Energy transfer and turbulence spectrum when the gradient instability is excited in the ionospheric plasma

S. S. Moiseev, R. Z. Sagdeev, A. V. Tur, and V. V. Yanovskiĭ

Khar'kov Physicotechnical Institute, Academy of Sciences of the Ukrainian SSR
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We study the strong turbulence excited by the gradient instability in the ionospheric plasma. We show that the energy transfer in such a turbulence has the character of spontaneous symmetry breaking with subsequent nonlinear interactions between the excited waves. We find the k^{-3} spectrum of the turbulence.

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INTRODUCTION

It is well known that turbulence appears in a liquid or a plasma because a system with a large number of degrees of freedom tends to get rid as soon as possible of a disequilibrium produced in it. In an incompressible liquid this leads to the excitation of a large number of degrees of freedom and to the appearance of the Kolmogorov cascade along the spectrum with a constant energy flux.[1,2] Plasma turbulence is more diversified, but in many cases it is similar to hydrodynamic turbulence. Examples in which deviations from the usual Kolmogorov scheme of energy transfer along the spectrum are possible are therefore of interest. One such example, which is considered in the present paper, is drift-wave turbulence with unmagnetized ions. Drift-wave turbulence is of interest from a thermodynamic point of view if only because these waves are produced by the disequilibrium of the system itself. As we show in the present paper, the energy transfer mechanism in the turbulence considered has then some similarity to spontaneous symmetry breaking in field theory. It is profitable for the system to break the initial symmetry thanks to an instability, after which conditions are produced in it for a very fast energy transfer to the dissipation region. Another reason for interest in drift turbulence is its prevalence in the ionospheric and laboratory plasmas and the important part played by drift waves in transport processes.

It is well known that the buildup of drift waves starts when we take into account dissipative mechanisms, for instance, by taking into account friction forces or Landau damping (see, e.g., Ref. 3 and references given there). As long-wavelength drift waves have a linear dispersion law, the turbulence which occurs turns out to be strong. In order to have a possibility to compare the results with experiments we shall consider the case of the ionospheric plasma in which, at heights of the order of hundreds of kilometers, there exists in the equatorial electro-jet a "pure" drift turbulence, encompassing wavelengths from hundreds of meters to a few meters (see, e.g., Refs. 4 and 5 and the references given there). The mechanism for the wave buildup in that case is the instability found in Ref. 6, which occurs in an inhomogeneous plasma with a current and with non-magnetized ions when $c_s^2 > v_d^2$ (c_s is the sound speed and v_d the drift velocity of the electrons in the electro-jet.) The vertical density gradient in the ionosphere dn_0/dz (the z-axis is along the vertical and the x-axis along the Earth's magnetic field) is the source for the disequilibrium. The drift waves which build up due to this instability (which we shall call the primary instability) propagate along the y-axis and have wavelengths of the order of 50–100 m.

The problem of the energy transfer to the observed meter waves is not a trivial one and is discussed in a large number of papers (see Ref. 7 and the references given there), since in the one-dimensional geometry, at realistic ionospheric parameters, the nonlinear terms in the equations of motion are very small and the transfer proceeds slowly. Under these conditions it is "profitable" for the system to choose a somewhat different, faster means of energy transfer into shorter waves. As we stated already, to do this the system breaks the one-dimensional symmetry spontaneously. This takes place thanks to the secondary instability of the primary wave in the density gradient. The instability of the one-dimensional propagation was found in Ref. 7. However, in contrast to the statement in Ref. 7, for realistic plasma parameters the secondary instability generates waves with wavelengths of the same order of magnitude as the primary one and, hence, it does not by itself explain the observed transfer. The role of the symmetry breaking and of the secondary waves manifests itself in the nonlinear theory, since the initial wave already interacts "with itself" through the secondary waves and this leads to energy transfer to meter waves. The short waves are stabilized by the viscosity and this leads to the formation of nonlinear packets which propagate obliquely close to the y-axis. In the long-wavelength region, where we can neglect the viscosity, a turbulence spectrum $\varepsilon_k \propto k^{-3}$ was obtained. The results agree well with the data from ionospheric experiments and radar measurements.[8] For instance, experiments show that amplitude fluctuations have a level of the order of 5 to 10%, and the theory gives 7%. The k^{-3} spectrum is also confirmed by ionospheric and radar measurements.[8]

