Nonlinear dynamics of rays in inhomogeneous media

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Nonlinear dynamics of rays in periodically inhomogeneous waveguide channels is investigated. Two fundamentally new effects that appear as a result of nonlinear interaction of the rays with the periodic inhomogeneity of the medium are considered. The first, modulation localization of the beam, is the analog of nonlinear resonance in classical mechanics. The second effect constitutes formation of a stochastic region of the waveguide channel on account of the stochastic instability of the beam. This phenomenon decreases the effective transverse dimension of the waveguide. The widths of the regions of the ray modulation localization and of the stochastic layer are calculated for a waveguide channel with a solitonlike refractive-index profile. The possibility of ray stochastization in three-dimensional waveguide channels in the absence of inhomogeneity along the ray propagation axis is demonstrated.

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1. INTRODUCTION

Sound waves in the ocean and in the atmosphere and radio waves in the ionosphere can propagate over very large distances.[1-3] The reason is that the wave phase velocity in the corresponding medium has a nonmonotonic dependence on a certain coordinate (e.g., on the depth in the ocean). The result is a natural waveguide channel in which the wave propagates. Many effects influence the propagation distance of a wave in a waveguide channel. An important factor that limits this distance is the inhomogeneity of the waveguide channel along its axis.

We present an example of a regular and moreover periodic large-scale inhomogeneity in the ionosphere. It is known that electro-magnetic waves propagate with different velocities in the region illuminated by the sun and in the shadow region. Thus, a wave propagating around the earth encounters a periodic inhomogeneity with a period of the order of the earth's radius.

If the wave propagation time is long, even small inhomogeneities can lead to cumulative effects, which manifest themselves in changes of the wave intensity and of the wavefront structure. It is clear that calculation of effects of this kind is beyond the scope of ordinary (in a certain sense) perturbation theory. The purpose of the present paper is to analyze cumulative effects of wave propagation in a periodically inhomogeneous medium. The real parameters of the medium are here such that geometric-optics approximation can be used.[1,2]

The problem of wave propagation in natural media can thus be reduced to a corresponding ray-dynamics problem, or to the equivalent problem of the motion of a nonlinear oscillator under the influence of a periodic perturbation. We consider in this paper two fundamentally new effects. The first can be called modulation localization of the ray as a result of a unique nonlinear interaction of the ray with the periodic inhomogeneity of the medium. This is the analog of nonlinear resonance in classical mechanics. The second effect can be called stochastic instability of the ray.[1)] Its gist is that under certain conditions the action of a periodic (not random!) inhomogeneity stochastizes the motion of the representative point of the ray in a plane perpendicular to the propagation direction. This phenomenon is analogous to stochastic instability in nonlinear mechanics.[5] It disturbs a certain region of the waveguide channel. The rays are radiated out of this region, and the effective length of the waveguide channel is decreased.

We note that the method developed below for the analysis of ray dynamics is one of the new applications of the stochastic-instability phenomenon first observed in cyclic accelerators. The phase-focusing mechanism proposed by Veksler and McMillan is an example of trapping of an accelerated-particle beam into a nonlinear resonance. The interaction of such resonances, which result from any kind of cyclic inhomogeneity, can lead to a stochastic departure of the particles from the volume.[5] Despite, however, the close analogy between stochastic instability of particles in cyclic accelerators and the corresponding ray instability in say, cyclic resonators, there is also a fundamental difference between these problems, owing to the wave nature of the field, for which ray dynamics is only a certain approximation.

It is helpful to point out one more analogy between the problem considered below and other quantum objects to which considerable attention is being paid of late. We have in mind the analysis of the quantum dynamics of a system that has in the classical limit a stochastic motion (with displacements in phase space).[12]

2. EQUATIONS OF RAY TRAJECTORY

We describe the ray trajectory using the Hamiltonian formalism.[6] Let the z axis coincide with that of the waveguide channel and let the coordinate $\rho=(x,y)$ lie in a plane transverse to the z axis. Then the ray coordinates (x,y,z) satisfy the Hamilton equations

$$d\mathbf{p}/dz=-\partial H/\partial\mathbf{r},\quad d\mathbf{r}/dz=\partial H/\partial\mathbf{p},\tag{2.1}$$

where H is the Hamilton function

$$H=-[n^2(\mathbf{r},z)-\mathbf{p}^2]^{1/2},\tag{2.2}$$

p is the momentum and equals

$$\mathbf{p}=n\dot{\mathbf{r}}/(1+\dot{\mathbf{r}}^2)^{1/2};\quad \dot{\mathbf{r}}=d\mathbf{r}/dz.\tag{2.3}$$

Here $n=n(\mathbf{r},z)$ is the refractive index.

