Green's functions

$$G_0^{\pm}(p,p'|\varepsilon)=\frac{\mp 1}{2\pi F}\exp\left[-\frac{i}{F}\left(\varepsilon(p'-p)-\frac{p'^3-p^3}{6m}\right)\right]\theta(\pm(p-p')). \tag{12}$$

We change over in (12) to the x-representation. We make the substitutions

$$\xi=(2m/F^2)^{1/3}(\varepsilon+Fx),\quad \xi'=(2m/F^2)^{1/3}(\varepsilon+Fx'),$$
$$\zeta=p+p',\quad z=i(1/2mF)^{1/3}(p-p'). \tag{13}$$

The integral with respect to ζ in (12) is now Gaussian. As a result we have

$$G_0^{\pm}(x,x'|\varepsilon)=\frac{\mp(2m)^{2/3}}{(2\pi)^{2/3}\times 2F^{1/3}}\int_{\Gamma_\pm}\frac{dz}{z^{1/3}}\exp\left(\frac{z^3}{12}+\frac{z}{2}(\xi'+\xi)-\frac{(\xi'-\xi)^2}{4z}\right), \tag{14}$$

where the contours Γ_+ and Γ_- pass along the positive and negative imaginary axes, respectively. The integral in (14) can be calculated by the saddle-point method. At $\xi\gg 1$ and $\xi'\gg 1$ the saddle point are

$$z_1^{\pm}=\pm i|\xi'^{1/2}+\xi^{1/2}|+O((\xi'\xi)^{-1/2}),\quad z_2^{\pm}=\pm i|\xi'^{1/2}-\xi^{1/2}|+O((\xi'\xi)^{-1/2}). \tag{15}$$

Shifting the integration contour Γ_+ in the manner shown in Fig. 1 and performing the calculation we obtain for G_+, and similarly for G_-,

$$G_0^{\pm}(x,x'|\varepsilon)=\frac{\mp i(2m)^{1/2}}{2F^{1/2}|\xi'-\xi|^{1/4}}\left\{\exp\left(\pm\frac{2i}{3}|\xi'^{3/2}-\xi^{3/2}|\right)\right.$$
$$\left.\pm i\exp\left(\pm\frac{2i}{3}|\xi'^{3/2}+\xi^{3/2}|\right)\right\}. \tag{16}$$

The Green's function in the classically accessible region ($\xi<0$) can be obtained in similar fashion. The electron motion here, however, is certainly limited, so that the region $\xi<0$ is of no importance for the subsequent calculations.

The first term in (16) corresponds to the amplitude of a direct transition of a particle from a point x' to a point x. The second term[1)] in (16) describes the amplitude of the transition when the electron first moves to the left, is reflected from the potential barrier ($\xi=0$), and arrives only then at the point x. The two processes are illustrated in Fig. 2. Taking into account the explicit form of the Green's function (16), we note that the contribution of the second process can be obtained from the first term in (16) in accord with Fig. 2. Thus, in a graphic construction we can use the following Green's function

$$G_0^{\pm}(x,x'|\varepsilon)=\frac{\pm i}{v^{1/2}(x)v^{1/2}(x')}\exp\left(\pm\frac{2i}{3}|\xi^{3/2}-\xi'^{3/2}|\right), \tag{17}$$

where $v(x)=[2(\varepsilon+Fx)/m]^{1/2}$ is the particle velocity at the point x. It is then necessary to take into account the particle reflection from the boundary of the classically inaccessible region. The turning point corresponds to the factor $\pm i$.

An important factor in the entire construction that follows is that the Green's function (17) breaks up into two factors pertaining to the points x and x':

$$\mp\frac{i}{v^{1/2}(x)}\exp\left(\pm\frac{2i}{3}\xi^{3/2}\right),\quad \mp\frac{i}{v^{1/2}(x')}\exp\left(\mp\frac{2i}{3}\xi'^{3/2}\right). \tag{18}$$

This makes it possible, just as in the absence of a field, to shift the entire x-dependence to the impurity vertices.

3. DERIVATION OF BASIC EQUATIONS

Electron scattering in the field of randomly distributed impurities will be considered in the Born approximation. Assume that the correlator that characterizes the impurity potential is narrow enough, so that

$$U(x-x')=\langle U(x)U(x')\rangle=U_0\delta(x-x'). \tag{19}$$

Including now factors of the form (18) in the impurity vertices and integrating with respect to the difference $x-x'$, we can obtain for the impurity lines the expressions

$$\frac{1}{l_+(x)}=\frac{2}{v(x)}\int_0^\infty U(x')\frac{dx'}{v(x')}, \tag{20}$$

$$\frac{1}{l_-(x)}\pm\frac{i}{l(x)}=\frac{2}{v(x)}\int_0^\infty\frac{U(y)}{v(x+y)}\exp\left[\pm\frac{4i}{3}(|\xi|^{3/2}-|\tilde{\xi}|^{3/2})\right]dy, \tag{21}$$

$$\xi=(2m/F^2)^{1/3}(\varepsilon+Fx),\quad \tilde{\xi}=(2m/F^2)^{1/3}(\varepsilon+F(x+y)). \tag{22}$$

The quantities $l_+(x)$ and $l_-(x)$ have here the meaning of the mean free path with respect to forward and backward scattering. Using (19), we obtain for $l_\pm(x)$

$$l_\pm(x)=l_\pm^0(1+Fx/\varepsilon),\quad l_\pm^0=\varepsilon U_0/m, \tag{23}$$

where $l_\pm^0$ corresponds to a particle with energy ε in the absence of a field.

The contribution of formulas (21) and (22) to the impurity vertex must be multiplied by an oscillatory factor that depends on the form of the attached electron

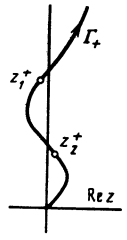

FIG. 1. Position of the contour Γ_+ in (14) in integration by the saddle-point method.

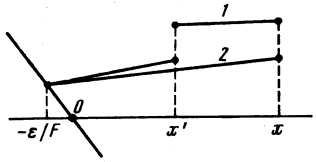

FIG. 2. Diagrammatic representation of various contributions to the Green's function (16).