§1. EQUATIONS OF MOTION

To derive the basic equations we shall start from two-fluid plasma hydrodynamics which, as is well known, is valid in the case considered for wavelengths of the order of several meters and larger:[7]

$$\frac{e}{m_e}\left(\mathbf{E}+\frac{1}{c}[\mathbf{v}_e\times\mathbf{B}]\right)+u_e^2\frac{\nabla n}{n}+\nu_e\mathbf{v}_e=0, \tag{1}$$

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$$\frac{d\mathbf{v}_i}{dt}=-u_i^2\frac{\nabla n}{n}+\frac{e}{m_i}\mathbf{E}-\nu_i\mathbf{v}_i=0, \tag{2}$$

$$\partial n/\partial t+\nabla(n\mathbf{v}_e)=0, \tag{3}$$
$$\partial n/\partial t+\nabla(n\mathbf{v}_i)=0, \tag{4}$$
$$\mathbf{E}=-\nabla\Phi,$$

where $\nu_{e,i}$ are the collision frequencies with neutrals.

Our assumptions are the following: 1) The ions are unmagnetized. 2) The electric fields are electrostatic. 3) The electron and ion densities are equal. 4) The ions and electrons are isothermal. 5) We neglect the electron inertia. 6) We neglect the ion inertia, for in real cases $c_s^2 > v_d^2$ and $\nu_i \gg \omega$. The two-stream instability does therefore not arise under these conditions and we consider the generation of type-II bursts (see Ref. 7).

We direct the z-axis along the inhomogeneity, vertically upwards, and the x-axis along the Earth's magnetic field. From (1) we find the electron velocities:

$$v_{ye}=v_d+\frac{\partial}{\partial z}\left(\frac{u_e^2}{\Omega_e}\ln n-\frac{e\Phi}{m_e\Omega_e}\right)-\frac{\nu_e}{\Omega_e}\frac{\partial}{\partial y}\left(\frac{u_e^2}{\Omega_e}\ln n-\frac{e\Phi}{m_e\Omega_e}\right) \tag{5}$$

$$v_{ze}=-\frac{\partial}{\partial y}\left(\frac{u_e^2}{\Omega_e}\ln n-\frac{e\Phi}{m_e\Omega_e}\right)-\frac{\nu_e}{\Omega_e}\frac{\partial}{\partial z}\left(\frac{u_e^2}{\Omega_e}\ln n-\frac{e\Phi}{m_e\Omega_e}\right), \tag{6}$$

$$(\mathrm{rot}\, v_e)_x=\frac{\nu_e}{\Omega_e}\mathrm{div}\, v_e,$$

v_d is the drift velocity of the electrons in the electro-jet ~ 10^2 m/s. Substituting (5) and (6) in the electron continuity equation we get

$$\frac{\partial n}{\partial t}+v_d\frac{\partial n}{\partial y}-\frac{\nu_e}{\Omega_e}\left(\frac{u_e^2}{\Omega_e}\nabla^2 n-\frac{en\nabla^2\Phi}{m_e\Omega_e}\right)$$
$$+\frac{e}{m_e\Omega_e}\left(\frac{\partial\Phi}{\partial y}\frac{\partial n}{\partial z}-\frac{\partial\Phi}{\partial z}\frac{\partial n}{\partial y}\right)+\frac{\nu_e}{\Omega_e}\frac{e}{m_e\Omega_e}\nabla\Phi\nabla n=0. \tag{7}$$

In (5) and (6) we neglected the electron inertia and terms containing $(\nu_e/\Omega_e)^2 \sim 10^{-4}$, in all other respects Eq. (7) is exact.

Neglecting the ion inertia in (2) we get

$$\mathbf{v}_{i\perp}=-\frac{1}{\nu_i}\nabla_\perp\left(u_i^2\ln n+\frac{e}{m_i}\Phi\right). \tag{8}$$

Substituting (8) in (4) we get

$$\frac{\partial n}{\partial t}-\frac{1}{\nu_i}\left(u_i^2\Delta n+\frac{e}{m_i}n\Delta\Phi\right)-\frac{e}{\nu_i m_i}\nabla\Phi\nabla n=0. \tag{9}$$

In the following estimates we shall use the experimental parameters given in Ref. 5 for the ionospheric plasma in the equatorial electro-jet:

$$\nu_e\sim 4\cdot 10^4\ s^{-1},\ \nu_i\sim 250\ s^{-1},\ \Omega_e\sim 5\cdot 10^6\ s^{-1},$$
$$\Omega_i\sim 90\ s^{-1},\quad L=\left(\frac{1}{n_0}\frac{dn_0}{dz}\right)^{-1}=600\ \mathrm{m},\quad v_d=10^2\ \mathrm{m/s}. \tag{10}$$
$$c_s^2=\frac{m_e}{m_i}u_e^2+u_i^2=k\frac{T_e+T_i}{m_i},$$
$$c_s^2\sim 10^5\ (\mathrm{m/s})^2.$$