We represent n in the form

$$n^2(\mathbf{r},z)=n^2(\mathbf{r})+\varepsilon v(\mathbf{r},z), \tag{2.4}$$

where $n(\mathbf{r})$ corresponds to the regular case (homogeneous in z), and the perturbation εv takes into account the influence of the inhomogeneity. The quantity $\varepsilon \ll 1$ is a dimensionless parameter of the perturbation. Taking its smallness into account, we can write (2.2) in the form

$$\begin{aligned} &H=H_0(\mathbf{r},\mathbf{p})+\varepsilon V(\mathbf{r},\mathbf{p},z),\\ &H_0(\mathbf{r},\mathbf{p})=-[n^2(\mathbf{r})-\mathbf{p}^2]^{1/2},\\ &V(\mathbf{r},\mathbf{p},z)=v(\mathbf{r},z)/2H_0. \end{aligned} \tag{2.5}$$

The unperturbed motion of the ray is determined by the Hamiltonian H_0. The problem of the ray trajectory is thus reduced in the inhomogeneous case to the equivalent problem of the action of a nonstationary perturbation on a particle that executes finite motion described by the Hamiltonian H_0. The role of the time is assumed in this case by the variable z, and it is the inhomogeneity along this variable which produces the perturbation.

We begin the investigation of the problem with the simplest planar case, when n is independent of y. Equations (2.1) and (2.5) then become

$$\begin{aligned} &\dot{p}=-\partial H/\partial x, \quad \dot{x}=\partial H/\partial p, \quad p=p_x,\\ &H=H_0+\varepsilon V(x,p,z), \quad H_0=-[n^2(x)-p^2]^{1/2}. \end{aligned} \tag{2.6}$$

We describe first the unperturbed motion of the ray. A typical plot of $n(x)$ vs. the transverse coordinate x is shown in Fig. 1. The values of $n_\pm$ determine the corresponding asymptotic forms of $n(x)$ as $x\to\pm\infty$. We assume next for simplicity

$$n_+=n_-=n_\infty. \tag{2.7}$$

It is easy to show that the unperturbed phase trajectories, defined by Eqs. (2.6) and (2.7) at $V\equiv 0$, take the form shown in Fig. 2. Trajectories of type 1 correspond to finite motions of the ray along the x axis. These are in fact the rays propagating in the natural waveguide channel. Trajectories of type 3 correspond to infinite motion of the ray along x. The two trajectory types are demarcated by the separatrix 2.

Let E be the energy of the equivalent particle corresponding to the value of the integral of motion $H_0(x,p)=E$. It follows then from (2.5) and (2.7) that on the separatrix we have

$$E=-n_\infty. \tag{2.8}$$

In the finite-motion region

$$-n_0<E<-n_\infty, \tag{2.9}$$

and in the region of infinite motion

$$-n_\infty<E<0. \tag{2.10}$$

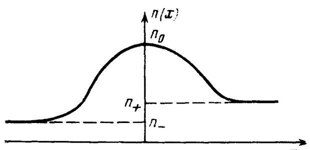

FIG. 1. Typical dependence of the refractive index $n(x)$ on x.

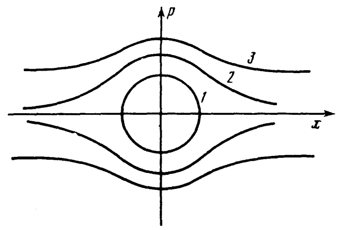

FIG. 2. Trajectory of rays in the phase plane (p,x) in the unperturbed case: 1—wave rays, 2—separatrix, 3—radiated rays.

