Removal of causal particle horizon at anisotropic cosmological singularities

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The problem of the existence of a causal particle horizon near the initial singularity in anisotropic homogeneous cosmology is examined. Some new types of null singularity (one vacuum and three material) in Bianchi type VI models are presented in which there is no causal particle horizon along a preferred direction of the homogeneous space V_3. It is established that the special vacuum metric of type VI_h, which is identical to the Lifshitz-Khalatnikov solution, has a wave algebraic structure II(N) of the curvature tensor, which has vanishing invariants. It contains a null Killing-Cauchy horizon which is regular in vacuum and includes in addition a stationary inhomogeneous L region of V_4. This Lifshitz-Khalatnikov vacuum asymptotic behavior in models of VI_h type determines a specific nonscalar null singularity, at which all the components of the Riemann and Ricci tensor are infinite (together with the physical characteristics of the dynamically unimportant matter), although the curvature invariants of the V_4 wave metric are zero.

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1. In an isotropic Friedmann cosmology there necessarily exists (if the cosmological constant is zero, $\Lambda=0$) an initial singularity in the past, at which the gravitating matter is collected at a point and its physical characteristics (energy density ε, pressure P, temperature T, etc.) and the curvature of the spacetime V_4 become infinite. This singular spacelike boundary $\tau=0$ of V_4 is impenetrable for causal influences and prevents the establishment of causal correlations in the initial singular state of the Universe through the physical processes one could presume in the preceding phase of cosmological contraction. Therefore, the initial singularity in classical cosmology must be a source of all matter and the necessary information that determines the subsequent evolution and physical processes for the expanding Universe.

In isotropic Friedmann models near the singularity $\tau=0$ there is always a particle horizon, so that at the time τ after the beginning of the expansion phase the causal connection can encompass only restricted spatial regions with scale $l\leq c\tau$. This generates the problem of the causal connection of the Universe, and it is rather difficult to explain the observed large-scale ($l \gtrsim c\tau$) correlation of its properties, namely, the high degree of isotropy and homogeneity extending beyond the Hubble radius [currently, $R_H\approx c\tau_H\approx 3\times 10^{28}$ cm (Ref. 1)]. In particular, it is hard to understand why the temperature $T\approx 2.7$ °K of the microwave background is the same to a high accuracy (~0.1%) over the whole of the celestial sphere despite the fact that on angular scales $\vartheta > 30°$ this radiation was emitted by causally disconnected regions of plasma at red shifts $z\sim 10$ (or even $\vartheta > 10'$ if there was no secondary ionization of the plasma and its interaction with the radiation ceased at the recombination era $z\sim 10^3$).[2]

The attempts to solve the problem of the causal horizon give up the cosmological postulate of isotropy and homogeneity in the early stages in the cosmological expansion and have been undertaken in the framework of anisotropic and, mainly, homogeneous cosmology (see Refs. 2–15). The main motive for investigating the problem of the existence and possible nature of cosmic singularities, and also the structure of their causal horizon[8-16] was the hope that in general relativity it will be possible to find a fairly large class of anisotropic models in which the initial singularity does not have a particle horizon, so that all the spatial regions of the Universe near it are causally connected. The existence of such singularities without causal particle horizon would justify Misner's conjecture of a random (mixmaster) cosmology[11] and would make it possible in principle to explain the high observed isotropy and homogeneity of the Metagalaxy by means of physical dissipative processes in the initial phase of the expansion (such as neutrino or graviton viscosity) rather than by the *a priori* hypothesis of maximal symmetry of the Universe.

In anisotropic cosmology it is also impossible to avoid an initial singularity, at which the invariants of the vacuum Weyl conformal curvature tensor and all the physical characteristics of the matter are infinite; in fact, the typical singularities are here of vacuum type,[3-10] determined by the free gravitational field and independent of the presence of material sources, in particular, a fluid with $P<\varepsilon$ (Refs. 12–15):

$$T_i^k=(\varepsilon+P)u_iu^k-P\delta_i^k,\quad P=P(\varepsilon).$$

Near general vacuum singularities with Kasner (as, for example, in the homogeneous Bianchi types I–VII)[12-16] and oscillator (in Bianchi types VIII and IX)[8,9] asymptotic behavior there exists a causal particle horizon,[10,12] as for a material quasi-isotropic singularity of Friedmann type.