Writing n in the form $n = n_0(z) + \tilde{n}$, we get from (7) for the quantity $n' = \tilde{n}/n_0(z)$

$$\frac{\partial n'}{\partial t}+v_d\frac{\partial n'}{\partial y}-\left(\frac{\Omega_e}{\nu_e L}+\frac{\partial}{\partial y}\right)\frac{\nu_e}{\Omega_e}\frac{1}{\Omega_e}\frac{\partial}{\partial y}W_e-\frac{\nu_e}{\Omega_e}\frac{1}{\Omega_e}\frac{\partial^2 W_e}{\partial z^2}$$
$$+\frac{1}{\Omega_e}\left(\frac{\partial W_e}{\partial z}\frac{\partial n'}{\partial y}-\frac{\partial W_e}{\partial y}\frac{\partial n'}{\partial z}\right)$$
$$+\frac{\nu_e}{\Omega_e}\frac{e}{m_e\Omega_e}\left(\frac{\partial}{\partial y}n'\frac{\partial\Phi}{\partial y}+\frac{\partial}{\partial z}n'\frac{\partial\Phi}{\partial z}\right)=0. \tag{11}$$

Here $W_e = u_e^2 n' - e\Phi/m_e$. Similarly we get from Eq. (9)

$$\frac{\partial n'}{\partial t}-\frac{1}{\nu_i}\left(c_s^2\Delta n'-\frac{\Omega_i}{\Omega_e}\Delta W_e\right)-\frac{1}{\nu_i}\frac{e}{m_i}\left(\frac{\partial}{\partial y}n'\frac{\partial\Phi}{\partial y}+\frac{\partial}{\partial z}n'\frac{\partial\Phi}{\partial z}\right)=0. \tag{12}$$

We neglected in Eqs. (11), (12) the small drift of the ions in the electro-jet and the gradient drifts. For the plasma parameters considered these drifts are two orders of magnitude smaller than the electron drift and are normally neglected. Moreover, taking these effects into account does not lead to any principally new features.

§2. LINEAR THEORY

For perturbations of the form $\exp(-i\omega t + ik_y y + ik_z z)$ we get in the linear approximation from (11) and (12) the dispersion equation

$$\omega-kv_d=\left(1-\frac{i\Omega_e}{\nu_e k_y L}\frac{k_y^2}{k^2}\right)(-\nu_i\omega-ik^2c_s^2)\frac{\psi}{\nu_i}, \tag{13}$$

$\psi = \nu_i\nu_e/\Omega_i\Omega_e$. For the plasma parameters of (10) we have $\psi = 1/45$. From (13) we find:[6]

$$\omega_k=\frac{k_y v_d}{1+\psi}\approx k_y v_d,\quad \gamma_k=\frac{\Omega_e}{\nu_e}\psi\frac{\omega_k}{k_y L}\frac{k_y^2}{k^2}-c_s^2\frac{\psi}{\nu_i}k^2, \tag{14}$$

$$k^2=k_y^2+k_z^2,\ \omega=\omega_k+i\gamma_k,\ |\gamma_k|\ll|\omega_k|.$$

It follows from (14) that waves propagating along the y-axis with $k_z = 0$ have the largest growth rate:

$$\gamma_{max}=\frac{\nu_i}{\Omega_i}\frac{v_d}{L}-c_s^2\frac{\psi}{\nu_i}k_y^2. \tag{15}$$

When the small ion drift in the electro-jet is taken into account

$$\omega_k=\mathbf{k}(\mathbf{v}_{0e}+\mathbf{v}_{0i}\psi),$$

and the maximum growth rate is reached for a small k_z, $k_z \sim k_y\psi v_{0i}/v_{0e}$. For the sake of simplicity we shall, as in Ref. 7, assume that $k_z = 0$. (Taking a small k_z into account does not change the results obtained.) Substituting the numerical values of (10) in (15), we find that the condition $\gamma > 0$ for the primary instability gives

$$\lambda_y>\lambda_{min}=\pi\cdot 10^2 v_d^{-1/2}, \tag{16}$$

$\lambda_y = 2\pi/k_y$, $\lambda_{min} \sim 30$ m. Thus, the primary instability results in buildup of waves with $\lambda_y \sim 50$ m or more.