We introduce in the region (2.9), where the motion is not periodic, the action and angle variables (I,ϑ):

$$\begin{aligned} &I=\frac{1}{2\pi}\oint p\,dx, \quad \vartheta=\frac{\partial S(x,I)}{\partial I},\\ &S(x,I)=\int^x p\,dx, \quad p=[n^2(x)-E^2]^{1/2}. \end{aligned} \tag{2.11}$$

In terms of the new variables $H_0=H_0(I)$, and

$$\omega(I)=dH_0(I)/dI \tag{2.12}$$

is the nonlinear frequency of oscillations of the ray along x relative to the z axis of the waveguide channel. In addition, we can write the Fourier expansion

$$x=\sum_m x_m e^{im\vartheta}, \quad p=\sum_m p_m e^{im\vartheta}. \tag{2.13}$$

Using the variables I and ϑ, we rewrite according to (2.11) the equations of motion in the form

$$\begin{aligned} &H=H_0(I)+\varepsilon V(I,\vartheta,z),\\ &\dot{I}=-\varepsilon\partial V/\partial\vartheta, \quad \dot{\vartheta}=\omega(I)+\varepsilon\partial V/\partial I. \end{aligned} \tag{2.14}$$

As already noted, we are interested in the case of a perturbation potential that is periodic in V. Then

$$V(I,\vartheta,z)=\frac{1}{2}\sum_{m,s=-\infty}^{+\infty} V_{ms}(I)e^{im\vartheta+is\varkappa z}+\text{c.c.}, \tag{2.15}$$

where $\varkappa$ is the "frequency" of the perturbation ($2\pi/\varkappa$ is the spatial period of the perturbation), and c.c. stands for terms that are the complex conjugates of the preceding ones.

It is seen from (2.14) and (2.15) that the strongest influence of the perturbation takes place in the resonant case, i.e., upon satisfaction of the condition

$$m\omega(I)+s\varkappa=0. \tag{2.16}$$

This case is called nonlinear resonance and was described in detail earlier.[5] We present directly the result for the motion of a ray in the vicinity of one resonance. Let I_0 be that value of I at which the condition (2.16) is satisfied for definite values of m and s. It follows then from (2.14) (Ref. 5) that ϑ satisfies the equation

$$\begin{aligned} &\ddot{\vartheta}+\Omega^2\sin\vartheta=0,\\ &\Omega^2=\varepsilon m^2\left|\frac{d\omega(I_0)}{dI}V_{ms}\right|. \end{aligned} \tag{2.17}$$

This is the pendulum equation and describes nonlinear periodic modulation of the phase ϑ of the ray with frequency Ω. The principal condition for the validity of (2.17) is of the form

$$\frac{\varepsilon}{\alpha} \ll 1, \quad \alpha = \frac{d\omega}{dI}\frac{I}{\omega}, \tag{2.18}$$

i.e., a sufficiently strong nonlinearity.

From (2.17) and (2.14) we get the region of localization of the nonlinear resonance[5]:

$$\Delta I = \max|I - I_0| = 4\left|\varepsilon V_{ms} \Big/ \frac{d\omega(I_0)}{dI}\right|^{1/2},$$
$$\Delta\omega = \max|\omega(I) - \omega(I_0)| = \Delta I \frac{d\omega}{dI} = 4\left|\varepsilon V_{ms} \frac{d\omega(I_0)}{dI}\right|^{1/2}. \tag{2.19}$$

The physical meaning of the foregoing results was the following. As already noted, in the absence of perturbation the ray trajectory oscillates along the x axis at a frequency $\omega(I)$. In the vicinity of the resonant frequency $\omega(I_0)$ there is superimposed on this motion an additional ray modulation along z. The modulation amplitude is determined by expressions in (2.19). The amplitude in turn determines also the region of ray localization in the plane perpendicular to z. An additional waveguide channel is thus produced along the trajectory of the unperturbed ray that corresponds to the action of I_0, and has an effective dimension ΔI. The rays trapped in this channel oscillate in it, with frequency Ω, about the unperturbed trajectory. This leads in turn to a periodic modulation of the group velocity of the wave field. To verify this, we consider the connection between the frequency $\omega(I)$ and the group velocity of the corresponding wave of the unperturbed problem.

In the employed notation

$$k_z = k_0|H_0|, \quad k_0 = \nu/c, \tag{2.20}$$

where ν is the cyclic frequency of the wave field in vacuum, and k_z is the wave number of the field along the z axis. From (2.20) we have

$$\frac{1}{v_g} = \frac{dk_z}{d\nu} = \frac{|H_0|}{c} + k_0 \frac{d|H_0|}{dI}\frac{dI}{d\nu}.$$

The quantization conditions in the waveguide yield

$$I = \frac{1}{2\pi}\oint p\,dx \approx m\frac{c}{\nu} \quad (m = 1, 2, \ldots).$$