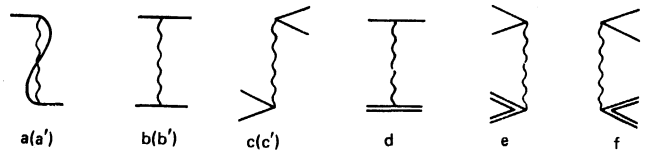

FIG. 3. Impurity vertices that form essential diagrams. Vertices a', b', and c' differ from a, b, and c in that the single lines are replaced by double ones. The vertices correspond to the following factors: a) $-1/2l_-(x)-1/2l_+(x)-i/2l(x)$, a') $-1/2l_-(x)-1/2l_+(x)+i/2l(x)$, b, b') $-1/l_+(x)$, c, c') $-1/l_-(x)$, d) $1/l_+(x)$, e) $[\exp(2i\omega\Delta)]/l_-(x)$.

lines. The choice of the essential diagrams can be based on neglect of the contributions of diagrams containing rapidly oscillating factors such as $\exp(\pm i\xi^{3/2}/3)$ compared with diagrams containing slowly varying factors such as $\exp[\pm 2i\omega\Delta(x)]$, where

$$\omega\Delta=\frac{2}{3}(|\xi_\varepsilon|^{3/2}-|\xi_{\varepsilon+\omega}|^{3/2})=\xi^{1/2}\omega\frac{\partial\xi}{\partial\varepsilon}=-\frac{(2m(\varepsilon+Fx))^{1/2}\omega}{F}. \tag{24}$$

If we forget for the time being the reflection from the boundary, then the structure of the diagrams chosen in this manner is the same as in Berezinskiĭ's paper and we can use his results directly. Figures 3 and 4 show the impurity and external vertices that make up the essential diagrams. A single line corresponds to G_- and the double to G_+. We put $x>x'$. Following Berezinskiĭ, we divide each of the diagrams into three parts: right (to the right of x), central, and left. The following equations hold for the left and central parts[2,3]:

$$\frac{d\tilde{R}_m}{dx}=\frac{m^2}{l_-(x)}\{\tilde{R}_{m-1}e^{-2i\omega\Delta(x)}+\tilde{R}_{m+1}e^{2i\omega\Delta(x)}-2\tilde{R}_m\}, \tag{25}$$

$$\frac{d\tilde{Z}_{\cdot m}}{dx}=i\omega\tilde{Z}_{\cdot m}\frac{d\Delta(x)}{dx}+\frac{1}{l_-(x)}\{(m+1)^2(\tilde{Z}_{\cdot m+1}e^{2i\omega\Delta(x)}-\tilde{Z}_{\cdot m}) \tag{26}$$
$$-m^2(\tilde{Z}_{\cdot m}-\tilde{Z}_{\cdot m-1}e^{-2i\omega\Delta(x)})\},$$

where the subscript m indicates that $2m$ double lines and $2m$ single lines enter the point x. The point x in the central part $Z_{\cdot m}$ has incoming $2m+1$ double lines and $2m+1$ single lines.

Equations (25) and (26) constitute the initial equations for the analysis of our problem. They differ from the corresponding equations of Refs. 2 and 3 in that the coefficients in the right part depend on x. In addition, it is necessary also to take into account in (25) and (26) the effects of reflection from the boundary of the classically accessible region. Changing from x to a new variable s given by

$$ds/dx=1/l_-(x) \tag{27}$$

and introducing

$$\tilde{R}_m(x)=R_m(s)e^{-2i\omega\Delta m},\quad Z_m(x,x')=Z_m(s',s)e^{-2i\omega\Delta m}, \tag{28}$$

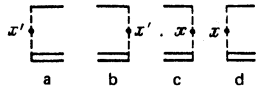

FIG. 4. External vertices that enter in the essential diagrams: they correspond to the following factors: a) $\exp(-iw\Delta(x')/v(x')$, b) $\exp(iw\Delta(x'))/v(x')$, c) $\exp(iw\Delta(x))/v(x)$, d) $\exp(-iw\Delta(x))/v(x)$.

we rewrite (25) and (26) in the form

$$(-imv(s)+d/ds)R_m=m^2(R_{m-1}+R_{m+1}-2R_m), \tag{29}$$

$$(-i(m+1/2)v(s)+d/ds)Z_m=((m+1)^2(Z_{\cdot m+1}-Z_m)-m^2(Z_{\cdot m}-Z_{\cdot m-1})), \tag{30}$$

$$v(s)=2\omega\frac{d}{ds}\Delta(s). \tag{31}$$

The correlator $X(x,x'|\omega)$, after summation over all types of external vertices and after allowance for the normalization factor in (7), is expressed in terms of the quantities defined above in the following manner:

$$X_s(x,x'|\omega)=\frac{2}{v(x)}\sum_{mm'}P_{m'}(s')Z_{m'm}(s',s)P_m(s), \tag{32}$$

$$P_m=1/2(R_m+R_{m+1}),\quad \bar{P}_m=1/2(\bar{R}_m+\bar{R}_{m+1}), \tag{33}$$

where $\bar{R}_m$ is the right-hand side. The equation for $\bar{R}_m$ differs from (29) in the sign of the derivative with respect to s.

Using (23) and (27), we can connect explicitly the variables

$$2\alpha s=\ln(1+2\alpha x/l_-^0),\quad \alpha=Fl_-^0/2\varepsilon, \tag{34}$$

where α is the principal dimensionless parameter of the problem and characterizes the external-field strength. The external frequency ω enters in (29) and (30) in the following dimensionless fashion:

$$\nu=\beta e^{\alpha s},\quad \beta=2\omega\tau_0,\quad \tau_0=l_-^0/v_0. \tag{35}$$

We investigate now the influence of the effects of reflection from the boundary of the classically accessible region. For the diagrams selected above it is expressed by a boundary condition that must be imposed on Eq. (29). According to the preceding section, the boundary condition for R_m can be written in the form

$$\tilde{R}_m(x)|_{x=-\varepsilon/F}=(-i)^m(i)^m e^{-2i\Delta(x)m},\quad R(s)|_{s=s_c\to-\infty}=1. \tag{36}$$

In the absence of an external field, the boundary condition for R_m takes the form[4]

$$R_m(x\to-\infty)=\delta_{n0}. \tag{37}$$

The condition (36) should actually be imposed far from the turning point, where the quasiclassical approximation used above is valid. It is natural to expect, however, the initial condition to influence the character of the dependence of R on m only in a small vicinity of s_c. Far from s_c, R as a function of m and s should assume a universal character.

Another manifestation of the reflection effects may be the appearance of external vertices of a new type, shown in Fig. 5. They did not occur at $F=0$, since a particle moving off to the left was unable to return to the point x. All the essential impurity vertices change the number of aparticles in the cross section only in even fashion. At $F\neq 0$ the particle can return to x after reflection from the boundary (see Fig. 2). To this end, however, it must reach the point x_c. The probability of this process decreases with the difference $(x-x_c)$ quite rapidly. To verify this, let us calculate the left-hand side with participation of the indicated vertices, for example for the case when there is only forward scattering $[l_t(x)=0]$. The numbers m_1 and m_2 of the single and double lines contained in $R_{m_1m_2}$ is then $m_1=m_2\pm 2$. Following the same principles which were used in the derivation of (25) and (26), we can obtain

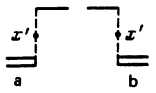

FIG. 5. Type of external vertices due to effects of reflection from the potential barrier, the vertices correspond to the factors: a) $\exp(2i\xi^{3/2}/3)/v(x')$, b) $\exp(-2i\xi^{3/2}/3)/v(x)$.

for $R_{m_1m_2}$ the following equation:

$$dR_{m_1m_2}/dx=-2(m_1-m_2)^2R_{m_1m_2}/l_+(x). \tag{38}$$

The coefficient in the right-hand side of (38) is determined by the number of methods of joining the impurity vertices shown in Fig. 3(a), 3(a'), 3(b), 3(b') and 3(d).

$$-2m_1\frac{1}{2l_+}-2m_2\frac{1}{2l_+}-m_1(2m_1-1)\frac{1}{l_+}-m_2(2m_2-1)\frac{1}{l_+}$$
$$+2m_1m_2\frac{1}{l_+}=-\frac{2(m_1-m_2)^2}{l_+}.$$

It follows from (38) that

$$R_{m_1m_2}(x)=R^0_{m_1m_2}\left(\frac{1+2\alpha x_c/l_+^0}{1+2\alpha x/l_+^0}\right)^\lambda,\quad \lambda=\frac{(m_1-m_2)^2}{\alpha}, \tag{39}$$

and at $x\gg l_+^0\gg x_c$

$$R_{m_1m_2}\approx R^0_{m_1m_2}\left(\frac{l_+^0}{x}\right)^\lambda\ll 1.$$

Consequently, far from the boundary we can neglect diagrams containing the vertices shown in Fig. 5.