The next question concerns nonlinear mechanisms whereby the energy of these long waves is transferred to the short ~ 5 m waves. (We have already mentioned that radar measurements and ionospheric experiments have revealed generation of such waves.) It is shown in Ref. 7 that for a one-dimensional wave moving along the y-axis and for the plasma parameters given by (10), the dispersion Eq. (13) is valid also in second order in the amplitude, which leads to a very slow steepening of the wave. We note also that the one-dimensional propagation of the wave is unstable and that gradients produced by the primary wave along the y-axis generate a secondary wave moving along the z-axis. This instability was obtained in Ref. 7 for the case when the mode with $k_y = 0$ was excited. One can then obtain the dispersion equation for the secondary instability, and hence Re ω and Im ω, from the appropriate expressions for the primary instability for the formal substitution

$$y\to z,\ z\to y,\ k_y\to k_z,\ \Omega_e\to-\Omega_e,$$
$$L^{-1}\to Ak_m\sin k_m y,\quad v_d\to v_{dz}=-\frac{1}{\Omega}\frac{\partial W^{(1)}}{\partial y}=-\frac{\Omega_e}{\nu_e}\psi v_d A\cos k_m y.$$

(We assume here that due to the primary instability the harmonic $A\cos k_m y$ is excited.) We get for the secondary instability[7]

$$\operatorname{Re}\omega^{(2)}=-\frac{\Omega_e}{\nu_e}\psi v_d A k_z\cos k_m y=-k_z v_{dz}, \tag{17}$$

$$\gamma^{(2)}=\psi\left\{\left(\frac{\Omega_e}{\nu_e}\right)^2\frac{k_m v_d}{2}\psi A^2\sin 2k_m y-\frac{c_s^2}{\nu_i}k_z^2\right\}. \tag{18}$$

From the condition $\gamma^{(2)}>0$ we find the minimum wavelength excited by the secondary instability. From (18) it follows that

$$\left(\frac{\Omega_e}{\nu_e}\frac{\nu_i}{\Omega_i}\right)\frac{k_m v_d}{2}A^2\frac{\nu_i}{c_s^2}\geqslant k_z^2. \tag{19}$$

For the plasma parameters given in (10) ($\Omega_e\nu_i/\Omega_i\nu_e\approx 300$) we get

$${}^1/_3\,k_m v_d A^2\geqslant k_z^2,\quad k_m=2\pi/\lambda_m\approx 10^{-1}\ \text{m},\quad \lambda_{min}^{(2)}\geqslant 3A^{-1}. \tag{20}$$

As the actually observed amplitude is of the order of 5 to 10%, (20) gives

$$\lambda_{min}^{(2)}\sim 30-60\ \text{m}, \tag{21}$$

in contrast to the statement of Ref. 7 where the value $\lambda_{min}^{(2)}\sim 5-6$ m is given for the secondary waves. The secondary instability thus produces modes of the same wavelength as the primary one. Since the characteristic dimension of the change in the density in the primary wave turned out to be of the order of the wavelength of the secondary wave, we must evaluate the secondary instability more correctly. Since the secondary instability is important in what follows, we check on the existence of localized modes.

The starting equations for the secondary instability follow from (9) and (11)

$$\frac{\partial n_2}{\partial t}-\left(\frac{\Omega_e}{\nu_e L}+\frac{\partial}{\partial y}\right)\frac{\nu_e}{\Omega_e}\frac{1}{\Omega_e}\frac{\partial W_e}{\partial y}-v_{dz}\frac{\partial n_2}{\partial y}-\left(-\frac{\Omega_e}{\nu_e}\frac{\partial n_1}{\partial y}+\frac{\partial}{\partial z}\right)\frac{\nu_e}{\Omega_e}\frac{1}{\Omega_e}\frac{\partial W_2}{\partial z}=0, \tag{22}$$

$$\frac{\partial n_2}{\partial t}=\frac{1}{\nu_i}\left(c_s^2\Delta n_2-\frac{\Omega_i}{\Omega_e}\Delta W_2\right). \tag{23}$$

For the sake of simplicity we shall assume k_z to be such that $\gamma^{(2)}(k_z)$ is close to

$$\gamma_{max}^{(2)}=\frac{\Omega_e}{\nu_e}\psi v_{dz}\frac{\partial n_1}{\partial y} \tag{24}$$