We get therefore, taking (2.12) into account,

$$v_g = c/[|H_0| + \omega(I)I]. \tag{2.21}$$

In the unperturbed case $I = 0$ and the action I, hence also v_g, does not depend on z. In the presence of a perturbation we have from (2.21) and (2.14), on account of the inhomogeneities,

$$\frac{dv_g}{dz} = \frac{dv_g}{dI}\dot{I} = \frac{\varepsilon}{c}v_g^2 I \frac{d\omega(I)}{dI}\frac{\partial V}{\partial\vartheta}$$
$$= \frac{m}{c}\varepsilon\alpha\omega(I_0)v_g^2 V_{ms}(I_0)\sin(m\vartheta + s\varkappa z). \tag{2.22}$$

It follows from (2.22) that in the inhomogeneous case v_g is also periodically modulated in z with a modulation period Ω and with a modulation amplitude

$$\Delta v_g = \frac{dv_g}{dI}\Delta I = \frac{4}{c}v_g^2 I_0\left[\varepsilon V_{ms}\frac{d\omega(I_0)}{dI}\right]^{1/2} = \frac{v_g^2 I_0 \Delta\omega}{c}. \tag{2.23}$$

A similar appearance of modulation oscillations can be postulated also for the phase velocity $v_p = c/H_0|$ of the wave. In the general case, the number of regions in which modulation localization of the beam takes place is connected with the number of possible resonances of type (2.16). We shall discuss this in greater detail in Sec. 4.

3. RAY TRAJECTORY FOR SOLITONLIKE $n(x)$ PROFILE

We consider by way of example a waveguide channel with the following refractive-index profile:

$$n^2(x) = n_\infty^2 + \mu^2/\mathrm{ch}^2(x/a), \tag{3.1}$$

where $\mu = (n_0^2 - n_\infty^2)^{1/2}$ characterizes the depth of the corresponding potential well, and a its effective width. Substituting (3.1) in (2.11) and (2.12) we have

$$\begin{gathered} I = I_s(1-\beta), \quad \beta = [H_0^2(I) - n_\infty^2]^{1/2}/\mu, \\ H_0(I) = -[n_\infty^2 + \mu^2(1 - I/I_s)^2]^{1/2}, \quad I_s = a\mu, \\ \omega(I) = (I_s - I)/[a^2|H_0(I)|], \\ x = a\,\mathrm{arcsh}\,[\beta^{-1}(1-\beta^2)^{1/2}\cos\vartheta], \\ p = \mu\beta(1-\beta^2)^{1/2}\sin\vartheta/(\cos^2\vartheta + \beta^2\sin^2\vartheta)^{1/2}, \\ \vartheta = \omega(I)z + \vartheta_0. \end{gathered} \tag{3.2}$$

It follows from (3.2) that the separatrix corresponds to the action I_s, with

$$\begin{gathered} H_0(I_s) = -n_\infty, \quad \omega(I_s) = 0, \\ H_0(0) = -n_0, \quad \omega(0) \equiv \omega_0 = \mu/an_0. \end{gathered} \tag{3.3}$$

At $|I_s - I| \ll I_s$ we obtain from (3.2) the behavior of the frequency near the separatrix:

$$\omega(I) \approx \frac{1}{a^2 n_\infty}(I_s - I) = \left(\frac{2}{a^2 n_\infty}\right)^{1/2}[H(I_s) - H_0(I)]^{1/2}. \tag{3.4}$$

The expansion of the momentum p in a Fourier series (see the Appendix) is of the form

$$p = \eta\beta(1-\beta^2)^{1/2}\sum_{m=0}^{\infty} A_m \sin[(2m+1)\omega z],$$
$$A_m = (-1)^m \frac{(2m-1)!!}{2^{3m}m!}(1-\beta^2)^m$$
$$\times F\left(m + \frac{1}{2}, m + \frac{3}{2}, 2m+2, 1-\beta^2\right). \tag{3.5}$$

From (3.1)–(3.4) we obtain asymptotic expressions for the spectrum:

$$\begin{gathered} A_m \sim (1-\beta)^m, \quad \beta \to 1 \quad (I \to 0), \\ A_m \sim \left(\frac{2}{\pi}\right)^{1/2}\frac{(1-\beta^2)^m}{m\beta}, \quad \beta \to 0 \quad (I \to I_s). \end{gathered} \tag{3.6}$$

According to the second equation of (3.6), the ray-oscillation spectrum is cut off exponentially near the separatrix at numbers $m \gtrsim N$, where

$$N \approx 1/\beta^2 = \mu^2/[H_0(I_s) - H_0(I)] = \omega_0^2/\omega^2(I) \gg 1. \tag{3.7}$$