We note that Eqs. (25)–(29) admit of a transition to $F\to 0$.

4. ASYMPTOTIC SOLUTION OF EQUATIONS AS $\omega\to 0$

Equations (29)–(32) can be analyzed with the aid of the generating function formalism. We denote by R without a subscript the generating function for R_m:

$$R(\rho,s)=\sum_1^\infty R_m(s)\rho^{m-1},$$
$$R_{m+1}(s)=\frac{1}{m!}\frac{\partial^m R(\rho,s)}{\partial\rho^m}\Big|_{\rho=0} \tag{40}$$

and define the function A as

$$A(u,s)=R(\rho,s)/(1+u)^2,\quad \rho=u/(1+u). \tag{41}$$

From (29) we can obtain the equation

$$\frac{\partial A}{\partial s}=\frac{1}{(1+u)^2}+\frac{\partial}{\partial u}\left(u(u+1)\frac{\partial A}{\partial u}\right)+iv(s)\frac{\partial}{\partial u}(u(u+1)A). \tag{42}$$

We must add to in initial conditions, which in the case of total reflection (36) take the form

$$A(u,s)|_{s=s_c}=(1-\rho)=1/(1+u), \tag{43}$$

where $s_c\to\infty$.

The dependence of A on s stems from the initial conditions and from the dependence of ν on s in (42). It can consequently be represented in the form

$$A(u,s)=A(u,\nu)+A(u,\nu,s). \tag{44}$$

Since we expect the dependence on the initial condition to hold only near s_c, we have at $|s-s_c|\gg 1$

$$A(u,\nu,s)\to 0,\quad A(u,s)\to A(u,\nu).$$

It will be shown below that this limit is exponential in s.

The behavior of the correlator (32) at long times is determined by the low frequencies. In the limit as $\nu\to 0$, the main contribution to (32) comes from diagrams with large $m\sim 1/\nu$. In accord with (40), they correspond to $A(u,s)$ as $u\to\infty$. We introduce therefore the variable $y=-iv(u+1)$ and rewrite (42) in the form

$$\alpha yA'+\partial A/\partial s=(-iv)^2/y^2+(y^2(A'-A))'+iv(y(A'-A))'. \tag{45}$$

The prime denotes here the partial derivative with respect to y. The solution of (45) in the region $y>y_0(u>u_0)$, where y_0 is chosen such that $\nu\ll y_0\ll 1$ ($1\ll u_0\ll 1/\nu$) can be sought in the form of an expansion in powers of ν:

$$A(y,s)=\sum_1^\infty(-iv)^nA_n(y,s). \tag{46}$$

To obtain the asymptotic form of (32) as $\nu\to 0$ it suffices to retain in (46) the first term of the expansion. We represent it, in accord with (44) in the form

$$A_1(y,s)=A_1(y)+a_1(y,s). \tag{47}$$

From (45) we have

$$\alpha yA_1'=(y^2A_1')'-(y^2A_1)', \tag{48}$$
$$\alpha ya_1'+\partial a_1/\partial s=(y^2a_1')'-(y^2a_1)'. \tag{49}$$

Equation (48) can be integrated directly:

$$A_1=c_1y^\alpha e^\nu+\frac{c_2}{y}\int_0^\infty\frac{e^{-\nu\xi}}{(1+\xi)^{2+\alpha}}d\xi. \tag{50}$$

The integration constant c_1 above must be set equal to zero by virtue of the condition that A_1 be regular at infinity. The constant c_2 is obtained from the condition that (51) must be matched with the region $u<u_0(y<y_0)$. Here $\nu u\ll 1$, therefore in the right-hand side of (42) we can neglect the last term

$$\frac{\partial A_0}{\partial s}=\frac{1}{(1+u)^2}+\frac{\partial}{\partial u}\left(u(u+1)\frac{\partial A_0}{\partial u}\right). \tag{51}$$

The solution of (51) is represented in the form

$$A_0(u,s)=A_0(u)+a_0(u,s), \tag{52}$$

where $A_0(u)$, which is the solution of (51) that is regular at zero, is of the form

$$A_0=1/(1+u)=-iv/y. \tag{53}$$

Consequently, the boundary condition for (51) is written in the form

$$A_1(y\to 0)=1/y. \tag{54}$$

From this we find that $c_2=1+\alpha$, and the solution (50) can be rewritten as

$$A_1=\frac{1+\alpha}{y}\Psi(1,-\alpha;y), \tag{55}$$

where Ψ is a confluent hypergeometric function.

We determine now $a_0(u,s)$ and $a_1(y,s)$. In accordance with Ref. 4 we can obtain for $a_0(u,s)$ in the region $u<u_0$

$$a_0(u,s)=8\int_0^\infty \lambda\, d\lambda\, \mathrm{th}\, 2\pi\lambda F\left(\frac{1+i\lambda}{2},\frac{1-i\lambda}{2};1,-u\right)$$
$$\times\exp\left[-(s-s_c)\frac{1+\lambda^2}{4}\right]\int_0^{u_0} du_1 F\left(\frac{1+i\lambda}{2},\frac{1-i\lambda}{2};1,-u_1\right)$$
$$\times(A(u_1,s_c)-A_0(u_1)), \qquad (56)$$

where F is a hypergeometric function. It follows from (56) that $a_0(u,s)\to 0$ as $s_c\to-\infty$, and hence the solution (51) at $|s-s_c|\gg 1$ does not depend on the initial conditions. In the case (43), according to (53), $A(u,s_c)$ is equal to $A_0(u)$ and consequently $a_0(u,s)\equiv 0$.