(n_1 is the primary wave). Neglecting small viscous terms we then easily get from (4) the equation for the eigenmodes; it has the form of a Schrödinger equation

$$\frac{\partial^2\Phi}{\partial y^2}-k_z^2\left[1+\frac{i(\Omega_e\omega/\nu_e k_z)\psi\,\partial n_1/\partial y}{\omega(1+\psi)+k_z v_d^2}\right]\psi=0, \tag{25}$$

$$\Phi=[\omega(1+\psi)+k_z v_{dz}]n_2. \tag{26}$$

Expanding the density perturbation near the inflection point we get the Schrödinger equation for a harmonic oscillator. Hence for $n_1>0$ and the inflection point (see Fig. 2) we have

$$E=-k_z^2\left(1+\frac{i(\Omega_e\omega/\nu_e k_z)\psi n_1 k_m}{\omega(1+\psi)+k_z v_d^2}\right),\quad \frac{E_n}{\alpha}=n+\frac{1}{2}. \tag{27}$$

In the quasiclassical approach

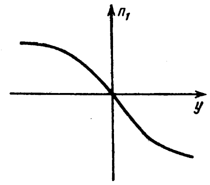

FIG. 1.

$$\omega(1+\psi)+k_z v_d=-i\frac{\Omega_e}{\nu_e}\frac{\omega}{k_z}\psi n_1 k_m, \tag{28}$$

$$\Phi=e^{-\alpha y^2}P_n(\sqrt{\alpha}y), \tag{29}$$

$P_n(\sqrt{\alpha y})$ are Hermite polynomials,

$$\alpha=k_z\left(\frac{\nu_e}{\Omega_e}\right)^{1/2}\left(1+\frac{i}{2}\frac{\gamma}{\omega_k}\right). \tag{30}$$

One sees easily that localized eigenmodes grow when $n_1<0$ and the density behavior is as in Fig. 1 and for $n_1>0$ and a density behavior as in Fig. 2. In that case Eq. (30) shows that the transverse derivative is small compared to the longitudinal one. The reason is that the wavelength of the inhomogeneity is large $\sim(k_m n_1)^{-1}$ and the quantity v_{dz} varying over distances $2\pi k_m^{-1}$ affects the density well weakly.

The density perturbation produced by the primary instability thus produces an instability of the vertically propagating waves. Positive perturbations with the density behaving as in Fig. 2 then excite a wave moving upwards, and a negative segment of a perturbation with the density behavior of Fig. 1 generates a wave moving downwards. Notwithstanding the fact that the modes appearing here have a wavelength of the order of that of the primary wave, i.e., by themselves they do not explain the formation of short-waves, their role in the nonlinear dynamics is very large, as shall become clear in what follows.

§3. INTERACTION BETWEEN VERTICAL AND HORIZONTAL WAVES

From (11) and (12) we can obtain a single rather unwieldy equation which was studied in Refs. 9–11. For studying the transfer we proceed differently, explicitly separating the vertical and the horizontal waves and their interactions. We turn to the original Eqs. (11) and (12) and we shall look for a solution in the form of a sum:

$$n=n_1+n_2,\ W=W_1+W_2. \tag{31}$$

We shall assume here the functions n_2 and W_2 to satisfy Eqs. (22) and (23) with nonlinear terms taken

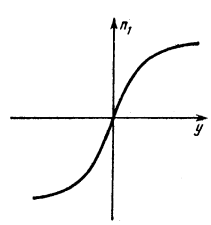

FIG. 2.

into account, i.e.,

$$\frac{\partial n_2}{\partial t}-\frac{1}{\Omega_e}\frac{\partial W_1}{\partial y}\frac{\partial n_2}{\partial z}-\frac{1}{\Omega_e}\frac{\partial W_2}{\partial y}\frac{\partial n_2}{\partial z}+\frac{1}{\Omega_e}\frac{\partial W_2}{\partial z}\frac{\partial n_2}{\partial y}$$
$$-\left(-\frac{\Omega_e}{\nu_e}\frac{\partial n_1}{\partial y}+\frac{\partial}{\partial z}\right)\frac{\nu_e}{\Omega_e}\frac{1}{\Omega_e}\frac{\partial W_2}{\partial z}=0, \qquad (32)$$

$$\frac{\partial n_2}{\partial t}-\frac{1}{\nu_i}\left(c_s^2\Delta n_2-\frac{\Omega_i}{\Omega_e}\Delta W_2\right)=0. \qquad (33)$$

Equations (33) and (32) indicate that n_2 is a nonlinear wave moving along the z-axis and changes at small amplitude into the secondary wave considered earlier. For that reason we can drop in (32) the unimportant terms with L and neglect in (32) and (33) second derivatives with respect to y in comparison with second derivatives with respect to z. Moreover, we shall show that the remaining nonlinear terms in (11) and (12) result in small corrections to the results obtained below.

