Degeneracy of one-dimensional acoustic turbulence at large Reynolds numbers

S. N. Gurbatov and A. I. Saichev

N. I. Lobachevskii Gor'kii State University
(Submitted 10 July 1980)
Zh. Eksp. Teor. Fiz. **80**, 689–703 (February 1981)

The statistical properties of a one-dimensional acoustic turbulence (OAT) are considered for time intervals at which the formation and merging of the shock fronts are important. One- and two-point probability densities are found for an OAT with an infinite Reynolds number. It is shown that for times such that the external turbulence scale is much greater than the correlation radius of the initial field, these distributions are self-similar: the one-point OAT probability density is Gaussian, while the two-point density is substantially non-Gaussian. The law of growth of the external scale of the OAT due to the merging of the shock fronts is found for these times. The energy spectrum which follows from the two-point OAT probability density is discussed. The last stage of OAT degeneracy, for which OAT damping is due only to linear dissipation, is also considered.

PACS numbers: 47.25. – c, 47.40.Nm

The problem of one-dimensional acoustic turbulence (OAT), which is described by the Burgers equation, is important for two reasons. First, the OAT is a simple hydrodynamic model turbulence that contains the basic features of any turbulence, namely the redistribution of the energy over the spectrum due to the nonlinear interaction, and the damping of the energy at small scales.[1,2] On the other hand, the Burgers equation adequately describes the one-dimensional acoustic waves in a compressible fluid[3-5] and, in particular, the statistics of an intense acoustical noise field. There are many papers in which the effect of nonlinearity and dissipation on the evolution of OAT has been investigated (see, for example, Refs. 6–14). However, they involve hypotheses whose validity is not obvious, and which lead sometimes to physically incorrect conclusions, such as a negative energy spectrum.

Along with this, the presence of an exact solution of Burgers equation[15,16] permits us to obtain some exact statistical results. On the basis of this solution, the statistical properties of the OAT have been found for an infinite Reynolds number, both in the initial stage, when discontinuities of the acoustic wave are absent,[17-20] and in the limit $t\to\infty$, when the OAT has the form of a random sequence of triangular pulses.[21,22] It is shown in Ref. 22 that as $t\to\infty$ and at infinite Reynolds numbers, the correlation function and the energy spectrum of the OAT are self-similar and depend only on $l(t)$—the average distance between the discontinuities (on the external scale of the turbulence).

In the present work, the one- and two-point probability densities are found on the basis of an exact solution of the Burgers equation for a OAT at infinite Reynolds number, and it is shown that as $t\to\infty$ they are also self-similar. In contrast to Ref. 22, where the initial field was replaced by a series of discrete independent quantities, the method of the present work allows us to connect the asymptotic properties of the OAT with the scales of the initial field and to estimate the characteristic time when the OAT becomes self-similar. Also discussed here is the effect of finite viscosity and, consequently, of a finite internal scale of the turbulence l_0—the mean thickness of the shock front—on the energy spectrum, and the arrival of the OAT at the final stage of the degeneracy in which the damping of the OAT is determined only by the linear dissipation, is traced.

1. INITIAL EQUATIONS AND QUALITATIVE ANALYSIS OF THE OAT

The Burgers equation which describes the OAT has the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\mu\frac{\partial^2 u}{\partial x^2},\quad u(x,t=0)=u_0(x), \tag{1.1}$$

where μ is the coefficient of viscosity. It solution is the following:[15]

$$u(x,t)=\int_{-\infty}^{\infty}(x-y)\exp\left\{\frac{1}{2\mu}G(y,x,t)\right\}dy\times\left[t\int_{-\infty}^{\infty}\exp\left\{\frac{1}{2\mu}G(y,x,t)\right\}dy\right]^{-1}, \tag{1.2}$$

$$G(y,x,t)=v(y)-(x-y)^2/2t,\quad v(y)=-\int^{y}u_0(x)\,dx. \tag{1.3}$$

We consider the case of infinite Reynolds numbers. At $\mu\to 0$, the contribution to the integrals in (1.2) is made only by the small neighborhood of the point y, where the function $G(y,x,t)$ has an absolute maximum and

$$\lim_{\mu\to 0}u(x,t)=(x-y(x,t))/t. \tag{1.4}$$

Here $y(x,t)$ is the coordinate of the absolute maximum of $G(y,x,t)$.[16] As is seen from (1.3), $y(x,t)$ is one of the roots of the equation

$$u_0(y)=(x-y)/t, \tag{1.5}$$

and the solution (1.4) is identical with one of the branches of the Riemann solution:[3]

$$u(x,\ t)=u_0(x-u(x,\ t)t). \tag{1.6}$$

We separate the basic stages of the development of OAT in time. If the curvature $1/t$ of the parabola in (1.3) is greater than the characteristic curvature $v(y)-\sigma\omega_1=[\langle(u_0')^2\rangle]^{1/2}$, then Eq. (1.5) has a single root and the wave is described by the Riemann solution (1.6). Physically, the condition $\sigma\omega_1 t<1$ means that the shift

0038-5646/81/020347-08$02.40

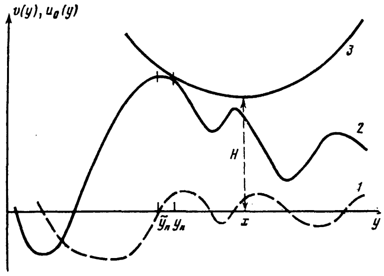

FIG. 1. Profile of the initial field $u_0(y)$—curve 1; integral of the initial field $v(y)=-\int^y u_0(x)dx$—curve 2, parabola $(x-y)^2/et+H$—curve 3; $\tilde{y}_n$—zero of $u_0(y)$; y_n—zero of $u(x,t)$—the coordinate of the absolute maximum of $G(y,x,t)$.

of the wave because of the nonlinearity $\sim\sigma t(\sigma^2=\langle u_0^2\rangle)$ is less than the scale of the initial field $u_0/u_0'\sim 1/\omega_1$. Analysis of the statistics of the OAT at this stage is given in Refs. 17–20, where the probability densities and the energy spectra of the Riemann waves are found. It is also shown there that the one-dimensional probability density of a statistically homogeneous OAT does not change at this stage, while the energy spectrum broadens because of the steepening of the wave. If the spectrum is maximal not at zero, then, because of the effect of self-detection, it is shifted in the direction of lower wave numbers.