We carry out in (49) the Laplace transformation:

$$a_\sigma(y)=\int_0^\infty \exp[-\sigma(s-s_c)]\,a_1(y,s-s_c)\,d(s-s_c),$$
$$y^2a_\sigma''+y(-y+2-\alpha)a_\sigma'+(-2y-\sigma-\alpha)a_\sigma=f. \qquad (57)$$

Here $f=-a_1(y,s_c)$ is obtained by expanding the initial conditions in powers of ν, and in the case of (43) we have

$$f=-1/y+A_1=-y^\alpha\Psi(1,1-\alpha;y). \qquad (58)$$

Substituting $a_\sigma=y^k w$, we reduce (57) to the form

$$yw''-(\delta-y)w'-\gamma w=fx^{-(k+1)}; \qquad (59)$$
$$2k=\alpha-1+i\lambda,\quad 2\gamma=\alpha+3+i\lambda,$$
$$\delta=1+i\lambda,\quad -\lambda^2=(1+\alpha)^2+4\sigma. \qquad (60)$$

The solution of (59) can be obtained with the aid of the confluent hypergeometric functions $\Phi(\gamma,\delta;y)$, which is regular at zero, and $\Psi(\gamma,\delta;y)$, which is regular at infinity. We have

$$w(y)=-\Psi(\gamma,\delta;y)\frac{\Gamma(\gamma)}{\Gamma(\delta)}\int_{y_0}^{y}\frac{dz}{z^{2+k-\delta}}f(z)\Phi(\gamma,\delta;z)e^{-z}$$
$$-\Phi(\gamma,\delta;y)\frac{\Gamma(\gamma)}{\Gamma(\delta)}\int_{y}^{\infty}\frac{dz}{z^{k+2-\delta}}f(z)\Psi(\gamma,\delta;z)e^{-z}. \qquad (61)$$

To obtain the inverse Laplace transform of (57) we must calculate an integral of the type

$$I=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\sigma\exp[\sigma(s-s_c)]\,(zy)^{i\lambda/2}$$
$$\times\Gamma\left(\frac{\alpha+3+i\lambda}{2}\right)\Big/\Gamma(1+i\lambda)\,\Psi\left(\frac{\alpha+3+i\lambda}{2},1+i\lambda;y\right)\Phi\left(\frac{\alpha+3+i\lambda}{2};1+i\lambda;z\right).$$

At any finite value of y and z, the only singularity of the integrand in the left half-plane is the branch point $\sigma=-(1+\alpha)^2/4$. Shifting the integration contour so that it follows the edges of the cut drawn from $-(\alpha+1)^2/4$ to $-\infty$, we obtain for I, after changing over to the variable λ,

$$I=\frac{1}{4\pi i}\int_0^\infty\lambda\, d\lambda\exp\left[-\frac{(s-s_c)}{4}((1+\alpha)^2+\lambda^2)\right]\Psi\left(\frac{\alpha+3+i\lambda}{2};1+i\lambda;y\right)$$
$$\times(xz)^{i\lambda/2}\left\{\Gamma\left(\frac{\alpha+3+i\lambda}{2}\right)\Phi\left(\frac{\alpha+3+i\lambda}{2};1+i\lambda;z\right)\Big/\Gamma(1+i\lambda)\right.$$
$$\left.-\Gamma\left(\frac{\alpha+3-i\lambda}{2}\right)z^{-i\lambda}\Phi\left(\frac{\alpha+3+i\lambda}{2};1+i\lambda;z\right)\Big/\Gamma(1-i\lambda)\right\}. \qquad (62)$$

We have used above the following identity for confluent hypergeometric functions [see formula (6.7) (6) of Ref. 8]:

$$z^{i\lambda/2}\Psi\left(\frac{\alpha+3+i\lambda}{2};1+i\lambda;z\right)=z^{-i\lambda/2}\Psi\left(\frac{\alpha+3-i\lambda}{2};1-i\lambda;z\right).$$

The expression in the curly brackets under the integral sign in (62) can be rewritten by using the properties of the Γ functions in the form

$$\frac{i}{\lambda}\left\{\Gamma(-i\lambda)\Phi\left(\frac{\alpha+3+i\lambda}{2};1+i\lambda;z\right)\Big/\Gamma\left(\frac{\alpha+3-i\lambda}{2}\right)\right.$$
$$\left.+\Gamma(-i\lambda)z^{-i\lambda}\Phi\left(\frac{\alpha+3-i\lambda}{2};1-i\lambda;z\right)\Big/\Gamma\left(\frac{\alpha+3+i\lambda}{2}\right)\right\}$$
$$\times\Gamma\left(\frac{\alpha+3+i\lambda}{2}\right)\Gamma\left(\frac{\alpha+3-i\lambda}{2}\right)\Big/\Gamma(i\lambda)\Gamma(-i\lambda).$$

The expression in the curly brackets is equivalent, according to [6.5(7)] of Ref. 8, to $\Psi[(\alpha+3+i\lambda)/2;\,1+i\lambda;z]$.

Gathering now everything together, we obtain ultimately

$$a_1(y,s)=\int_0^\infty\lambda\, d\lambda\,\mathrm{sh}\,\pi\lambda\exp\left[-\frac{(s-s_c)}{4}(\lambda^2+(1+\alpha)^2)\right]$$
$$\times y^{(\alpha-1)/2+i\lambda/2}\Gamma(\gamma)\Gamma(\gamma+1-\delta)\Psi(\gamma,\delta;y)\cdot$$
$$\times\int_{y_0}^\infty dz\, z^{-(\alpha+1-i\lambda)/2}e^{-z}f(z)\Psi(\gamma,\delta;z). \qquad (63)$$

It follows hence that $a_1(y,s)$ decreases exponentially as $s_c\to-\infty$ and consequently $A_1(y,s)$ does not depend on the initial conditions and is equal to $A_1(y)$ from (55). In the case (58), the parameter y_0 in (63) can tend to zero; the resultant integral with respect to z in (63) remains converging.

This property is possessed also by the succeeding terms of the expansion (47), since the corresponding equations have the same structure as (48) and (49). Let us obtain, in particular, $A_2(y)$:

$$2\alpha A_2+\alpha y A_2'=(y^2(A_2'-A_2))'-\varphi',$$
$$\varphi=1/y-yA_1+yA_1'. \qquad (64)$$

When $c_2=1+\alpha$ is chosen in (50), the singularity $1/y$ in φ is cancelled and the solution of (64), which is integrable at zero, takes the form

$$A_2=-\frac{\Gamma((\alpha+3)/2+\mu)}{\Gamma(1+2\mu)}y^{(\alpha-1)/2+\mu}\left\{\Psi\left(\frac{\alpha+3}{2}+\mu;1+2\mu;y\right)\right.$$
$$\times\int_0^y dz\, e^{-z}z^{-(\alpha+1)/2+\mu}\varphi'\Phi\left(\frac{\alpha+3}{2}+\mu;1+2\mu;z\right)$$
$$\left.+\Phi\left(\frac{\alpha+3}{2}+\mu;1+2\mu;y\right)\int_y^\infty dz e^{-z}z^{-(\alpha+1)/2+\mu}\varphi'\Psi\left(\frac{\alpha+3}{2}+\mu;1+2\mu;z\right)\right\}, \qquad (65)$$

where $\mu=(3\alpha+1)/2$. We note that $A_2=0$ in the absence of a field ($\varphi'=0$).

An expression for the left-hand side can be obtained from the following considerations. It was shown that far from the boundary the left-hand side depends on s via ν, i.e., $A=A(y,\nu,\alpha)$. The dependence of A on α is due to the presence in the equation for A of a derivative with respect to s ($\nu'=\alpha\nu$). The equation for the right-hand side of $\bar A$ differs from the corresponding equation for the left-hand side only in the sign of the derivative with respect to s, and therefore $\bar A=A(y,\nu,-\alpha)$ and, in particular

$$\bar A_1=(1-\alpha)\Psi(1,\alpha,y)=\frac{1}{y}-\frac{1}{y^2}\int_0^\infty\frac{e^{-\zeta}d\zeta}{(\zeta+y)^{1-\alpha}} \qquad (66)$$

We note that the boundary condition (54) for $\bar A$ from (66) will be satisfied only at $\alpha<1$. The solution (50),

which satisfies expression (54) in the case $\alpha>1$, is of the form

$$\bar{A}_1=\frac{1}{y}\Phi(1,\alpha,y)=(\alpha-1)\int_0^1 e^{y\zeta}(1-\zeta)^{\alpha-2}d\zeta. \tag{67}$$

Formula (67), however, is a solution that increases exponentially as $y\to\infty$ ($A_1\approx\Gamma(\alpha)e^y/y^\alpha$), and this leads to diverging terms of the expansion for the correlator (32). Similar divergences appear in (32) at $\alpha>1$ also on account of the right-hand side of A_1. Thus, the expansion (47) becomes meaningless at $\alpha>1$.