Subtracting Eqs. (32) and (33) from the equations for the sums n_1+n_2 and W_1+W_2 we get equations for n_1 and W_1:

$$\frac{\partial n_1}{\partial t}-\left(\frac{\Omega_e}{\nu_e L}+\frac{\partial}{\partial y}\right)\frac{\nu_e}{\Omega_e}\frac{1}{\Omega_e}\frac{\partial W_1}{\partial y}+\frac{1}{\Omega_e}\frac{\partial n_2}{\partial y}\frac{\partial W_1}{\partial z}$$
$$-\frac{1}{\Omega_e}\frac{\partial W_2}{\partial y}\frac{\partial n_1}{\partial z}+\frac{1}{\Omega_e}\frac{\partial n_1}{\partial y}\frac{\partial W_1}{\partial z}-\frac{1}{\Omega_e}\frac{\partial W_1}{\partial y}\frac{\partial n_1}{\partial z}=0, \qquad (34)$$

$$\frac{\partial n_1}{\partial t}-\frac{1}{\nu_i}\left(c_s^2\frac{\partial^2 n_1}{\partial y^2}-\frac{\Omega_i}{\Omega_e}\frac{\partial^2 W_1}{\partial y^2}\right)=0. \qquad (35)$$

All equations are written in a reference frame moving with the drift. Moreover, we have neglected in (35) small transverse derivatives $\partial^2/\partial z^2$ in comparison with $\partial^2/\partial y^2$ [see Eq. (30)].

Using linear theory we can transform the nonlinear terms in Eqs. (32) and (34). We evaluate

$$\frac{1}{\Omega_e}\frac{\partial W_1}{\partial y}=i\frac{1}{\Omega_e}\frac{m_i}{m_e}\frac{\partial}{\partial y}\left(\frac{-c_s^2k_y^2-\nu_i\omega}{k_y^2}\right)n_1.$$

From the dispersion equation (13) we find $-\nu_i\omega-ik_y^2c_s^2=-\nu_i\omega_k$, and then

$$-\frac{1}{\Omega_e}\frac{\partial W_1}{\partial y}=-\frac{\Omega_e}{\nu_e}\psi v_d n_1. \qquad (36)$$

As the dispersion relation for the secondary instability is obtained from the dispersion relation for the primary one by the substitution

$$k_y\to k_z,\quad \Omega_e\to-\Omega_e,\quad \omega\to k_z v_{dz},\quad v_{dz}=-\frac{\Omega_e}{\nu_e}\psi v_d n_1,$$

we can obtain directly from (36)

$$\frac{1}{\Omega_e}\frac{\partial W_2}{\partial z}=\left(\frac{\Omega_e}{\nu_e}\right)^2\psi^2 v_d n_1 n_2. \qquad (37)$$

From this we find for the nonlinear terms

$$\frac{1}{\Omega_e}\frac{\partial n_2}{\partial y}\frac{\partial W_1}{\partial z}=\frac{\Omega_e}{\nu_e}\psi v_d n_2\frac{\partial n_1}{\partial z},$$
$$-\frac{1}{\Omega_e}\frac{\partial W_2}{\partial y}\frac{\partial n_1}{\partial z}=-\left(\frac{\Omega_e}{\nu_e}\right)^2\psi v_d n_1\frac{\partial n_1 n_2}{\partial y}. \qquad (38)$$

As a result we get the set of equations:

$$\frac{\partial n_1}{\partial t}-\left(\frac{\Omega_e}{\nu_e L}+\frac{\partial}{\partial y}\right)\psi v_d n_1+\frac{\Omega_e}{\nu_e}\psi v_d n_2\frac{\partial n_1}{\partial z}$$
$$-\left(\frac{\Omega_e}{\nu_e}\right)^2\psi^2 v_d n_1\frac{\partial n_1 n_2}{\partial y}=\frac{c_s^2}{\nu_i}\psi\frac{\partial^2 n_1}{\partial y^2}, \qquad (39)$$

$$\frac{\partial}{\partial z}\left(\frac{\partial n_2}{\partial t}-\frac{\Omega_e}{\nu_e}\psi v_d n_1\frac{\partial n_2}{\partial z}+\left(\frac{\Omega_e}{\nu_e}\right)^2\psi^2 v_d n_1 n_2\frac{\partial n_1}{\partial y}\right.$$
$$\left.-\frac{c_s^2}{\nu_i}\frac{\partial^2 n_2}{\partial z^2}\right)=\left(\frac{\Omega_e}{\nu_e}\right)^2\psi^2 v_d\left\{\frac{\partial n_1 n_2}{\partial y}\frac{\partial n_2}{\partial z}-\frac{\partial}{\partial z}(n_1 n_2)\frac{\partial n_2}{\partial y}\right\}$$
$$\approx\left(\frac{\Omega_e}{\nu_e}\right)^2\psi^2 v_d n_2\frac{\partial n_1}{\partial y}\frac{\partial n_2}{\partial z}. \qquad (40)$$

