Effective-range approximation for scattering of a particle with nonzero orbital angular momentum

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A study is made of scattering of a slow particle by a potential well capable of holding a weakly bound particle with given orbital angular momentum. On this basis, an effective-range approximation is constructed for nonzero orbital angular momentum.

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The aim of this paper is to generalize the well-known approximate expression in the theory of scattering of a slow particle by a short-range central force field in the s state,

$$k \operatorname{ctg} \delta_0 = -1/a + r_0 k^2/2, \tag{1}$$

to nonzero values of the orbital angular momentum. In Eq. (1), a is the scattering length, δ_0 is the phase shift of the particle wave with zero orbital angular momentum, r_0 is the effective range, which in the special case of a rectangular well is equal to its radius, and $\hbar k$ is the momentum of the particle.

When the force field is a potential well capable of holding a weakly bound particle with energy

$$\varepsilon = -\hbar^2\alpha^2/2m, \quad \alpha r_0 \ll 1, \tag{2}$$

the scattering length is $1/\alpha$ (in the first approximation in the small parameter αr_0).

A characteristic property of the expression (1) is, as is well known, the fact that it is independent of the form of the potential. A dependence on the form of the potential appears only in the terms of higher order in k, which we shall not take into account. Therefore, to derive formula (1) in the presence of a weakly bound state, it is in fact sufficient to calculate scattering by a rectangular potential well of radius r_0 whose depth only slightly exceeds the critical value at which a bound state with angular momentum $l=0$ appears. Such a calculation is made in Problem 3 in §133 in Landau and Lifshitz's book.[1] Our aim is to extend this calculation to the case of nonzero orbital angular momentum l.

The main details of the calculation are contained in the Appendix to the paper. Here, we give the final result:

$$k^{2l+1} \operatorname{ctg} \delta_l = -c_l \frac{k^2+\alpha^2}{2r_0^{2l-1}}, \quad c_l = \frac{2l+1}{2l-1}[(2l-1)!!]^2. \tag{3}$$

For a potential well of arbitrary form, the expression (3) constitutes an effective-range approximation analogous to (1). Note that the case $l=0$ cannot be obtained directly from (3) (see the Appendix). The expression for the scattering amplitude to which (3) leads has the form

$$f_l = -\frac{2r_0^{2l-1}k^{2l}}{c_l(k^2+\alpha^2) + i\cdot 2k^{2l+1} r_0^{2l-1}}. \tag{4}$$

In its energy dependence, this expression is analogous to Eq. (133.15) in Ref. 1. A difference is that (4) contains the effective range r_0, whereas the corresponding expression in Ref. 1 contains the undetermined parameter b, whose nature is not considered. Comparison shows that the parameter b is related to the effective range r_0 by

$$b = \frac{c_l}{2(2m/\hbar^2)^{l-1} r_0^{2l-1}}.$$

We note in passing that in the case considered here when a bound state with energy $-\hbar^2\alpha^2/2m$ exists the expression $k^2+\alpha^2$ does not vanish for real k. Under these conditions, we must ignore the imaginary term in the denominator of (4) and write

$$f_l = -2r_0^{2l-1} k^{2l}/c_l(k^2+\alpha^2). \tag{5}$$

The presence in the denominator of (5) of the expression $k^2+\alpha^2$ is directly related to the circumstance that at the coordinate origin of the complex k plane the Jost function for $\alpha=0$ has a double zero for all $l \neq 0$.[2] If the depth of the well is increased slightly compared with the case when $\alpha=0$, then, as is shown in Ref. 2, the double zero of the Jost function is split into two imaginary zeros at $i\alpha$ and $-i\alpha$. If the well is made shallower compared with the case when $\alpha=0$, then in accordance with Ref. 2 the double zero of the Jost function is split into two complex zeros in the lower half-plane of k situated symmetrically with respect to the imaginary axis and corresponding to a quasi-stationary state with energy $E_0 - i\Gamma/2$. Instead of α^2, the negative quantity $-2mE_0/\hbar^2$ appears in (4).

Under these conditions, the real part of the denominator (4) vanishes at $E=E_0$. Near this value of the energy, it is necessary to take into account the imaginary part of the denominator, since it gives the width. Namely,

$$\Gamma = \frac{4}{c_l} E_0^{l+1/2} \left(\frac{2m}{\hbar^2}\right)^{l-1/2} r_0^{2l-1}. \tag{6}$$

In its energy dependence, this expression is equivalent to Eq. (133.16) in Ref. 1. A new feature in our result compared with the quoted expression is the dependence of Γ on the effective range r_0.

APPENDIX

We consider as a preliminary a particle in a bound state in a rectangular well of depth $|V|$ and radius r_0.