At $\sigma\omega_1 t\gg 1$, there are several roots of Eq. (1.5), the formal solution (1.6) is multivalued, and discontinuities appear in the wave; this leads to dissipation of its energy. At $\sigma\omega_1 t\gg 1$, the curvature of the parabola is much less than the characteristic curvature $v(y)$ (Fig. 1) and the coordinates of the absolute maxima of $G(y,x,t)$ (1.3) are close to the maxima of $v(y)$, i.e., to the points of vanishing of the initial field $-\tilde{y}_k[u_0(\tilde{y}_k)=0,\ u_0'(\tilde{y}_k)>0]$. Solving (1.5) by perturbation methods, we find that the deviation of the coordinate of the absolute maximum from nearest vanishing point is equal to

$$\Delta y_k=y_k-\tilde{y}_k\simeq\frac{|x-\tilde{y}_k|}{tu_0'(\tilde{y}_k)}\sim\frac{|x-\tilde{y}_k|}{\sigma\omega_1 t}\ll|x-\tilde{y}_k|\,. \quad (1.7)$$

Therefore, the solution (1.4) outside the discontinuities is an almost linear function of x. Here the function $y(x,t)$ has the form of steps with jumps at the points of the discontinuities $x=\xi_k$, where the transition from y_k to y_{k+1} takes place. Here y_k is a subset of the set of vanishing points of the initial field $\tilde{y}_k$. Correspondingly, the field $u(x,t)$ has the form of a sequence of triangular pulses with slope $1/t$ and discontinuities at $x=\xi_k$ (Fig. 2). Thus, at $\mu\to 0$, the width of the shock front δ and the internal scale of turbulence $l_0=\delta$ tend to 0. The coordinates of the discontinuities are found from the condition $G(y_k,\xi_k,t)=G(y_{k+1},\xi_k,t)$:[22]

$$\xi_k={}^1/_2(y_k+y_{k+1})+V_k t, \quad (1.8)$$

$$V_k=\frac{1}{\eta_k}(v_{k+1}-v_k)=\frac{1}{\eta_k}\int_{y_k}^{y_{k+1}}u_0(x)dx,\quad \eta_k=y_{k+1}-y_k. \quad (1.9)$$

Here η_k is the distance between the "zeros," V_k is the velocity of the discontinuity.

We now discuss the possible cases of OAT development in the discontinuity stage. It can happen that $V_k\equiv 0$. This situation is realized, for example, for an initial field of the form $u_0=a[\cos\psi(x)]'_\alpha$, where $\psi(x)$ is a monotonic function of x. Here, the discontinuities are immobile and do not merge, the form of the wave is unchanged, its amplitude decreases as $l(0)/t$, while the energy does not depend on the initial energy and falls off as $l^2(0)/t^2$, where $l(0)$ is the scale of the initial field. In the general case, $V_k\neq 0$, and the discontinuities merge. For not very long times, the evolution of the OAT depends on the fine structure of the initial spectrum. Thus, in the case of a narrow-band initial field, when the "carrier frequency" k_0 ($k_0\approx\omega_1$) is much greater than the width of the spectrum $\Delta k=1/\rho_c$, at times $1\ll\sigma\omega_1 t\ll(\omega_1\rho_c)^2$ the discontinuities do not merge, since the velocities of the neighboring discontinuities are strongly correlated and they move with practically identical velocities. In particular, at $1\ll\sigma\omega_1 t\ll\omega_1\rho_c$, when we can neglect the motion of the discontinuities, the one-dimensional distribution is close to uniform, with boundaries $\pm\pi/k_0 t$, while the energy falls off as $1/k_0^2 t^2$.[23-25] If, however the characteristic scale of the initial field is $\sim 1/\omega$ and its correlation length ρ_c is of the same order, then the merging of the discontinuities begins even at $\sigma\omega_1 t\gtrsim 1$.

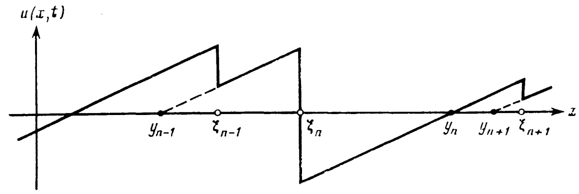

FIG. 2. Field of the acoustic turbulence in the discontinuous stage: y_n—coordinates of the "zeros"; ξ_n—coordinates of the discontinuities.

Because of the merging, the distance between the discontinuities—the external scale of turbulence $l(t)$—increases. At $l(t)\gg\rho_c$, the statistics of the OAT no longer depend on the fine structure of the initial spectrum. Setting $\eta_k\sim l$ in (1.9), we have in this case, for the dispersion of the discontinuity velocity,

$$\langle V^2\rangle\approx\frac{1}{l^2}\int_0^l(l-\rho)B_0(\rho)d\rho,\quad B_0(\rho)=\langle u_0(x)u_0(x+\rho)\rangle, \quad (1.10)$$

$$\langle V^2\rangle\approx\frac{1}{l}\int_0^\infty B_0(\rho)d\rho=\frac{\pi}{l}S_0(0),\quad S_0(0)\neq 0, \quad (1.11)$$

$$\langle V^2\rangle\approx-\frac{1}{l^2}\int_0^\infty\rho B_0(\rho)d\rho=\frac{1}{l^2}\int_{-\infty}^{\infty}\frac{S_0(k)}{k^2}dk,\quad S_0(0)=0,$$

where $S_0(k)$ is the spectrum of the energy of the initial field. The rates of merging of the discontinuities at $S_0(0)\neq 0$ and $S_0(0)=0$ are different: at $S_0(0)\neq 0$, the discontinuity velocities fall off more slowly with increase in $l(t)$ and the rate of merging is higher, which leads to a more rapid increase in $l(t)$ and to a slower attenuation of the energy than at $S_0(0)=0$. We estimate the law of growth of the external scale of the OAT by writing the kinetic equation for $l(t)$. The increase of Δl in the interval Δt is proportional to l and to the ratio of the distance traversed by the discontinuity $(\langle V^2\rangle)^{1/2}\Delta t$ to the length l:

$$\Delta l\sim l((\langle V^2\rangle)^{1/2}\Delta t/l)\sim(\langle V^2\rangle)^{1/2}\Delta t.$$

Taking the limit as $\Delta t \to 0$, we obtain

$$dl/dt \sim (\langle V^2 \rangle)^{1/2}. \quad (1.12)$$

Thus, from (1.11) and (1.12) we have $l(t) \sim ([S_0(0)]^{1/2} t)^{2/3}$ at $S_0(0) \neq 0$ and

$$l(t) \sim \left(\left(\int_{-\infty}^{\infty} S_0(k) k^{-2} dk \right)^{1/2} t \right)^{1/2}$$

at $S_0(0) = 0$.