We take the perturbation to be periodic deviations of the waveguide axis from the straight z axis. In this case

$$n(x, z) = n(x - f(z)),$$

where $f(z)$ describes the deviation of the axis from the coordinate x in the plane $z = \mathrm{const}$. At small deviations ($\varepsilon = |f|/a \ll 1$) we have

$$n^2(x - f(z)) = n^2(x) + f(z)\frac{\partial n^2(x)}{\partial x} + \ldots.$$

Retaining the first two terms, we obtain for the perturbation the expression

$$\varepsilon V(x, z) = \frac{\varepsilon}{2H_0}v(x, z) = \frac{1}{2H_0}\frac{\partial H_0^2}{\partial x} = -\frac{dp}{dz}f(z).$$

Using the expansion (3.5), we represent the perturbation in the form

$$\varepsilon V(x,z)=-\mu\omega\beta(1-\beta^2)^{1/2}f(z)\sum_{m=0}^{\infty}(m+{}^1/_2)A_m e^{i(2m+1)\omega z}+\text{c.c.}$$

Let $f(z)=f_0\cos\varkappa z$, where $2\pi/\varkappa$ is the spatial period of the perturbation. For the matrix elements in (2.15) we then obtain

$$V_{m_s}=-I_s\omega\beta(1-\beta^2)^{1/2}(m+{}^1/_2)A_m,\quad \varepsilon=f_0/a. \tag{3.8}$$

The resonance condition takes in this case the form

$$(2m+1)\omega=\varkappa. \tag{3.9}$$

The distance between the nearest resonances is

$$\delta\omega=|\omega_{m+1}-\omega_m|=2\varkappa/(2m+1)(2m+3). \tag{3.10}$$

In particular, near the separatrix, where large values of m are possible, we have

$$\delta\omega\sim 2\omega^2/\varkappa. \tag{3.11}$$

We estimate now the width of the modulation localization of the ray in two limiting cases: near the region of small ray oscillations and near the separatrix. At small oscillations, the greatest influence on the behavior of the ray is exerted according to (3.9) by the resonance with $m=0$, i.e., $\omega(I)=\varkappa$. Taking into account the asymptotic form (3.6) for the spectrum A_m as $\beta\to 1$, as well as (2.18), we obtain

$$\Delta I/I_s=4\varepsilon^{1/2}\varkappa a n_0^2(1-\varkappa/\omega_0)^{1/4}/(\mu n_\infty),$$
$$\frac{\Delta\omega}{\omega_0}=\frac{\Delta I}{\omega_0}\frac{d\omega(I)}{dI}=4\varepsilon^{1/2}\frac{\varkappa a n_\infty}{\mu}\left(1-\frac{\varkappa}{\omega_0}\right)^{1/4}. \tag{3.12}$$

We note that nonlinear resonance sets in under condition (2.18) when the nonlinearity is large enough. This means according to (3.2) and (2.18) that

$$\alpha\approx 1-\omega(I)/\omega_0=1-\varkappa/\omega_0>\varepsilon. \tag{3.13}$$

If the values of $\varkappa$ are so close to ω_0 that condition (3.13) does not hold, then the main resonance must be considered in a different manner. It follows thus from (3.12) and (3.13) that the regions of modulation localization of the ray have a certain lower bound on the order of $\varepsilon^{3/4}$.

Near the separatrix it is possible also for high harmonics of the ray oscillations to enter into resonance. For a resonance $m\gg 1$ we can obtain the following estimates of ΔI and $\Delta\omega$

$$\frac{\Delta I}{I_s}=4\left(\frac{2}{\pi}\right)^{1/4}\left(\varepsilon\frac{\omega}{\omega_0}\frac{n_\infty}{n_0}\right)^{1/2},$$
$$\frac{\Delta\omega}{\omega_0}=4\left(\frac{2}{\pi}\right)^{1/4}\left(\varepsilon\frac{\omega}{\omega_0}\frac{n_0}{n_\infty}\right)^{1/2}. \tag{3.14}$$

We shall use these expressions in the next section.