We turn now to the central part of Z. We rewrite first expression (32) in the form

$$X_\omega(s,s'|\omega)=\frac{2e^{-\alpha s}}{\nu_0}\sum_m Q_m(s',s)\bar{P}_m(s),$$
$$Q_m(s,s')=\sum_{m'}P_{m'}(s')Z_{m'm}(s',s). \tag{68}$$

Equation (30) for Z can then be directly rewritten for Q_m. The initial condition for Z was

$$Z_{m'm}|_{s=s'+0}=\delta_{mm'}$$

and consequently

$$Q_m|_{s=s'+0}=P_m(s).$$

We introduce the generating function

$$Q(r;s,s')=\sum_0^\infty r^m Q_m(s,s'),\quad B(u;s,s')=\frac{1}{1+u}Q\left(\frac{u}{1+u};s,s'\right). \tag{69}$$

To determine B we obtain from (30) and (69) the following equations:

$$\alpha yB'+\frac{\partial B}{\partial s}=(y^2B')'-y^2B'-yB+i\nu(yB')'-i\nu yB'-\frac{i\nu}{2}B, \tag{70}$$

$$B(y;s',s)|_{s=s'+0}=\frac{-i\nu}{2y}+\left(\frac{-y}{i\nu}-\frac{1}{2}\right)A(y,s'). \tag{71}$$

We seek the solution of (70) at $\nu\ll1$ in the form of a series in powers of ν, retaining only the zeroth term of the expansion [$B=\Phi+O(\nu)$]. Denoting the Laplace transform of Φ by $\Phi_\omega(y)$, we obtain from (70) [see (57)]

$$y^2\Phi_\omega''+y(-y+2-\alpha)\Phi_\omega'+(-y-\sigma)\Phi_\omega=-(1+\alpha)\Psi(1,-\alpha,y), \tag{72}$$

where the right-hand side is the zeroth term of the expansion of (71) in powers of ν.

Equation (72) is similar to (57) and therefore its solution can be obtain in similar fashion. We write down the result [cf. (63)]

$$\Phi(y,s-s')=\frac{1}{4\pi^2}\int_0^\infty\lambda\,\mathrm{sh}\,\pi\lambda d\lambda\exp\left[-\frac{(s-s')}{4}((1-\alpha)^2+\lambda^2)\right]$$
$$\times y^{(\alpha-1+i\lambda)/2}N\Gamma\left(\frac{\alpha+1+i\lambda}{2}\right)\Gamma\left(\frac{\alpha+1-i\lambda}{2}\right)\Psi\left(\frac{1+\alpha+i\lambda}{2};1+i\lambda;y\right), \tag{73}$$

$$N=(1+\alpha)\int_{y_0}^\infty dz z^{-(\alpha+1-i\lambda)/2}e^{-z}\Psi\left(\frac{\alpha+1+i\lambda}{2};1+i\lambda;z\right)\Psi(1,-\alpha,z). \tag{74}$$

Here $y_0\to0$. The integral (74) remains converging in this case if $\alpha<1$.

To calculate (74) we use the integral representation {formula [6.5(12)] of Ref. 8}:

$$\Psi(u,v;z)=\frac{1}{\Gamma(u)}\int_0^\infty e^{-z\zeta}\zeta^{u-1}(1+\zeta)^{v-u-1}d\zeta. \tag{75}$$

Substituting it in (74) and integrating with respect to z, we get

$$N=(1+\alpha)\left[\Gamma\left(\frac{1-\alpha+i\lambda}{2}\right)/\Gamma\left(\frac{1+\alpha-i\lambda}{2}\right)\right]\int_0^\infty\frac{d\zeta}{(1+\zeta)^{2+\alpha}}\int_0^\infty d\zeta_1$$
$$\times\zeta_1^{(\alpha-1)/2+i\lambda/2}(1+\zeta_1)^{-(1+\alpha)/2-i\lambda/2}(1+\zeta+\zeta_1)^{(\alpha-1)/2-i\lambda/2}. \tag{76}$$

The integral with respect to ζ_1 reduces to a confluent hypergeometric function {see [2.1(12)] from Ref. 8}:

$$(1+\zeta)^{(\alpha-1)/2-i\lambda/2}\Gamma\left(\frac{1-\alpha-i\lambda}{2}\right)\Gamma\left(\frac{1+\alpha+i\lambda}{2}\right)$$
$$\times F\left(\frac{\alpha+1+i\lambda}{2},\frac{1-\alpha+i\lambda}{2};1;\frac{\zeta}{1+\zeta}\right).$$

Making in (76) the substituting $(1+\xi)=(1-\zeta)^{-1}$, we rewrite N in the form

$$N=(1+\alpha)\Gamma\left(\frac{1-\alpha+i\lambda}{2}\right)\Gamma\left(\frac{1-\alpha-i\lambda}{2}\right)\int_0^1 d\xi(1-\xi)^{(1+\alpha+i\lambda)/2}$$
$$\times F\left(\frac{1+\alpha+i\lambda}{2},\frac{1-\alpha-i\lambda}{2};1;\xi\right).$$

The resultant integral with respect to ξ can be calculated exactly {see Ref. 9, formula [7.512(4)]}:

$$N=(1+\alpha)\Gamma\left(\frac{1-\alpha+i\lambda}{2}\right)\Gamma\left(\frac{1-\alpha-i\lambda}{2}\right)\Gamma\left(\frac{1+\alpha+i\lambda}{2}+1\right)$$
$$\times\Gamma\left(\frac{1+\alpha-i\lambda}{2}+1\right)\frac{\pi^2((1+\alpha)^2+\lambda^2)}{2\Gamma(1+\alpha)\Gamma(2+\alpha)(\mathrm{ch}\,\pi\lambda+\cos\pi\alpha)}. \tag{77}$$

We turn now to the correlator (68) and express it in terms of the generating functions A and B. Following Abrikosov and Ryzhkin,[4] we have

$$\sum_0^\infty(\bar{R}_m+\bar{R}_{m+1})Q_{m+1}=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\sum_n(\bar{R}_n+\bar{R}_{n+1})e^{in\varphi},$$
$$\sum_0^\infty Q_m e^{-im\varphi}=\frac{1}{2\pi}\int_0^{2\pi}Q(r)(1+(1+\rho)\bar{R}(\rho))d\varphi, \tag{78}$$

where $\rho=\exp(i\varphi)$ and $r=\exp(-i\varphi)$. Substituting (78) in (68) and changing from $\bar{R}$ and Q to $\bar{A}$ and B, and to the variable

$$u=r/(1-r)=(1-e^{i\varphi})^{-1}$$

we express the function $X(s,s')$ at $s>s'$ in the form

$$X_\omega(s,s'|\omega)=\frac{ie^{-\alpha s}}{\pi\nu_0}\int_\Gamma du\left(-\frac{1}{2u}-\frac{1+2u}{2}\bar{A}(-(u+1))\right)B(u;s,s), \tag{79}$$

where the contour Γ is the straight line $\mathrm{Re}\,u=-\frac{1}{2}$.