Everywhere below, we consider the case $S_0(k) \sim k^n$, $k \to 0, n \geq 2$. Here the field $v(x)$ (13) is statistically homogeneous with a variance

$$\sigma_*^2 = \int_{-\infty}^{\infty} S_0(k) k^{-2} dk = \sigma^2/\lambda^2, \quad \sigma^2 = \int_{-\infty}^{\infty} S_0(k) dk, \quad (1.13)$$

where λ is the characteristic scale of the initial field. In this case the times at which the external scale of turbulence $l(t) \gg \rho_c$ can be estimated from the condition $\bar{V}t \gg \rho_c$, where $\bar{V}$ is the average velocity of the discontinuity over the time t. From (1.11) we have $\bar{V} \sim \sigma/\lambda\rho_c$, meaning that $l(t) \gg \rho_c$ at

$$\sigma t/\lambda\rho_c^2 \gg 1. \quad (1.14)$$

Let us estimate the external scale of the OAT, assuming (1.14) to be satisfied. It is seen from (1.3) and (1.4) that $u(x,t)$ is determined by the coordinate of the absolute maximum of the function $G(y,x,t)$, which is equal to the sum of the statistically homogeneous field $v(y)$ with variance σ_*^2 (1.13) and the decaying parabola $-(x-y)^2/2t$. Under the condition (1.14) $(x-y)^2/2t \sim \rho_c^2/2t \ll \sigma_*$ the parabola is a smooth function in the scale ρ_c and the absolute maximum of $G(y,x,t)$ coincides with one of the maxima $v(y)$. The only one of them that can be absolute for $G(y,x,t)$ is the one which has not been displaced strongly downward by the decaying parabola. Assuming that the characteristic value of the maximum of $v(y)$ is of the order σ_*, we obtain from the condition $(x-y)^2/2t \sim \sigma_*$ estimates for the external scale of the OAT $l(t) \sim |x-y|$ and its energy:

$$l(t) \sim (\sigma_* t)^{1/2} \sim (\sigma t/\lambda)^{1/2}, \quad \langle u^2 \rangle \sim (x-y)^2/t^2 \sim l^2/t^2 \sim \sigma/\lambda t. \quad (1.15)$$

The growth of the scale of the field leads to a shift of the spectrum in the direction of small wave numbers. We now discuss the effect of the external scale of the turbulence on the form of the energy spectrum of the OAT. At $\mu \to 0$, when the width of the shock front $\delta \to 0$, we have for the Fourier transform of the field $u(x,t)$, using the equality $C_u = C_{u'}/ik$,

$$C_u(k,t) = \frac{1}{2\pi ikt} \left[\sum_n \eta_n \exp(ik\xi_n) - \delta(k) \right], \quad (1.16)$$

where $\eta_n = y_{n+1} - y_n$, η_n is the distance between the "zeros" of the field $u(x,t)$, ξ_n are the coordinates of the discontinuities, and k is the wave number. The sum in (1.16) is the sum of the Fourier transforms of the individual discontinuities. Correspondingly, the energy spectrum of the OAT is equal to the energy spectrum of the discontinuities without account of their interference and of the infinite sum which describes their interference. At $k \gg l^{-1}(t)$ the interferences can be neglected and the energy spectrum has a universal asymptote:

$$S(k,t) = \langle \eta^2 \rangle / 2\pi k^2 t^2, \quad (1.17)$$

where $\langle \eta^2 \rangle$ is the average distance between the zeros per unit length. Thus, at $\mu > 0$, because of the discontinuities, the energy spectrum has a power-law asymptote. In the case of sufficiently large but finite Reynolds numbers, the OAT as before has the form of a random sawtooth wave, but with a finite width of the shock front $\delta_n = 4\pi t/\eta_n$.[22] The finiteness of δ_n leads to the result that the power-law decay of the energy spectrum at $k > 1/\delta_n \sim l/\mu t$ changes to exponential.

2. ONE-POINT PROBABILITY DENSITY AND THE OAT ENERGY

In this and the next sections, we find the one- and two-point OAT distributions, using the solution (1.4) and assuming $u_0(x)$ to be a Gaussian random field with energy spectrum $S_0(k) \sim k^n$, $k \to 0, n \geq 2$.

We introduce the integral function and the probability density of the absolute maximum H of the function $G(y,x,t)$ in the interval $[a,b]$:

$$F(H;[a,b]) = P\left(\left[v(y) - \frac{(x-y)^2}{2t} \right] < H; y \in [a,b] \right), \quad (2.1)$$

$$W(H;[a,b]) = F_H'(H;[a,b]). \quad (2.2)$$

The probability that the absolute maximum has the value $H_1 < H < H_1 + \Delta H$, and that its coordinate $y(x,t)$ lies in the range $L = [y, y+\Delta y]$ is equal to the probability that the absolute maximum in L is located between H_1 and $H_1 + \Delta H$, while in the remaining regions, $\bar{L} = (-\infty, \infty) \backslash L$, the maxima are smaller than H. Let $\Delta y \gg \rho_c$; then the absolute maxima in L and $\bar{L}$ are statistically independent and their joint probability defactorizes into a product. Integrating it over all H, we obtain for the probability of finding $y(x,t)$ in the region L:

$$P(y(x,t) \in [y, y+\Delta y]) = \int_{-\infty}^{\infty} W(H;L) F(H;\bar{L}) dH. \quad (2.3)$$

Thus, the probability density $y(x,t)$ is expressed in terms of $F(H;[a,b])$ (2.1), which is equal to the probability of the absence of interactions of the level H with the function $G(y,x,t)$ in the interval $[a,b]$. We first find the average number of intersections of $G(y,x,t)$ (1.3) with the level H in this interval: $N(H;[a,b])$. It follows from the Gaussian nature of $v(y)$ that[26,27]

$$N(H;[a,b]) = \frac{\lambda}{2\pi} \int_a^b \exp\left\{ -\frac{(H+(x-y)^2/2t)^2}{2\sigma_*^2} \right\}$$

$$\times \left[\exp\left\{ -\frac{(x-y)^2}{2\sigma^2 t} \right\} + \frac{y-x}{\sigma t} \Phi\left(-\frac{y-x}{\sigma t} \right) \right] dy, \quad (2.4)$$

$$\Phi(x) = \int_{-\infty}^{x} \exp\left\{ -\frac{t^2}{2} \right\} dt. \quad (2.5)$$

We consider the OAT over sufficiently long times, when the characteristic dimension $|x-y| \sim l(t) \sim (\sigma t/\lambda)^{1/2}$ of the region in which $G(y,x,t)$ has a maximum $v(y)$ is much greater than ρ_c (1.14). Many local maxima compete here for the right to be the absolute maximum; therefore, the absolute maximum $H \gg \sigma_*$. As is shown below, the integrand in (2.3) is concentrated at $H \gg \sigma_*$ and apparently this inequality is valid. At $H \gg \sigma_*$ and $|b-a| \gg \rho_c$, the number of intersections m of the level