Equations (39) and (40) describe the interaction between the nonlinear waves that move along the y- and z-axis. [We discarded in (39) the nonlinear term

$$\frac{\partial n_1}{\partial y}\frac{\partial W_1}{\partial z}-\frac{\partial W_1}{\partial y}\frac{\partial n_1}{\partial z},$$

which is small compared to those retained.] The second term in (39) describes the primary instability, and the next one the transfer from the wave n_1 to the wave n_2. The last term on the left-hand side of (39) is the one of greatest interest to us, as it represents the self-action of the wave n_1 with participation of the wave n_2. Similarly, in (40) we take into account the secondary instability, the transfer from the wave n_2 to the wave n_1, and the self-action of n_2 occuring with the wave n_1 participating.

§4. ENERGY TRANSFER AND TURBULENCE SPECTRUM

It is seen from (39) and (40) that the energy transfer from the long-wave to the short-wave modes occurs in the wave n_1 thanks to the term

$$\left(\frac{\Omega_e}{\nu_e}\right)^2\psi v_d n_1\frac{\partial n_1 n_2}{\partial y},$$

and in the wave n_2 thanks to the last term in (40). Qualitatively Eq. (39) is already close to the Burgers equation with an instability $\gamma=(\Omega_e v_d/\nu_e L)\psi$.

We find now the characteristic dimensions and wave amplitudes for which the instability is stabilized. We consider first the wave n_1. Viscosity stabilizes the distortion of the nonlinear packet whose size is given by the condition:

$$\left(\frac{\Omega_e}{\nu_e}\right)^2\psi^2 v_d\frac{\partial n_1 n_2}{\partial y}n_1\sim\frac{c_s^2\psi}{\nu_i}\frac{\partial^2 n_1}{\partial y^2}.$$

Substituting here the plasma parameters from (10) we get

$$\lambda_y n_1 n_2\sim 0.02\ \text{m}. \qquad (41)$$

Due to the development of the instability the nonlinear term grows to the amplitude:

$$\frac{\Omega_e}{\nu_e}\frac{\psi}{L}v_d n_1\sim\left(\frac{\Omega_e}{\nu_e}\right)^2\psi^2 v_d\frac{\partial n_1 n_2}{\partial y}n_1,$$
$$\lambda_y\sim 10^3 n_1 n_2\ [\text{m}].$$

Hence

$$\lambda\sim\sqrt{20}\sim 4.4\ \text{m}. \qquad (42)$$

We now consider Eq. (40). Comparing the transfer with the instability we get

$$\frac{\partial}{\partial z}\left(\frac{\Omega_e}{\nu_e}\right)^2\psi^2 v_d n_1 n_2\frac{\partial n_1}{\partial y}\sim\left(\frac{\Omega_e}{\nu_e}\right)^2\psi^2 v_d\frac{\partial n_1}{\partial y}n_2\frac{\partial n_2}{\partial z},\quad n_1\sim n_2. \qquad (43)$$

Comparing transfer and viscosity we find

$$\frac{\partial}{\partial z}\frac{c_s^2\psi}{\nu_i}\frac{\partial^2 n_2}{\partial z^2}\sim\left(\frac{\Omega_e}{\nu_e}\right)^2\psi^2 v_d n_2\frac{\partial n_1}{\partial y}\frac{\partial n_2}{\partial z},$$
$$\lambda_z n_1 n_2\sim 0.02\lambda_y/\lambda_z\ [\text{m}]. \qquad (44)$$

From (41) and (44) follows the value $\lambda_y \sim \lambda_z$. As a result we get $n \sim 1/14$ for $n \sim n_1 \sim n_2$. The experiments indicate for n an order of 5 to 10%, which agrees well with the value obtained. For λ experiment gives a value of the order of a few meters, which also agrees with the result (42).