The asymptotic form of (79) is $\nu\to0$ can be obtained by using the first terms of the expansions of $\bar{A}$ and B in terms of ν:

$$X_\omega(s,s'|\omega)=\frac{ie^{-\alpha s}}{\pi\nu_0}(1-\alpha)\int_\Gamma duB_0(u)\int_0^\infty\frac{d\zeta e^{-i\nu u\zeta}}{(1+\zeta)^{2-\alpha}}$$
$$=(1-\alpha)\frac{2e^{-\alpha s}}{\nu_0}\int_0^\infty d\zeta e^{i\nu\zeta}\int_\Gamma duB_0(u)\frac{(-u)^{1-\alpha}}{(\zeta-u)^{2-\alpha}}. \tag{80}$$

The function $B_0(u)=\Phi[-i\nu(u+1)]$ is, in accord with the definition (77), regular in the right half-plane from the straight line $\mathrm{Re}\,u=-\frac{1}{2}$. Consequently the contour Γ in (80) can be turned to the right (Fig. 6). A contribution will then be made by integration along the upper and lower edges of the cut drawn from zero to $u=\zeta$, and integration along a circle with radius $\rho\to0$ and with a

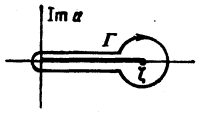

FIG. 6. Integration contour Γ that ensures contributions to expressions (80) and (81).

center at the point $a=\zeta$. The result is

$$X_+(s,s'|\omega)=\frac{2e^{-\alpha s}}{\pi\nu_0}\sin\pi\alpha\int_0^\infty d\zeta\, e^{i\nu\zeta}\Big\{\frac{\zeta^{1-\alpha}}{\rho^{2-\alpha}}B_0(\zeta)-\int_0^{\zeta-\rho}\frac{du\,u^{1-\alpha}}{(\zeta-u)^{2-\alpha}}B_0(u)\Big\}=\frac{4e^{-\alpha s}}{\pi\nu_0}\sin\pi\alpha\int_0^1\frac{du}{(1-u)^{1-\alpha}}\times\frac{d}{du}\Big(u^{-\alpha}\int_0^\infty d\zeta\, e^{i\nu\zeta/u}B_0(\zeta)\Big). \tag{81}$$

Rotating now the contour of integration with respect to ζ along the imaginary axis and making the change of variable $\zeta=y/\nu$, we reduce (81) to the form

$$X_+(s,s'|\omega)=\frac{4e^{-\alpha s}\sin\pi\alpha}{(-i\nu)\pi\nu_0}\int_0^1\frac{du}{(1-u)^{1-\alpha}}\frac{d}{du}\Big(u^{-\alpha}\int_0^\infty dy e^{-y/u}\Phi(y)\Big), \tag{82}$$

where $\Phi(y)$ is given by (73). Substitution of (73) in (82) results in an integral of the form

$$\int_0^\infty dy\, e^{-y/u}y^{(\alpha-1+i\lambda)/2}\Psi\Big(\frac{1+\alpha+i\lambda}{2};\,1+i\lambda;\,y\Big)=\Gamma\Big(\frac{1+\alpha+i\lambda}{2}\Big)\Gamma\Big(\frac{1+\alpha-i\lambda}{2}\Big)\frac{u^{(1+\alpha+i\lambda)/2}}{\Gamma(1+\alpha)}F\Big(\frac{1+\alpha+i\lambda}{2},\frac{1+\alpha+i\lambda}{2};\,1+\alpha;\,1-u\Big).$$

We have used in its calculation the integral representation (75).

The next integral in (82) is with respect to u. Making the substitution $u=1-\eta$, we have

$$J=\int_0\frac{d\eta}{\eta^{1-\alpha}}\frac{d}{d\eta}\Big((1-\eta)^{(1-\alpha+i\lambda)/2}F\Big(\frac{1+\alpha+i\lambda}{2},\frac{1+\alpha+i\lambda}{2};\,1+\alpha;\,\eta\Big)\Big)$$
$$=\int_0 d\eta\,\eta^{\alpha-1}\Big\{\frac{-1+\alpha-i\lambda}{2}(1-\eta)^{(-1-\alpha+i\lambda)/2}F\Big(\frac{1+\alpha+i\lambda}{2},\frac{1+\alpha+i\lambda}{2};\,1+\alpha;\,\eta\Big)+\frac{1}{1+\alpha}\Big(\Big(\frac{1+\alpha}{2}\Big)^2+\Big(\frac{i\lambda}{2}\Big)^2\Big)(1-\eta)^{(1-\alpha+i\lambda)/2}F\Big(\frac{1+\alpha+i\lambda}{2}+1,\frac{1+\alpha+i\lambda}{2}+1;\,2+\alpha;\,\eta\Big)\Big\}. \tag{83}$$

In the calculation of the derivative of the confluent hypergeometric function above, we used the identity (2.17) of Ref. 8.

The integrals in (83) can be calculated exactly {see Ref. 9, formula [7.512(3)]}:

$$J=\frac{\alpha-1-i\lambda}{2}\Gamma(1+\alpha)\Gamma(\alpha)\Gamma\Big(\frac{1-\alpha+i\lambda}{2}\Big)\Gamma\Big(\frac{1-\alpha-i\lambda}{2}\Big)\times\Big(\Gamma\Big(\frac{1+\alpha+i\lambda}{2}\Big)\Big)^{-1}\Big(\Gamma\Big(\frac{1+\alpha-i\lambda}{2}\Big)\Big)^{-1}$$
$$+\Big(\Big(\frac{1+\alpha}{2}\Big)^2+\Big(\frac{i\lambda}{2}\Big)^2\Big)\frac{1}{1+\alpha}\Gamma(2+\alpha)\Gamma(\alpha)\Gamma\Big(\frac{3-\alpha+i\lambda}{2}\Big)\Gamma\Big(\frac{1-\alpha-i\lambda}{2}\Big)\times\Big(\Gamma\Big(\frac{3+\alpha+i\lambda}{2}\Big)\Big)^{-1}\Big(\Gamma\Big(\frac{1+\alpha-i\lambda}{2}\Big)\Big)^{-1}=-\Gamma(\alpha)\Gamma(1+\alpha)\Gamma\Big(\frac{1-\alpha+i\lambda}{2}\Big)$$
$$\times\Gamma\Big(\frac{1-\alpha-i\lambda}{2}\Big)\Big(\Gamma\Big(\frac{1+\alpha+i\lambda}{2}\Big)\Big)^{-1}\Big(\Gamma\Big(\frac{1+\alpha-i\lambda}{2}\Big)\Big)^{-1}\Big(\Big(\frac{1+\alpha}{2}\Big)^2+\frac{\lambda^2}{4}\Big)$$