The radial wave function for $r \leq r_0$ is

$$R_l = A j_l(\varkappa r), \quad \varkappa = (U-\alpha^2)^{1/2}, \quad U = 2m\hbar^{-2}|V|, \tag{7}$$

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where j_l is a spherical Bessel function.

Let $\varkappa_0 = (U_0)^{1/2}$ be the critical value at which the level with the given angular momentum appears. The condition for the appearance of the level is

$$j_{l-1}(\varkappa_0 r_0)=0. \quad (8)$$

We set $U = U_0 + \lambda$ and assume $\lambda \ll U_0$. Then

$$\varkappa \approx \varkappa_0 + (\lambda - \alpha^2)/2\varkappa_0. \quad (9)$$

We form the expression

$$F = \left[\frac{1}{r^{l+1} R_l}\frac{d}{dr}(r^{l+1} R_l)\right]_{r=r_0}. \quad (10)$$

Taking into account (8) and (9) and using the well-known properties of the Bessel functions, we obtain in the first approximation

$$F = (\alpha^2 - \lambda) r_0/2. \quad (11)$$

In the region outside the well, $r \geq r_0$, the wave function is

$$R_l = B h_l^{(1)}(i\alpha r), \quad (12)$$

where $h_l^{(1)}$ is a spherical Hankel function. The value of F when (12) is substituted in (10) under the condition $\alpha r_0 \ll 1$ is

$$F = -\alpha^2 r_0/(2l-1). \quad (13)$$

Equating (11) and (13), we find the connection between λ and α:

$$\lambda = \alpha^2 (2l+1)/(2l-1). \quad (14)$$

Note that the expression (13) is valid only for $l \neq 0$. In the case $l = 0$ we have $F = -\alpha$, and therefore the connection between λ and α in the first approximation in αr_0 has the form $\alpha = \lambda r_0/2$. We now consider the scattering problem in which we are interested.

Within the well for $r \leq r_0$ the wave function has, as before, the form (7), but in the expression (9) we have k^2 instead of $-\alpha^2$. Accordingly, instead of (11) we obtain $F = -(k^2 + \lambda) r_0/2$. Substituting the expression (14) for λ, we find

$$F = -\frac{r_0}{2}\left[k^2 + \alpha^2 \frac{2l+1}{2l-1}\right]. \quad (15)$$

Outside the well, the wave function can be represented in the form

$$R_l = B[j_l(kr)\,\mathrm{ctg}\,\delta_l - n_l(kr)] \quad (16)$$

(n_l is a spherical Neumann function).

Calculating F with the function (16) and using the condition $kr \ll 1$, we obtain

$$F = \frac{k^{2l+1} r_0^{2l}\,\mathrm{ctg}\,\delta_l}{[(2l-1)!!]^2} + \frac{k^2 r_0}{2l-1}. \quad (17)$$

Equating (15) and (17), we find

$$k^{2l+1}\,\mathrm{ctg}\,\delta_l = -\frac{2l+1}{2l-1}[(2l-1)!!]^2 \frac{k^2+\alpha^2}{2r_c^{2l-1}}. \quad (18)$$

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[1] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika, Nauka (1974); English translation: Quantum Mechanics, Pergamon Press, Oxford (1974).

[2] Yu. N. Demkov and G. F. Drukarev, Zh. Eksp. Teor. Fiz. 49, 691 (1965) [Sov. Phys. JETP 22, 479 (1966)].

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Motion of atoms and molecules in a resonant light field

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The effect of resonant light pressure on the motion of atoms and molecules is investigated. A kinetic equation is obtained that describes the action exerted on the particles by both the average light-pressure forces and by their fluctuations. The forces due to spontaneous and induced transitions are considered, as well as those due to some combination of both. The kinetic equation is used to describe atom scattering under conditions close to those in experiment [A. Arimonodo, H. Lew, and T. Oka, Phys. Rev. Lett. 43, 753 (1979)]. Also obtained is the temperature of atoms cooled in a standing wave by a light-pressure force of mixed type.

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§1. INTRODUCTION

Resonant particles are acted upon in the field of laser emission by rather appreciable light-pressure forces.[1-3] These forces depend on the intensity of the external field, on the proximity of its frequency to resonance, and on its spatial structure. The light pressure on an individual particle is determined in final analysis by the rate of scattering of the external-field photons.

In a traveling light wave, only spontaneous transitions can contribute to the pressure. In non-uniform fields (e.g., in a standing wave), a stimulated light-pressure force (gradient force) is also produced. Inasmuch as the absorption and emission of external-field quanta are correlated processes in stimulated transitions, the