4. FORMATION OF STOCHASTIC LAYER NEAR THE SEPARATRIX

It was shown earlier[5,7,8] that the perturbation produces in the vicinity of the separatrix a so-called stochastic layer, in which the particle trajectories are random. The main feature of this phenomenon is that the stochastic layer is produced under periodic perturbations of arbitrary form and magnitude, and only the width of the layer is determined by the character of the perturbation. No analogous property was observed up to now for ray trajectories. At the same time, formation of a stochastic layer in which random motion of the rays takes place can have, as will be made clear

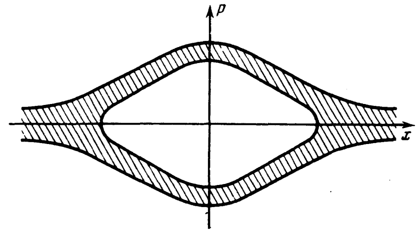

FIG. 3. Stochastic layer in phase plane of rays (shaded region).

below, important physical consequences.

It was already noted in Sec. 3 that the analysis of modulation localization of a ray is valid when other resonances are far enough from the considered one. Near the separatrix, the distance $\delta\omega$ between the resonances is determined by (3.11). As the separatrix is approached $\omega\to 0$ and $\delta\omega$ decreases rapidly. In this case the regions of the nonlinear resonances can overlap. It is known (the Chirikov criterion[5,9,10]) that overlap of resonances causes the trajectories to become stochastic. This condition is written in the form

$$K=(\Delta\omega/\delta\omega)^2>1. \tag{4.1}$$

Let us consider inequality (4.1), using formulas (3.14) and (3.11). We have

$$K=4\left(\frac{2}{\pi}\right)^{1/2}\varepsilon\left(\frac{\varkappa}{\omega_0}\right)^2\left(\frac{\omega_0}{\omega}\right)^3\frac{n_0}{n_\infty}. \tag{4.2}$$

It follows therefore that at all ε and $\varkappa$ the value of K increases as the separatrix is approached, $\omega\to 0$. Therefore, starting with a certain value $\bar{\omega}$, K reaches unity and the criterion (4.1) begins to be satisfied. Thus, a stochastic layer (shaded region in Fig. 3) is produced in the vicinity of the separatrix and its boundary is determined from the condition $K=1$. Hence

$$\bar{\omega}/\omega_0=[4(2/\pi)^{1/2}\varepsilon(\varkappa/\omega_0)^2(n_0/n_\infty)]^{1/3}. \tag{4.3}$$

With the aid of (4.3) and (3.4) we obtain the width of the stochastic layer in terms of variables I and H

$$\frac{\overline{\delta I}}{I_s}=\frac{n_\infty}{n_0}\frac{\bar{\omega}}{\omega_0}=\left[4\left(\frac{2}{\pi}\right)^{1/2}\varepsilon\left(\frac{\varkappa}{\omega_0}\right)^2\left(\frac{n_\infty}{n_0}\right)^2\right]^{1/3},$$
$$\frac{\overline{\delta H}}{H_0(I_s)-H_0(I)}=\left(\frac{\overline{\delta I}}{I_s}\right)^2=\left[4\left(\frac{2}{\pi}\right)^{1/2}\varepsilon\left(\frac{\varkappa}{\omega_0}\right)^2\left(\frac{n_\infty}{n_0}\right)^2\right]^{2/3}. \tag{4.4}$$

If the initial state of the ray is such that its action I lies in the region (4.4), this means that its motion in space along z is of the diffusion type. The diffusion causes the ray to reach the region near the unperturbed separatrix and to be "radiated" out of the waveguide region. The described phenomenon is similar to the existence of a "loss cone" of particles in magnetic traps. Thus, the action of an inhomogeneity as a perturbation leads to a decrease of the effective width of the waveguide channel. We note also that field modes with higher numbers end up in the region of the stochastic layer. Radiation of the field from the stochastic layer means therefore also the filtering of the high waveguide-channel modes.

It was shown earlier[7,8] that the particle traverses the stochastic-layer width within a short characteristic time of the order of the period of the small oscillations. In our case it means that the emission of the ray from the stochastic layer of the waveguide region takes place over a length

$$l \sim 2\pi/\omega_0 = 2\pi a n_0/\mu = 2\pi a n_0/(n_0^2 - n_\infty^2)^{1/2}. \tag{4.5}$$

In real situation, μ reaches values $\sim 10^{-1} n_0$, and the emission length exceeds by an order of magnitude the width a of the waveguide channel. It must also be noted that over the length l defined by (4.5), information is lost concerning that part of the initial wave front which is produced by rays that are far enough from the axis.