Gathering now everything together and using the properties of the Γ functions, we obtain for Eq. (82)

$$X(s,s'|\omega)=\frac{1}{i\omega}\frac{\sin\pi\alpha}{16\pi\alpha}\frac{\pi^2}{l_-^0}e^{-2\alpha s}\exp\Big[-\frac{(s-s')}{4}(1-\alpha)^2\Big]\times\int_0^\infty\lambda\,\mathrm{sh}\,\pi\lambda\frac{(1+\alpha^2+\lambda^2)^2-4\alpha^2}{(\mathrm{ch}\,\pi\lambda+\cos\pi\alpha)^2}\exp\Big[-\frac{(s-s')\lambda^2}{4}\Big]d\lambda. \tag{84}$$

It follows therefore that the asymptotic behavior of the density correlator at long times is indeed determined by the singularity at $\omega=0$, and the form of the singularity corresponds to localized states.[6] For the distribution function of the electronic density of these states we obtain from (84)

$$p_\infty(x|x')=\frac{\pi\sin\pi\alpha}{16\alpha l_-(x)}\int_0^\infty\lambda d\lambda\,\mathrm{sh}\,\pi\lambda\times\frac{((1-\alpha)^2+\lambda^2)((1+\alpha)^2+\lambda^2)}{(\mathrm{ch}\,\pi\lambda+\cos\pi\alpha)^2}\Big|\frac{\varepsilon+Fx'}{\varepsilon+Fx}\Big|^{[(1-\alpha)^2+\lambda^2]/8\alpha}. \tag{85}$$

We recall that here $x>x'$ and the point x' corresponds to the position of the particle at the instant when the electric field is applied.

The form of p_∞ at $x<x'$ can be determined from the symmetry of the problem

$$p_\infty(x|x';\,\alpha)=p_\infty(-x|-x';\,-\alpha).$$

From this we obtain for p_∞ an expression suitable for any ratio of x and x':

$$p_\infty(x|x')=\frac{\pi\sin\pi\alpha}{16\alpha l_-(x)}\exp\Big[-\frac{1+\alpha^2}{4}|\tau|+\frac{\tau\alpha}{2}\Big]\times\int_0^\infty\lambda\,\mathrm{sh}\,\pi\lambda\frac{(1+\alpha^2+\lambda^2)^2-4\alpha^2}{(\mathrm{ch}\,\pi\lambda+\cos\pi\alpha)^2}\exp\Big(-\frac{|\tau|\lambda^2}{4}\Big)d\lambda,\qquad \tau=\frac{1}{2\alpha}\ln\frac{\varepsilon+Fx}{\varepsilon+Fx'}. \tag{86}$$

The function $p_\infty(x|x')$ should satisfy, according to (2) the normalization condition

$$\int_{-\infty}^\infty p_\infty(x|x')dx=l_-^0\int_{-\infty}^\infty e^{2\alpha s}p_\infty(s|s')ds=1.$$

Substituting here Eq. (86) and integrating, we verify this equality.

5. CONCLUSION

It is of interest to study the dependences of (84)–(86) on the electric field. We note first that at $F\neq0$ the problem is homogeneous in energy space, and therefore the dependence on the energy and on the coordinate can enter only in the form $\varepsilon+Fx$. This property indeed holds for Eqs. (84)–(86), since $l_-(x)$ can be expressed, according to (23), in the form

$$l_-(x)=l_-^0e^{2\alpha s}=(\varepsilon+Fx)/mU_0 \tag{87}$$

and the parameter α can be expressed in a form with ε excluded:

$$\alpha=Fl_-^0/2\varepsilon=F/2mU_0. \tag{88}$$

Turning on the electric field alters substantially the form of the distribution function of the localized states.

We calculate the asymptotic form of (86) at large $|\tau| \gg 1$:

$$p_\infty(s|s') = \frac{C}{l_-^0}\exp\left[-2\alpha s - \frac{1+\alpha^2}{4}|\tau| + \frac{\tau\alpha}{2}\right]\int_0^\infty \lambda^2 d\lambda \exp\left(\frac{-|\tau|\lambda^2}{4}\right)$$

$$= \frac{2\pi^{1/2}C}{l_-^0|\tau|^{3/2}}\exp\left[-2\alpha s - \frac{1+\alpha^2}{4}|\tau| + \frac{\tau\alpha}{2}\right], \qquad (89)$$

$$C = \frac{\pi^2 \sin\pi\alpha}{16\alpha}\frac{(1-\alpha^2)^2}{(1+\cos\pi\alpha)^2}.$$

We see therefore that the decrease of p_∞ at large positive x (the field is directed along x) is not exponential as in the absence of a field, but follows the slower power law:

$$p_\infty(x|x') = \frac{2(2\pi\alpha)^{1/2}C}{x}\left|\ln\frac{x}{x'}\right|^{-3/2}\left|\frac{x'}{x}\right|^{(1-\alpha)^2/8\alpha}. \qquad (90)$$

In the direction counter to the field, the decrease of p_∞ is faster, so that p_∞ vanishes at finite x:

$$p_\infty(x|x') = \frac{2(2\pi\alpha)^{1/2}C}{x-x_c}\left|\ln\frac{x-x_c}{x'}\right|^{-3/2}\left(\frac{x-x_c}{x'}\right)^{(1+\alpha)^2/8\alpha}, \qquad (91)$$

where $x_c = -l_-^0/2\alpha$.

This difference in the behavior of p_∞ means the appearance of an average displacement of the electron, or of an induced dipole moment:

$$\bar{x} = \int_{-\infty}^{\infty} p_\infty(x|x')(x-x')\,dx = \frac{(l_-^0)^2}{2\alpha}\int_{-\infty}^{\infty} e^{2\alpha s}ds\,(e^{2\alpha s} - e^{2\alpha s'})\,p_\infty(s|s').$$

Substituting here (86), we find that $\bar{x}$ exists only at $\alpha \leq \alpha_1 = 5 - 2\sqrt{6} \approx 0.11$ and is equal to

$$\bar{x} = 24\pi \sin\pi\alpha I_\alpha l_-(x), \qquad (92)$$

$$I_\alpha = \int_0^\infty \lambda d\lambda \frac{\operatorname{sh}\pi\lambda}{(\operatorname{ch}\pi\lambda + \cos\pi\alpha)^2}\frac{1+\alpha^2+\lambda^2}{(1+\alpha^2+\lambda^2)^2 - (10\alpha)^2},$$

where I_α is a slowly varying function of α and can be approximately written as $I_\alpha \approx I_0 = \zeta(3)/2\pi$. It is seen from (92) that the average displacement increases with increasing distance from the boundary of the classically accessible region, and remains finite at $\alpha = \alpha_1$ because of the additional logarithmic factor $\bar{x}$ in (90).

The higher moments of $p_\infty(x|x')$ can be determined in similar fashion. The power-law rather than exponential decrease of p_∞ as $|x| \to \infty$ causes then all the moments above a certain order to diverge. Further, as follows from (90), as $\alpha \to 1$ all the moments will diverge. Therefore the localization length $\bar{l}_n$, defined as

$$\bar{l}_n = \left(\frac{1}{\Gamma(n+1)}\int_{-\infty}^{\infty}(x-\bar{x})^n p_\infty(x|x')\,dx\right)^{1/n}, \qquad (93)$$

diverges for an $n > 0$ as $\alpha \to 1$.