H in the interval $[a, b]$ obeys the Poisson law

$$Q_m = (N^m/m!)\exp\{-N(H;\ [a,\ b])\}.$$

In this case, we have for $F(H;[a,b])$

$$F(H;\ [a,\ b]) = Q_0 = \exp\{-N(H;\ [a,\ b])\}. \tag{2.6}$$

Substituting (2.6) and (2.4) in (2.3) and integrating by parts, we get

$$P(y(x,t)\in[y,y+\Delta y]) = \int_{-\infty}^{\infty} N(H;\ [y,y+\Delta y])F_\infty'(H)dH. \tag{2.7}$$

Here $F_\infty(H) = F[H;(-\infty;\infty)]$. At $\sigma\lambda t \gg 1$, the quantity P is substantially different from zero only in the region

$$|x-y| \sim (\sigma t/\lambda)^{1/2} \sim \sigma t/(\sigma\lambda t)^{1/2} \ll \sigma t$$

(1.15), which allows us to assume the expression in the square brackets in (2.4) to be equal to unity. Moreover, at $H \gg \sigma_*$

$$N(H;[a,b]) = \frac{\lambda}{2\pi}\exp\left\{-\frac{H^2}{2\sigma_*^2}\right\}\int_a^b \exp\left\{-\frac{H(x-y)^2}{2\sigma_*^2 t}\right\}dy, \tag{2.8}$$

$$F_\infty(H) = \exp\left\{-\left(\frac{\sigma_*\gamma(t)}{H}\right)^{1/2}\exp\left\{-\frac{H^2}{2\sigma_*^2}\right\}\right\};\quad \gamma(t) = \frac{\sigma\lambda t}{2\pi}. \tag{2.9}$$

It is then seen that at $\gamma \gg 1$ the distribution $W_\infty(H) = F_\infty'(H)$ is concentrated at $H \gg \sigma_*$. Representing H in the form

$$H = H_0 + \sigma_*^2 z/H_0, \tag{2.10}$$

where H_0 is the solution of the transcendental equation

$$\left(\frac{\sigma_*\gamma(t)}{H_0}\right)^{1/2}\exp\left\{-\frac{H_0^2}{2\sigma_*^2}\right\} = 1,\quad H_0 \approx \sigma_*(\ln\gamma(t))^{1/2}, \tag{2.11}$$

we find that the integral function of z is equal to $F_\infty(z) = \exp\{-e^{(-z)}\}$, meaning that the absolute maximum lies in the narrow range

$$\Delta H/H_0 \sim \sigma_*^2/H_0^2 \sim 1/\ln\gamma(t)$$

near

$$H = H_0 \approx \sigma_*(\ln\gamma(t))^{1/2};$$

therefore,

$$\exp\left\{-\frac{(H+(x-y)^2/2t)^2}{2\sigma_*^2}\right\} \approx \exp\left\{-\frac{H_0^2}{2\sigma_*^2} - \frac{H_0(x-y)^2}{2\sigma_*^2 t} - z\right\}. \tag{2.12}$$

It is taken into account here that $|x-y| \sim (\sigma_* t)^{1/2}$. Under the condition (1.14), the characteristic width of the distribution of the coordinate of the absolute maximum $w(y;x,t)$ is much greater than the correlation length ρ_c, which allows us to go in (2.7) to the limit as $\Delta y \to 0$. Transforming to integration of (2.10) with respect to z, and taking into account (2.11) and (2.12), we get from (2.7)

$$w(y;x,t) = \frac{1}{(2\pi l^2)^{1/2}}\exp\left\{-\frac{(x-y)^2}{2l^2}\right\}, \tag{2.13}$$

$$l^2(t) = \sigma_*^2 t/H_0 \approx \sigma t/\lambda(\ln\gamma(t))^{1/2}. \tag{2.14}$$

Here $l(t)$ is the characteristic distance between the "zeros" of the field $u(x,t)$ and is, in essence, the external scale of the turbulence. It is seen from (1.4) that the one-dimensional distribution of $u(x,t)$ is also Gaussian:

$$W(u;t) = \frac{1}{(2\pi b^2(t))^{1/2}}\exp\left\{-\frac{u^2}{2b^2(t)}\right\}, \tag{2.15}$$

$$b^2(t) = l^2(t)/t^2 = \sigma/\lambda t(\ln\gamma(t))^{1/2}.$$

Thus, at long times, the energy of the OAT falls off according to the power law $\langle u^2\rangle \sim t^{-1}$ with a logarithmic correction. The slower rate of energy decrease in comparison with the harmonic signal is due to the increase in the external scale of the turbulence because of the merging of the discontinuities. The rate of merging becomes greater with increase in the energy of the initial field. Therefore, at fixed t and λ, the energy of the wave increases with increase in the energy of the initial field.

The Gaussian nature of the one-dimension distribution of the OAT is a rather natural result, which is valid not only in the one-dimensional case.[2] This is connected with the fact that the field at any point is due to the joint action of a large number of factors. Thus, in our case the field is determined by the values of the initial field from a region that is much greater than the initial correlation radius. We note that the distribution (2.15) allows us to find immediately all the higher moments of the field $u(x,t)$: $\langle u^{2n}\rangle = (2n-1)!!\,b^{2n}$, the calculation of which, within the framework of the method proposed in Ref. 22, is a very cumbersome, almost unsolvable problem.

3. TWO-POINT DISTRIBUTIONS AND THE ENERGY SPECTRUM OF OAT

We begin by discussing some two-point-distribution properties, which follow from (1.4). Let

$$u_1 = u(-x,\ t),\ u_2 = u(x,\ t),\ y_1 = y(-x,\ t),\ y_2 = y(x,\ t). \tag{3.1}$$

Since $y_{1,2}$ is the coordinate of the absolute maximum, the inequalities

$$G(y,\ -x,\ t) \leqslant G(y_1,\ -x,\ t),\quad G(y,\ x,\ t) \leqslant G(y_2,\ x,\ t)$$

are valid. Substituting y_2 and y_1 respectively in the left sides and adding, we get

$$y_2 \geqslant y_1,\quad u_2 - u_1 < 2xt, \tag{3.2}$$

i.e., $y(x,t)$ is a nondecreasing function of x. As $t\to\infty$, as is shown in Sec. 1, $y(x,t)$ is a step function of x with jumps at $x = \xi_k$. Two situations are possible: in the first, there can be no discontinuities in the interval $2x$ and $y(-x,t) = y(x,t)$, and in the second, there can be one or more discontinuities. For the two-point distribution of the coordinates of the absolute maxima $w(y_1,y_2;2x,t)$ and of the field $W(u_1,u_2;2x,t)$ we have here

$$w(y_1,\ y_2;\ 2x,\ t) = \delta(y_1-y_2)w_a + w_p, \tag{3.3}$$

$$W(u_1,\ u_2;\ 2x,\ t) = \delta(u_2-u_1-2xt)W_a + W_p, \tag{3.4}$$

where the first term describes the statistical properties of the wave in the absence, and the second, in the presence of discontinuities in the interval $2x$, while, by virtue of (3.2) $w_p \equiv 0$ at $y_2 < y_1$, and $W_p \equiv 0$ at $u_2 - u_1 > 2xt$.