The following scheme for the energy transfer along the spectrum therefore arises. For the primary wave, propagating along the y-axis it is advantageous to break the one-dimensional symmetry and excite a wave propagating vertically. Through this secondary wave the primary wave undergoes a strong "self-action" which leads to its steepening and energy transfer along the spectrum. The spontaneous breaking of the primary symmetry which occurs here is very strongly related to the similar effect in field theory. From a formal point of view the crossing nonlinearity

$$\frac{\partial n}{\partial y}\frac{\partial \Phi}{\partial z}=-\frac{\partial n}{\partial z}\frac{\partial \Phi}{\partial y}$$

vanishes for a single harmonic. However, there is no superposition principle for it. The latter would mean that this instability would vanish identically. One verifies easily that, in general, the crossing nonlinearity does not vanish even for a sum of two different harmonics. Therefore, if the coefficient of such a non-linearity is very large it may be advantageous for the system to excite two or more waves in order to have the possibility to eliminate the disequilibrium very rapidly. The resultant energy transfer along the turbulence spectrum is strong in the present case, so that we are forced to restrict ourselves to qualitative estimates when determining the spectrum.

Since the excited modes propagate along the y-axis, we shall consider the spectrum as a function of k_y, and assume that k_y is small enough to neglect viscosity. We consider Eq. (39). In the stationary case the term with the instability must be proportional to the nonlinear term with transfer, as there are no other terms. As a result we get

$$\gamma n_1 \sim \left(\frac{\Omega_e}{\nu_e}\right)^2 \psi^2 v_d n_1 n_2 \frac{\partial n_1}{\partial y},$$
$$n_1 \sim \left(\frac{\Omega_e}{\nu_e}\right)^2 \frac{1}{n_2}\frac{\gamma}{v_d k_y} \sim \frac{\Omega_i}{\nu_i}\frac{1}{n_2 L k_y}, \qquad (45)$$
$$\langle n_1^2\rangle = \int_0^\infty \varepsilon_{n_1}(k_y)\,dk_y, \quad \varepsilon_{n_1}(k_y) \sim \left(\frac{\Omega_i}{\nu_i}\right)^2 \frac{1}{n_2^2 L^2} k_y^{-3}$$

(n_2 depends weakly on y). As the spectrum (45) is obtained in a frame of reference fixed in the drift, the large-scale transport of density pulsations with speed v_d is automatically eliminated. For the spectrum v_{ze} we get from (36)

$$v_{ze} \sim \frac{v_d}{n_2 L k_y}, \quad \varepsilon_{ve}(k_y) \sim \frac{v_d^2}{L^2 n_2^2} k_y^{-3}. \qquad (46)$$

The spectrum $\varepsilon \propto k_y^{-3}$ agrees rather well with experiments and has a simple physical meaning. It follows from Eqs. (39) and (40) that the characteristic time for energy exchange between the vertical and the horizontal components is, in the region where the viscosity can be neglected, proportional to the growth rate of the secondary instability $\gamma_{(2)}^{-1}$ [see (18)]. (Together with this energy exchange there is also a transfer to short waves.) In a stationary turbulent state the characteristic time $\gamma_{(2)}^{-1}$ must be of the same order as the characteristic pumping time, i.e., $\gamma_{(1)}^{-1}$. The condition $\gamma_{(1)}^{-1} \sim \gamma_{(2)}^{-1}$ yields a spectrum $\propto k^{-3}$. We note that in Refs. 9 to 11 they studied the spectra of the gradient instability in the framework of a modified hypothesis that the correlation is relaxed (in the spirit of Kraichnan's scheme) and they showed that the spectrum can be in the range from k^{-3} to k^{-4}.

We turn to the nonlinear terms which we neglected. For instance, in (35) we neglected the term

$$-\frac{1}{\nu_i}\frac{e}{m_i}\frac{\partial}{\partial y}\left(n_1 \frac{\partial \Phi_1}{\partial y}\right).$$

Taking this term into account gives in Eq. (39) a term $2\psi v_d n_1 \partial n_2/\partial y$ and, apart from that, a nonlinear viscosity which is not at all important for the consideration of the transfer. One sees easily that the correction obtained is small compared to the nonlinear terms retained in (39). Similarly, taking into account the nonlinear term

$$-\frac{1}{\nu_i}\frac{e}{m_i}\frac{\partial}{\partial z}\left(n_2 \frac{\partial \Phi_2}{\partial z}\right)$$

in Eq. (33) leads to a correction in (40) of the form $2\psi^2(\Omega_e/v_e)v_d n_1 n_2 \partial n_2/\partial y$ which is small compared to the nonlinear terms which were retained.

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