It was indicated above that the presented analysis procedure is not valid in the region $\alpha > 1$. The solution of the initial equations should have in this case a substantially different character, which we have not succeeded in establishing. Nevertheless, when account is taken of the results obtained for $p_\infty(x|x')$ as $\alpha \to 1$ we can conclude that the electronic states will be delocalized. To investigate correctly this problem at $\alpha > 1$, we must introduce a mechanism that ensures the electron energy relaxation.

We now present a physical interpretation of the results. As indicated by Berezinskiĭ, the reason for the localization of the electrons in a one-dimensional disordered system is the multiple interference of the electrons when they are scattered by impurity centers. In the one-dimensional case the electron returning to a given point after a number of scattering events has a zero phase shift, since the phase of the electron wave function can be defined at each point of a one-dimensional filament.[1] Since the electric field is irrotational, the interference effects remain in force when it is applied. The additional phase shift that the electron acquires as it moves along the field is cancelled out on the return trip counter to the field. However, since the electric field influences the electron motion, the scattering characteristics themselves are altered. The electron moving along the field acquires an additional velocity. Therefore, if it is now scattered by an impurity, the corresponding scattering amplitude is already smaller. The farther the electrons move away to the right of the given point, the weaker the scattering [see (87)]. At a certain field value, the mean free path decreases rapidly enough, so that the electron can go off to infinity.

For impurity scattering, the parameter U_0 in (19) can be written in the form

$$U_0 = cV^2(2p_0), \qquad (94)$$

where c is the density of the scattering centers and $V(q)$ is the Fourier transform of the potential of an individual center. For a short-range potential we have $V(q) = V_0 = \text{const}$. When (94) is taken into account, the parameter α in (88) can be written in the form

$$\alpha = F/2mcV_0^2 = F/F_{imp},$$

where F_{imp} has the meaning of the force exerted on the electron by the impurities. In fact, F_{imp} can be represented in the form

$$F_{imp} = \frac{\Delta p}{\Delta t}w = 2p_0 wcv_0 = 2mcV_0^2, \quad w = \left(\frac{V_0}{v_0+V_0}\right)^2 = \left(\frac{V_0}{v_0}\right)^2 \ll 1,$$

where Δp is the characteristic change of the electron momentum during the time Δt between two collision events, and w is the backscattering probability. Delocalization takes place when $F > F_{imp}$.

In conclusion, the author thanks Yu. A. Firsov for interest in the work.

[1)] We note that it remains also as $F \to 0$ (free motion)

$$G_0^{\pm}(x, x'|\varepsilon) = \mp i(m/2\varepsilon)^{1/2}\{\exp[\pm i(2m\varepsilon)^{1/2}|x-x'|] \mp i\exp[\pm i(2m\varepsilon)^{1/2}|x'+x|]\}. \qquad (16')$$

The turning point $x_c \to \infty$ and the form of (16) correspond to stationary states of the standing-wave type.

[1]N. F. Mott and W. D. Twose, Adv. Phys. 10, 107 (1961). N. F. Mott, *ibid.* 16, 49 (1967).
[2]V. L. Berezinskiĭ, Zh. Eksp. Teor. Fiz. 65, 1251 (1973) [Sov. Phys. JETP 38, 620 (1973)].
[3]A. A. Gogolin V. I. Mel'nikov, and E. I. Rashba, *ibid.* 69, 327 (1975) [42, 168 (1975)].
[4]A. A. Abrikosov and I. A. Ryzhkin, *ibid.* 71, 1204 (1976) [44, 630 (1976)].

[5]A. A. Gogolin, *ibid.* 71, 1912 (1976) [44, 1003 (1976)].
[6]V. L. Berzinskiĭ and L. P. Gor'kov, *ibid.* 77, 2498 (1979) [50, 1209 (1979)].
[7]L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics, Nonrelativistic Theory), Fizmatgiz, 1963, 24 [Pergamon, 1968].
[8]A. Erdelyi, ed., Higher Transcendental Functions, McGraw, 1953.
[9]I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov (Tables of Integrals), Nauka, 1971.

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Statistical features of avalanche ionization of wide-gap insulators by laser radiation under conditions of shortage of initiating electrons

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A theoretical investigation is made of the characteristics of the process of laser damage to transparent insulators as a result of development of an electron avalanche when this avalanche is retarded by a shortage of initiating electrons. The process of simultaneous formation of initiating electrons and of an avalanche is considered from the statistical point of view. The expected dependence of the breakdown threshold on the volume of the interaction zone is analyzed.

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There are several mechanisms of photoconductivity in wide-gap solid insulators. Free carriers may appear as a result of interband transitions involving the absorption of one or more photons (photoionization of the matrix), as a result of ionization of impurities and defects, and also because of development of impact avalanche ionization. This last mechanism has a number of important features. Firstly, an avalanche-like increase in the number of free carriers practically always results in optical breakdown of an insulator. Secondly, this mechanism requires the presence of a certain initial number of free carriers in the zone of interaction of radiation with matter. These carriers may form in wide-gap insulators at reasonable temperatures only as a result of photoionization of the matrix or defects.

The process of impact avalanche ionization is usually considered beginning from a certain initial density $N_0 \approx 10^7 - 10^{12}$ cm^{-3}, the actual value depending on the laser radiation frequency. It is found that the breakdown threshold field depends very weakly on N_0 (Refs. 1 and 2). However, it is obvious that this conclusion can only be valid if a sufficient number of initiating carriers is present in the interaction zone of volume V. Since in many laser damage experiments[3,4] the size of the interaction region is very small, $V \lesssim 10^{-10}$ cm^3, there may be a situation when the threshold field is governed not so much by the rate of development of an avalanche but by the rate of creation of initiating electrons. However, it should be stressed particularly that the situation under discussion here is radically different from the usual case of breakdown as a result of multiphoton ionization, although in both cases the damage threshold is governed by the photoionization rate. In fact, in the case under discussion, we have

$$\gamma > W,$$

where γ is the avalanche growth constant and W is the frequency of formation of carriers as a result of photoionization, whereas in the case of damage as a result of multiphoton ionization the opposite inequality is satisfied.

We shall consider the influence of an insufficient number of initiating electrons on the threshold (critical) damage field and the associated statistical features of the optical breakdown process. In particular, we shall discuss the temperature and size dependences of the breakdown threshold under these conditions.

1. ELECTRON AVALANCHE IN THE PRESENCE OF A SMALL NUMBER OF INITIATING ELECTRONS

Let us assume that a sequence of impact ionizations by some specific electron is a Poisson process, namely that the probability of a certain electron making k ionizations in a time t is

$$P_k(t) = \frac{(t/\tau)^k}{k!} \exp\left(-\frac{t}{\tau}\right),$$

where $\tau \approx \gamma^{-1}$. An analysis of the usual conditions for the process to be of the Poisson type leads to the conclusion that this is a reasonable assumption if

$$\tau \gg 1/\nu,$$

where ν is the frequency of electron–phonon collisions.

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