5. STOCHASTIZATION OF RAYS IN A THREE-DIMENSIONAL WAVEGUIDE CHANNEL

We consider now a waveguide channel in which the refractive index depends on both transverse coordinates x and y. The ray motion is then analogous to the motion of a particle with two degrees of freedom in a potential well. In the presence of periodic perturbations along the z axis, just as in the case of a flat waveguide channel, modulation localization of the beam will be observed, as well as stochastization of the rays near the separatrix. In contrast to the planar case, the presence of two degrees of freedom in the considered waveguide channel can give rise to another ray-stochastization mechanism, which is possible even in the absence of perturbations along the z axis. The last phenomenon is connected with the stochastic disturbance of one of the integrals of motion of the ray because of interaction of different degrees of freedom[11] (see also the review[12]). Interaction of two degrees of freedom is possible at certain dependences of the refractive index $n(\mathbf{r})$ on the transverse coordinates x and y.

We consider by way of example a waveguide channel with a refractive index of the form

$$n^2(x,y) = n_0^2 - [(x^2+y^2)/a^2 + 2(x^2y - {}^1/_3 y^3)/a^3], \tag{5.1}$$

where a has the dimension of length and is of the order of the width of the waveguide channel. The profile (5.1) can be used to approximate the refractive index near the waveguide axis, where $n(x,y)$ is a maximum. For rays propagating near the waveguide axis and at small angles to this axis, the following conditions are satisfied:

$$\begin{gathered} |(x^2+y^2)/a^2 + 2(x^2y - y^3/3)/a^3| \ll n_0^2, \\ |\mathbf{p}| \ll n_0, \quad \mathbf{p} = n_0\dot{\mathbf{r}}. \end{gathered} \tag{5.2}$$

When the conditions (5.2) are satisfied, the Hamiltonian (2.2) can be expressed in the form (the paraxial approximation)

$$\begin{gathered} H = -n_0 + H', \\ H' = {}^1/_2 n_0 (\dot{x}^2 + \dot{y}^2) + {}^1/_2 [(x^2+y^2)/a^2 + 2(x^2y - {}^1/_3 y^3)/a^3]. \end{gathered} \tag{5.3}$$

At $n_0 = 1$ and $a = 1, H'$ coincides with the Hamiltonian of the Henon-Heiles model.[11] A numerical analysis of the model motion in that paper shows that at values $E' = H'$ smaller than a certain critical E'_c the particle trajectories correspond to a periodic stable motion. Starting with energy values $E' > E'_c$ the trajectories become stochastic, owing to the absence of a second integral of the motion.

The latter means that the trajectories of the rays corresponding to high modes with wave numbers

$$k_z = k_0|H| < k_z^c = k_0|n_0 - E_c'|, \tag{5.4}$$

are stochastic, while trajectories of rays for which

$$k_0|n_0 - E_c'| < k_z < k_0 n_0,$$

are periodic functions of the longitudinal coordinate z.

The region of stochastization of the waveguide rays with respect to H are determined by the condition

$$-|n_0 - E_c'| < H < -n_\infty, \quad n_\infty = \lim_{|\mathbf{r}|\to\infty} n(x,y). \tag{5.5}$$

Just as in the case of a planar waveguide channel with periodic inhomogeneities, the stochastization causes the rays to be diffusively "radiated" out of the waveguide-channel stochastization region, and decreases the effective width of the latter.

6. CONCLUSION

The presented results were obtained within the framework of geometric optics, so that we must discuss the question of the restrictions imposed by wave effects. The main condition for the applicability of geometric optics is a smooth variation of the refractive index $n(\mathbf{r})$ over distances of the order of the wavelength. An additional restriction is imposed by the nonlinearity of the ray oscillations about the waveguide-channel axis.

We note that the system considered by is equivalent (from the wave viewpoint) to a quantum nonlinear system. The coordinate z of the waveguide channel corresponds to the time of the quantum system. The simplest effect of violation of the quasiclassical description of a nonlinear system is connected with the spreading of the wave packet as a result of the nonlinearity. The time of this spreading is

$$t_c \sim 2\pi/[\hbar d\omega(I)/dI].$$

For wave optics this means that the geometric-optics approximation ceases to be valid starting with the distance

$$z > z_c \sim 2\pi k_0/[d\omega(I)/dI].$$

For the example considered in Sec. 3 we have

$$z_c \sim 2\pi k_0 a^2 [n_\infty^2 + \mu^2(1 - I/I_s)^2]^{3/2}/n_\infty^2.$$

The minimum value of z_c is of the order of the diffraction length for a beam with a radius of the order of the effective width of the waveguide channel. The wavelengths considered by us are much shorter than the waveguide-channel width: $a \gg \lambda = 2\pi/k_0$, therefore $z_c \gg l$. Thus, the stochastization of the rays sets in over distances shorter than z_c. However, the question of the range of applicability of the quasiclassical approximation in quantum mechanics, or of geometric optics in wave optics, in those cases when a stochastic instability develops, is at present quite complicated[12] and will not be discussed here.