At finite but sufficiently large $t(\sigma\omega_1 t \gg 1)$ the coordinates of the absolute maxima y_n differ from the points at which $u_0(x) - \tilde{y}_n$ vanish by a quantity Δy_n for which, by virtue of the estimate $|x-y| \sim (\sigma t/\lambda)^{1/2}$ (1.15), we have from (1.7):

$$\Delta y \sim \frac{|x-\tilde{y}_n|}{tu_0'} \sim \frac{(\sigma t/\lambda)^{1/2}}{\sigma\omega_1 t} = \frac{1}{\omega_1(\sigma t\lambda)^{1/2}}. \tag{3.5}$$

The difference of y_n from $\tilde{y}_n$ is connected with the devia-

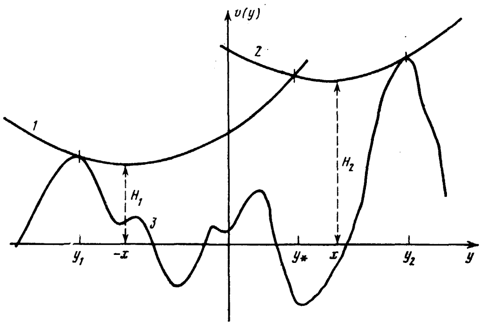

FIG. 3. Parabola $\alpha_1(y)=(x+y)^2/2t$—curve 1; parabola $\alpha_2(y)=(x-y)^2/2t+H_2$—curve 2; $v(y)=-\int^y u_0(x)dx$—curve 3. y_1 and y_2 are the coordinates of the absolute maxima of $G(y,-x,t)$ and $G(y,x,t)$, respectively.

tion of the law of field growth from linear in the intervals between the discontinuities. As is seen from (3.5), Δy is much smaller than the characteristic period $1/\omega$, of the initial field and, in turn, is much less than ρ_k. The difference of y_n from $\tilde{y}_n$ leads to the result that the terms in (3.3) and (3.4) corresponding to the case in which there are no discontinuities in the interval $2x$, are not delta functions. However the smearing of delta functions of the order of (3.5) is much less than the characteristic width w_p of the order of $l(t)\sim(\sigma t/\lambda)^{1/2}$.

We find $w_p(y_1,y_2;2x,t)$. Let H_1 and H_2 by the values of the absolute maxima of $G(y,-x,t)$ and $G(y,x,t)$, the coordinates of which are y_1 and y_2, i.e.,

$$v(y)\leqslant\alpha_1(y)=H_1+(x+y)^2/2t,\quad v(y)\leqslant\alpha_2(y)=H_2+(x-y)^2/2t$$

(see Fig. 3). The parabolas $\alpha_1(y)$ and $\alpha_2(y)$ intersect at the point $y^*=(H_2-H_1)$ while, from the fact that the maxima are $t/2x$, located at different points, the condition $y_1\leqslant y^*\leqslant y_2$ are satisfied, or

$$2y_1x/t\leqslant H_2-H_1\leqslant 2y_2x/t. \tag{3.6}$$

We now introduce the intervals $L_1=[y_1,y_1+\Delta y_1]$, $L_2=[y_2,y_2+\Delta y_2]$, such that $\Delta y_{1,2}\gg\rho_c$. Then the probability of finding $y(-x,t)$ and $y(x,t)$ in L_1,L_2 with the maxima in the intervals $H_1,H_1+\Delta H_1$ and $H_2,H_2+\Delta H_2$ is equal to the probability that in L_1,L_2 the values of the absolute maxima lie in the intervals $H_1,H_1+\Delta H_1$ and H_2, $H_2+\Delta H_2$, while the values of the absolute maxima outside the intervals L_1,L_2 are less than H_1,H_2, i.e.,

$$v(y)\leqslant\min(\alpha_1(y),\alpha_2(y)).$$

According to the condition $\Delta y_{1,2}\gg\rho_c$, these events are statistically independent. Moreover, at $\sigma t/\lambda\rho_c^2\gg 1$ the condition $H\gg\sigma$ is valid and the integral function of the absolute maximum is equal to (2.6). Integrating the probability of the absolute maxima and of their coordinates in the region L_1,L_2, $H_1,H_1+\Delta H_1$ and $H_2,H_2+\Delta H_2$ over all possible values of H_1 and H_2, with account of (3.6), we get for the probability of finding $y(-x,t)$ and $y(x,t)$ in L_1,L_2,

$$P(y(-x,t)\in L_1,y(x,t)\in L_2)=\int_0^\infty dH_1\int_{H_1+2y_1x/t}^{H_1+2y_2x/t}dH_2N_H'(H_1;[y_1,y_1+\Delta y_1])$$
$$\times N_H'(H_2;[y_2,y_2+\Delta y_2])\exp\{-N(H_1;(-\infty,(H_2-H_1)t/2x))-N(H_2;((H_2-H_1)t/2x,\infty))\}. \tag{3.7}$$

As earlier, we now take the limit as $\Delta y_{1,2}\to 0$ and use the expression (2.8) for $N(H;[a,b])$. As in the one-dimensional distribution, the basic contribution to the integral (3.7) is made by the regions near $H_1\sim H_2\sim H_0\gg\sigma_*$. Transforming in (3.7) to integration over the variables u and v:

$$H_1=H_0+\sigma_*^2u/H_0,\quad H_2=H_0+\sigma_*^2v/H_0$$

and using (2.11) and (2.12), we obtain

$$w_p(y_1,y_2;2x,t)=\left(\frac{\lambda}{2\pi}\right)^2\exp\left\{-\frac{H_0^2}{\sigma_*^2}-\frac{(y_1+x)^2}{2l^2}-\frac{(y_2-x)^2}{2l^2}\right\}$$
$$\times\int_{-\infty}^{\infty}du\int_{u+2y_1x/l^2}^{u+2y_2x/l^2}dv\exp\left\{-u-v-e^{-u}\Phi\left(\frac{1}{l}\left(x+\frac{l^2}{2x}(v-u)\right)\right)-e^{-v}\Phi\left(\frac{1}{l}\left(x-\frac{l^2}{2x}(v-u)\right)\right)\right\}. \tag{3.8}$$

Here and below $l=l(t)$ is the external scale of turbulence (2.14). Transforming to the dimensionless variables $s=x/l$ and $z_{1,2}=y_{1,2}/l$, taking it into account that according to (2.11) and (2.14),

$$(\lambda/2\pi)^2\exp\{-H_0^2/2\sigma_*^2\}=1/l^2,$$

making the change of variables $u=u$ and $z=(v-u)/l$, we get after integration over u

$$\tilde{w}_p(z_1,z_2;2s)=\exp\left\{-\frac{(z_1+s)^2}{2}-\frac{(z_2-s)^2}{2}\right\}$$
$$\times\int_{z_1}^{z_2}\frac{2sdz}{[\Phi(s+z)\exp\{sz\}+\Phi(s-z)\exp\{-sz\}]^2}. \tag{3.9}$$