Let us also touch upon certain questions encountered in very-long-distance wave propagation in the ionosphere. Under certain conditions, around-the-world signals can propagate in the E and F layers.[2] The ion density gradient on the illuminated and obscured sides is an elementary example of periodic inhomogeneity. If the stochasticity conditions are satisfied, the ray should depart from the channel by diffusion, and the characteristic departure length can equal one or two periods.[5] A ray instability (and attenuation) of this type can fully compete with ray scattering by large-scale (200–1000 m) random inhomogeneities. In fact, the diffusion coefficient is in the latter case[2]

$$D_0 \sim 2\cdot 10^{-11} - 10^{-14}\ \mathrm{cm}^{-1}. \tag{6.1}$$

The relative diurnal oscillation of the density ranges from 0.01 at a height of 200 km to 2 at 450 km. Recognizing that the ionosphere channel passes over a height 200–300 km, it is easy to obtain a diffusion coefficient $D \sim 10^{-11} - 10^{-14}$ cm^{-1}, i.e., a value of the same order as (6.1).

Another possibility of stochastic instability of rays with rebounding trajectories is connected with their periodic transitions from the E to the F channel and back. The fact that a nonlinearity, even a small one, can substantially influence in this case the divergence of a wave beam was already noted by Gurevich and Tsedilina.[2]

The mechanism of formation and suppression of the conditions for very-long-distance and around-the-world wave propagation in the ionosphere is at present a rather complicated research object. The arguments advanced above show that the onset of stochastic dynamics of rays in regularly inhomogeneous media may turn out to be significant alongside the other factors that limit the distance over which waves can propagate.

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APPENDIX

Using the expansion

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!\,\Gamma(1/2)} x^n,$$

we write

$$U(x) = \frac{\sin x}{[\cos^2 x + \beta^2 \sin^2 x]^{1/2}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!\,\Gamma(1/2)} (1-\beta^2)^n \sin^{2n+1} x.$$

Next, using the expansion

$$\sin^{2n+1} x = \frac{1}{2^{2n}} \sum_{m=0}^{n} (-1)^m \frac{(2n+1)!}{(n-m)!\,(n+m+1)!} \sin[(2m+1)x],$$

we obtain

$$U(x) = \sum_{m=0}^{\infty} A_m \sin[(2m+1)x],$$

$$A_m = (-1)^m \frac{2(1-\beta^2)^m \Gamma(m+1/2)\Gamma(m+3/2)}{[\Gamma(1/2)]^2 \Gamma(2m+2)} F(m+1/2, m+3/2, 2m+2, 1-\beta^2).$$

We investigate the asymptotic form of A_m as $\beta \to 0$. We use for this purpose the integral representation of the hypergeometric function

$$F(\alpha,\beta,\gamma,x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-ux)^{-\alpha}\, du.$$

Then

$$A_m = \frac{2}{\pi}(1-\beta^2)^m \int_0^1 \left[\frac{u(1-u)}{1-u+\beta^2 u}\right]^{m+1/2} \frac{du}{1-u}.$$

At $\beta = 0$ the integral diverges. We calculate the integral in A_m by the saddle-point method in the assymptotic limit $\beta \to 0$. We write it in the form

$$\int_0^1 e^{\varphi(u)}\, du;$$

$$\varphi(u) = \ln\left[\frac{1}{1-u}\left(\frac{u(1-u)}{1-u+\beta^2 u}\right)^{m+1/2}\right].$$

At $m \gg 1$ and $m\beta \ll 1$ we get

$$\int_0^1 e^{\varphi(u)}\, du \approx (\pi/2)^{1/2}/m\beta.$$

Thus

$$A_m \approx (2/\pi)^{1/2}(1-\beta^2)^m/m\beta.$$

Starting with certain $m \gtrsim N$, the spectrum of A_m decreases exponentially. For the characteristic number N we have

$$N \approx |\ln^{-1}(1-\beta^2)| \approx \beta^{-2}.$$

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