The discrete part of the distribution is found in similar fashion. Finally, for the two-point distribution of the coordinates of the absolute maxima we have

$$\tilde{w}(z_1,z_2;2s)=\frac{\delta(z_1-z_2)\exp\{-z_1^2/2-z_2^2/2\}}{\Phi(s+z_1)\exp\{sz_1\}+\Phi(s-z_2)\exp\{-sz_2\}}$$
$$+\exp\left\{-\frac{(z_1+s)^2}{2}-\frac{(z_2-s)^2}{2}\right\}\int_{z_1}^{z_2}\frac{2sdz}{[\Phi(s+z)\exp\{sz\}+\Phi(s-z)\exp\{-sz\}]^2}. \tag{3.10}$$

Transforming from $y(-x,t)$ and $y(x,t)$ to $u(-x,t)$ and $u(x,t)$ (1.4), and introducing

$$U_1=\frac{t}{l}u(-x,t)=\frac{1}{b}u(-x,t),\quad U_2=\frac{t}{l}u(x,t)=\frac{1}{b}u(x,t), \tag{3.11}$$

we obtain the expression for the dimensionless two-point probability density of the OAT

$$W(U_1,U_2;2s)=\langle\delta(U_1+s+z_1)\delta(U_2-s+z_2)\rangle=\tilde{w}(-U_1-s,-U_2+s;2s)$$
$$=\frac{\delta(U_2-U_1-2s)}{\Phi(-U_1)\exp\{U_1^2/2\}+\Phi(U_2)\exp\{U_2^2/2\}}$$
$$+\exp\left\{-\frac{U_1^2}{2}-\frac{U_2^2}{2}\right\}\int_{-s-U_1}^{s-U_2}\frac{2sdz}{[\Phi(s+z)\exp\{sz\}+\Phi(s-z)\exp\{-sz\}]^2}. \tag{3.12}$$

It is seen from (3.12) that the two-point probability density, as in the one-point case (2.15), depends at $l(t)\gg\rho_c$ only on the external scale of turbulence $l(t)$ (2.14) and is self-similar. The first term in (3.12) describes the properties of $\tilde{W}(U_1,U_2;2s)$ in the absence of discontinuities in the interval $2x$, and its normalization is equal to the probability that there are no discontinuities in the interval $2x$. It is seen from (3.12) that as $s\to 0$ and $s\to\infty$, the two-point distribution transforms into

$$W(U_1,U_2;2s)=\begin{cases}\delta(U_1-U_2)\dfrac{1}{(2\pi)^{1/2}}\exp\left\{-\dfrac{U_1^2}{2}\right\}, & s\to 0\\ \dfrac{1}{2\pi}\exp\left\{-\dfrac{U_1^2}{2}-\dfrac{U_2^2}{2}\right\}, & s\to\infty,\end{cases} \quad (3.13)$$

i.e., it tends to become Gaussian. This is connected with the fact that as $s\to 0$, $U_2\to U_1$ and the two-point distribution degenerates into a one-dimensional one, while as $s\to\infty$, the fields U_1 and U_2 become statistically independent and the two-point probability density factorizes into a product of one-dimensional probabilities. At $s\sim 1$, the distribution (3.12) is essentially non-Gaussian.

The distribution (3.12) determines all the two-point moments of the turbulence functions. In particular, we have for the correlation coefficient of the dimensionless fields U_1 and U_2 with the dimensionless spectrum from (1.4), (3.10) and (3.12),

$$R(2s)=\langle U_1U_2\rangle=\langle(z_1+s)(z_2-s)\rangle$$
$$=\frac{\partial}{\partial s}\left[s\int_{-\infty}^{\infty}\frac{dz}{\Phi(s+z)\exp\{(s+z)^2/2\}+\Phi(s-z)\exp\{(s-z)^2/2\}}\right] \quad (3.14)$$
$$(s>0),$$

$$S(\varkappa)=\frac{1}{\pi}\int_0^{\infty}R(2s)\cos(2s\varkappa)\,d2s. \quad (3.15)$$

Correspondingly, the correlation function and the energy spectrum of the OAT are

$$B(\rho,t)=\langle u(x,t)u(x+\rho,t)\rangle=\frac{l^2}{t^2}R\left(\frac{\rho}{l}\right),$$

$$S(k,t)=\frac{1}{\pi}\int_0^{\infty}B(\rho,t)\cos\rho k\,d\rho=\frac{l^3}{t^2}S(kl). \quad (3.16)$$

These same formulas for the spectrum and correlation function of the OAT were found by another method in Ref. 22, where the initial conditions were approximated by a discrete sampling, so that it was impossible to connect the external scale of turbulence $l(t)$ with the scales of the initial field in unambiguous fashion. From (3.14)–(3.16) it is not difficult to find the asymptotic behavior of the correlation function and of the energy spectrum of the OAT. By virtue of discontinuous character of the OAT field, the correlation function is not analytic at $s\to 0$, while the energy spectrum decays according to a power law:

$$R(2s)\approx 1-\frac{4}{\sqrt{\pi}}|s|, \quad |s|\ll 1, \quad (3.17)$$

$$S(k,t)=\frac{2l}{k^2\pi^{1/2}t^2}=\frac{2(\sigma_*/\lambda)^{1/2}}{(\pi t)^{3/2}(\ln\gamma(t))^{1/4}}\frac{1}{k^2}, \quad kl\gg 1. \quad (3.18)$$

In the region of small wave numbers, we have for the spectrum

$$S(k,t)\sim k^2l^5/t^2\sim k^2t^{1/2}, \quad (kl)^2\ll 1. \quad (3.19)$$

According to (3.14) and (3.16), the maximum of the spectrum is displaced in the direction of small wave numbers in proportion to $1/l(t)$. Physically, this picture of the evolution of the spectrum is due to the following: the appearance of discontinuities in the wave leads to dissipation of the energy of the wave and to the appearance of slowly decaying components of the spectrum in the region of large wave numbers. The growth of the low-frequency components is connected with the growth of the external scale of turbulence because of the merging of the discontinuities.

4. ACOUSTIC TURBULENCE AT FINITE REYNOLDS NUMBERS. FINAL STAGE OF DEGENERACY OF THE OAT

At finite viscosity μ (in the case of finite Reynolds numbers), the shock fronts of a sawtooth wave have finite duration, which leads to the appearance of a nonzero internal turbulence scale l_0. Moreover, because of the growth in the width of the shock front and the dissipation of energy over rather long times, the wave propagation becomes linear, the nonlinear effects are unimportant, and the degeneracy of the turbulence is due only to linear dissipation.

We limit ourselves below to the case of sufficiently large Reynolds numbers, when the OAT at $\sigma\omega_1 t\gg 1$ has the form of a random sequence of sawtooth pulses with a shock-front width $\delta_n=4\mu t/\eta_n$, with $\delta_n\ll\eta_n$.[22] In the analysis of the OAT characteristics associated with the finiteness of the internal scales, we need to know $g(\eta)$—the distribution of distances between neighboring "zeros" over a unit length. We find the statistics of η, assuming that $l(t)\gg\rho_c$, i.e., $\sigma t/\lambda\rho_c^2\gg 1$. Let H_1 and H_2 be values of the absolute maxima of $v(y)$ in the intervals $y_1, y_1+\Delta y_1$ and $y_2, y_2+\Delta y_2$, and let $w(H;\Delta)$ be is their probability density, which, at $H\gg\sigma_*$, is equal to

$$w(H;\Delta)=\frac{\partial}{\partial H}\exp\left\{-\frac{\lambda\Delta}{2\pi}\exp\left\{-\frac{H^2}{2\sigma_*^2}\right\}\right\}. \quad (4.1)$$

Under the condition $H_1-(x-y_1)^2/2t=H_2-(x-y_2)^2/2t=H$ the function $G(y,x,t)$ has two maxima, equal to H, while x here is the coordinate of the discontinuity. Taking into account the independence of H_1 and H_2, we have for the joint distribution of H and x

$$w(H,x;\Delta_1,\Delta_2)$$
$$=\frac{y_2-y_1}{t}w\left(H+\frac{(y_1-x)^2}{2t};\Delta_1\right)w\left(H+\frac{(y_2-x)^2}{2t};\Delta_2\right). \quad (4.2)$$

Multiplying (4.2) by the probability that $G<H$ in the remaining regions, and integrating over H and x under the assumptions similar to those made in Secs. 2 and 3, we obtain

$$P(y_1\in\Delta_1,y_2\in\Delta_2)=\frac{(y_2-y_1)\Delta_1\Delta_2}{2\pi^{1/2}l^3}\exp\left\{-\frac{(y_2-y_1)^2}{4l^2}\right\}. \quad (4.3)$$

Obviously, the equality $P(y_1\in\Delta_1,y_2\in\Delta_2)=g(y_2-y_1)\Delta_1\Delta_2$, is valid, where $g(\eta)\Delta\eta$ is the mean number of distances between the "zeros" per unit length in the range from η to $\eta+\Delta\eta$. Therefore, in accord with (4.3),

$$g(\eta)=\frac{\eta}{2\pi^{1/2}l^3}\exp\left\{-\frac{\eta^2}{4l^2}\right\}, \quad \eta>0. \quad (4.4)$$

By definition,

$$\int_0^{\infty}g(\eta)\,d\eta=\nu=1/\pi^{1/2}l$$

is the mean number of zeros per unit length, while

$$\int_0^{\infty}\eta g(\eta)\,d\eta=1$$

is the normalization condition.

For as analysis of the energy spectrum, it is convenient to introduce the gradient of the acoustic field $J = u_x'$. From the fact that the shape of the shock front is identical with the shape of the stationary shock front, we obtain

$$J=u_x'=\frac{1}{t}-\frac{1}{t}\sum_n \mathrm{ch}^{-2}\left(\frac{x-\xi_n}{\delta_n}\right)\frac{\eta_n}{2\delta_n},\quad \delta_n=\frac{4\mu t}{\eta_n}. \tag{4.5}$$

For the ratio of the shock front width $\delta_n = 4\pi t/\eta_n$ to the distance η_n, between zeros we have

$$\frac{\delta_n}{\eta_n}=\frac{4\mu t}{\eta_n^2}\sim\frac{\mu t}{l^2}\sim\frac{(\ln\gamma(t))^{1/2}}{\mathrm{Re}},\quad \mathrm{Re}=\left(\frac{\mu\lambda}{\sigma}\right)^{-1}. \tag{4.6}$$

Here Re is the acoustic Reynolds number. At $\delta/l \ll 1$, which is satisfied at large Re and not very long times, the field $J(x,t)=u_x''$ constitutes a sequence of narrow, practically non-overlapping pulses, the amplitude of which is inversely proportional to the viscosity coefficient μ. The energy spectrum here has the form of the spectra of the individual discontinuities and of an infinite sum describing their interference.[28,25] At $k \gg l^{-1}$, the interference can be neglected, and we have for the energy spectrum of the OAT

$$S(k,t)=2\pi\mu^2\int_0^\infty\frac{g(\eta)\,d\eta}{\mathrm{sh}^2(2\pi\mu tk/\eta)}. \tag{4.7}$$

Introducing the internal scale of turbulence in correspondence with (4.6),

$$l_0=\frac{\mu t}{l}=\frac{\mu tl}{l^2}=\frac{(\ln\gamma)^{1/2}}{\mathrm{Re}}l, \tag{4.8}$$

we find that at $k \gg l_0^{-1}$ the viscosity does not affect the behavior of the spectrum even in the inertial interval

$$S(k,t)=\frac{2\pi\mu^2}{(2\pi\mu t)^2k^2}\int_0^\infty\eta^2g(\eta)\,d\eta=\frac{2l}{\pi^{1/2}t^2k^2}, \tag{4.9}$$

which is identical with the asymptote obtained above for the spectrum (3.8) by a different method. At $k \gg l_0^{-1}$, (4.7) transforms into

$$S(k,t)=8\pi\mu^2\int_0 g(\eta)\exp\left\{-\frac{4\pi\mu tk}{\eta}\right\}d\eta. \tag{4.10}$$

Here the exponential factor describes the decay of the spectrum of a single discontinuity, while the averaging is carried out over the distance between the zeros η, which determines the width of the shock front and, consequently, the rate of falloff of the spectrum of a single discontinuity. At $k \gg l_0^{-1}$ we have from (4.4), (4.10)[29]

$$S(k,t)=\frac{16\pi^{1/2}\mu^2}{l}\int_0^\infty z\exp\left\{-z^2-\frac{2\pi l_0k}{z}\right\}dz$$

$$\approx\frac{16\pi\mu^2}{l\sqrt{3}}(\pi l_0k)^{1/3}\exp\{-3(\pi l_0k)^{2/3}\}. \tag{4.11}$$

We note that in the dissipative interval $k \gg l_0^{-1}$, the energy spectrum falls off more slowly than for a periodic wave, where the damping decrement is proportional to k. This is connected with the fluctuations of the width of the shock front, which change the law of energy decay.

As is seen from (4.8), at finite Reynolds numbers and $(\ln\gamma)^{1/2} \ll \mathrm{Re}$ the internal l_0 and external l scales of turbulence are substantially different and in this case there is an inertial interval in the energy spectrum $l^{-1} \ll k \ll l_0^{-1}$, where the behavior of the spectrum is determined only by the nonlinear effects and $S(k,t) \sim k^{-2}$. However, as t increases, the relative width of the shock front (4.6) increases logarithmically and at $(\ln\gamma)^{1/2} \sim \mathrm{Re}$ the characteristic width of the shock front becomes comparable with the external scale of the turbulence. Here the role of the nonlinear effects becomes insignificant and the wave propagation becomes linear with its damping due only to linear dissipation. An estimate of the time at which the OAT enters the linear regime can be obtained from the condition $l_0 \approx l$, which leads to the following estimate:

$$t_* \sim \frac{1}{\sigma\lambda}\exp\{\mathrm{Re}^2\}. \tag{4.12}$$

The entry of the wave into the linear regime changes qualitatively the picture of the degeneracy of the turbulence in the final stage. At $t > t_*$, the evolution of the energy spectrum of the wave is determined only by the factor $\exp\{-2\mu k^2 t\}$, which, in accord with (1.1), describes the linear damping of the energy of the wave. Since the damping in the linear stage does not change the behavior of the spectrum at $k \to 0$, the slope of the energy spectrum as $k \to 0$ and $t > t^*$ is an invariant. For an estimate of the behavior of the spectrum as $k \to 0$, we can use the expression (3.19), assuming that at $t > t^*$ the nonlinear effects are no longer significant and, setting $t = t^*$ in (3.19),

$$S(k,t>t_*)\underset{k\to 0}{\sim}\frac{k^2l^3(t_*)}{t_*^2}\approx\frac{k^2\mu^2\exp\{\mathrm{Re}^2/2\}}{\mathrm{Re}^{1/2}\lambda}. \tag{4.13}$$

A more rigorous analysis of the final stage of the degeneracy of the OAT, given in Ref. 30 on the basis of the exact Hopf solution (1.2) of the Burgers equation, shows that the estimate (4.13) is qualitatively correct. Namely, in the concluding stage we have, for the energy spectrum and for the energy of the OAT itself[30]

$$S(k,t)=4\mu^2k^2C\exp\{-2\mu k^2t\},\ \langle u^2\rangle=(\pi\mu/2)^{1/2}Ct^{-3/2},$$

$$C=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\exp\left\{\frac{1}{4\mu^2}B_0(\rho)\right\}-1\right]d\rho\approx\frac{\exp\{\mathrm{Re}^2\}}{(2\pi)^{1/2}\mathrm{Re}\,\lambda},$$

where $\mathrm{Re}=\sigma/2\mu\lambda$. The exponential increase in the slope of the spectrum as $k \to 0$ with increase in Re is due to the multiple merging of the shock fronts and to the growth of the external scale during the nonlinear stage of development of the OAT.

The authors are grateful to A. N. Malakhov for useful discussion of the research.

[1]L. M. Burgers, Roy. Neth. Acad. Sci. 17, 1 (1939).
[2]A. S. Monin and A. M. Yaglom, Statisticheskaya gidromekhanika (Statistical Hydrodynamics), Vol. 1, Nauka, Moscow, 1965.
[3]O. V. Rudenko and S. I. Soluyan, Teoreticheskie osnovy nelineinoi akustiki (Theoretical Foundations of Nonlinear Acoustics), Nauka, Moscow, 1975)
[4]T. Tatsumi and H. Tokunaga, J. Fluid Mech. 65, 581 (1974).
[5]L. A. Ostrovskiĭ, Ixv. vuzov. Radiofizika 19, 661 (1976).
[6]W. H. Reid, Appl. Sci. Res. A6, 85 (1975).
[7]W. C. Meecham and A. Siegel, Phys. Fluids 7, 1178 (1964).

[8]W. C. Meecham, P. P. Iyer and W. C. Clever, Phys. Fluids **18**, 1610 (1975).
[9]S. A. Orszag and L. R. Bissonette, Phys. Fluids **10**, 2603 (1967).
[10]M. Yano and T. Imamura, Phys. Fluids **15**, 108 (1972).
[11]R. H. Kraichnan, Phys. Fluids **11**, 265 (1968).
[12]J. J. Walton, Phys. Fluids **13**, 1364 (1970).
[13]S. S. Moiseev, A. V. Tur, and V. V. Yanovskiĭ, Izv. vuzov. Radiofizika **20**, 1032 (1977)
[14]S. S. Moiseev, A. V. Tur, and V. V. Yanovskiĭ, Izv. vuzov. Radiofizika **23**, 68 (1980).
[15]E. Hopf, Comm. Pure Appl. Math. **6**, 707 (1950).
[16]R. Courant, Partial Differential Equations (Russ. Transl.), Mir, 1964.
[17]A. N. Malakhov and A. M. Saichev, Zh. Eksp. Teor. Fiz. **67**, 940 (1974). [Sov. Phys. JETP **40**, 467 (1975)].
[18]A. I. Saichev, Izv. vuzov. Radiofizika **17**, 1025 (1974).
[19]O. V. Rudenko and A. S. Chirkin, Dokl. Akad. Nauk SSSR **214**, 1045 (1974) [Sov. Phys. Doklady **19**, 64 (1975)].
[20]O. V. Rudenko and A. S. Chorkin, Zh. Eksp. Teor. Fiz. **67**, 1903 (1974) [Sov. Phys. JETP **40**, 950 (1975)].
[21]T. Tatsumi and S. Kida, J. Fluid Mech. **55**, 659 (1972).
[22]S. Kida, J. Fluid Mech. **93**, 337 (1979).
[23]O. V. Rudenko and A. S. Chirkin, Dokl. Akad. Nauk SSSR **225**, 520 (1975) [Sov. Phys. Doklady **20**, 748(1975)].
[24]S. N. Gurbatov, A. N. Malakhov, Akust. Zh. **23**, 569 (1977) [Sov. Phys. Acoustics **23**, 325 (1977)].
[25]S. N. Gurbatov and L. G. Shepelevich, Izv. vuzov. Radiofizika **21**, 1627 (1978).
[26]V. I. Tikhonov, Vybrosy sluchaĭnykh protseeov (Problems of Random Processes) Nauka, Moscow, 1970).
[27]H. Cramer and M. R. Leadbetter, Stationary and Related Stochastic Processes, Wiley, New York, 1967.
[28]S. M. Rytov, Vvedenie v statisticheskuyu radiofizika (Introduction to Statistical Radiophysics) Nauka, Moscow, 1966.
[29]Handbook of Special Functions (M. Abramowitz and I. Stegun, eds.), Dover, 1964.
[30]S. N. Gurbatov, A. A. Dubkov and A. N. Malakhov, Zh. Eksp. Teor. Fiz. **22**, 456 (1977) [Sov. Phys. JETP **45**, 239 (1977)].

Translated by R. T. Beyer