There is only one known example[1)] of partial elimination of the causal horizon near an anisotropic special singularity, which, according to the Lifshitz–Khalatnikov classification,[3,7] corresponds in a comoving, synchronous coordinate frame to degeneracy of the V_4 metric with simultaneous vanishing of one of its principal values and determinant quadratically with re-

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spect to the proper time τ. The singularity is manifested as a simultaneous collapse of the space V_3 (τ = const) into a pancake, the world lines being focused, as it were, on a two-dimensional focal caustic with $\varepsilon > P \to \infty$. Such a material singularity is realized for axisymmetric homogeneous metrics of the Bianchi types I, II, III, VII_0, VIII, IX (excluding VI_0), which form the family of so-called T ($N=0$) (Ref. 17) and Taub–NUT models with matter at rest and $P < \varepsilon$ (for details, see Refs. 14 and 15):

$$-ds^2=-d\tau^2+X^2(\tau)\left[dx_1+2Nh(x_2)dx_3\right]^2+Y^2(\tau)\left[dx_2^2+\Sigma^2(x_2)dx_3^2\right],$$

$$\Sigma(x)=\begin{cases}\sin x\\ x\ ;\\ \sinh\end{cases}\quad h(x)=\begin{cases}\cos x\\ x^2/2\ ;\\ \cosh x\end{cases}\quad N=\begin{cases}0 & (\mathrm{I,\ VII_0;\ III})\\ 1 & (\mathrm{II,\ VIII,\ IX}).\end{cases} \tag{1}$$

This singularity in the axisymmetric T and Taub-NUT models (1) is characterized by a kinematic Kasner asymptotic behavior of the collapse of V_3 (τ = const) into a pancake of the following form for $P=n\varepsilon$ ($0 \leq n < 1$) (Ref. 14):

$$X(\tau)\propto\tau(1+\alpha\tau^{1-n})\to 0,\quad Y(\tau)\propto[1+\beta\tau^{1-n}]\to\mathrm{const},$$
$$\varepsilon,\ P\propto V^{-(1+n)}\propto\tau^{-(1+n)}\to\infty,\ \alpha,\ \beta=\mathrm{const}. \tag{2}$$

Near such a singularity there is no causal horizon along the symmetry axis, since a light ray ($ds^2=0$) traverses an infinite spatial distance $x_1 \propto |\ln\tau| \to \infty$ when $\tau \to 0$, so that all events in an infinite cylinder of radius $c\tau \to 0$ can be causally connected.

For vacuum Einstein fields (and also in the presence of a free electromagnetic field and $\Lambda \neq 0$), such an asymptotic behavior of simultaneous collapse of V_3 into a pancake with kinematic set of Kasner exponents $p_1 = 1, p_2 = p_3 = 0$ (Refs. 3, 4, and 7) corresponds to a fictitious singularity of the metric (1) like the Schwarzschild pseudosingularity,[17] which can be eliminated by coordinate transformations and can be observed only from the nonstatic homogeneous T region within the gravitational radius (see Refs. 14 and 15). Such pseudosingularities of the metrics (1) and (2) are actually regular but physically singular null hypersurfaces with isotropic normal ($n_i n^i = 0$) that are tangent to the local light cones and are Cauchy event horizons in V_4, i.e., semipermeable causal membranes through which light rays and all causal signals can pass in only one direction. They are invariant Killing horizons on which the V_4 field changes its symmetry—from being homogeneous to be static—so that these regular null boundaries in vacuum separate causally connected homogeneous T and stationary L regions of a space-time V_4 of fairly complicated global structure (for details, see Refs. 14 and 15).[2)]

If a gravitating fluid with $P=n\varepsilon$ ($0 \leq n < 1$) is present in the T regions (1), the fictitious singularity (2) on the boundary null horizons is transformed into a physical material singularity at which $\varepsilon, P \propto V^{-(1+n)} \propto \tau^{-(1+n)} \to \infty$. If $P < \varepsilon$, it preserves its previous null orientation, which is why there is no causal horizon along the symmetry axis of V_3. Note that a maximally hard fluid with $P = \varepsilon$ in the T regions (1) completely destroys the vacuum null horizons and replaces them by singular spacelike boundaries of V_4, which do have a causal horizon; this is because the gravitation of such a fluid leads to an asymptotic behavior of anisotropic point collapse different from (2).[14,15]

It should be emphasized that the kinematic Kasner asymptotic behavior (2) can be regarded locally as the metric of a flat space-time in a noninertial frame of reference of moving test particles. For example, the axisymmetric metric of Bianchi type I with homogeneous Euclidean space V_3 (τ = const) of the form

$$-ds^2=-d\tau^2+\tau^2dx^2+dy^2+dz^2 \tag{3}$$

can be reduced by the coordinate transformation $X = \tau \sinh x$ and $T = \tau \cosh x$ to the Minkowski metric. Therefore, the removal of the causal horizon in (2) can be regarded as a kinematic effect independent of gravitation in the framework of special relativity alone. Namely, the free motion of the test particles used to construct the synchronous frame (3) is such that there is an infinite number of ultrarelativistic particles within the light cone as $\tau \to 0$.

2. In the present paper, we wish to point out that a similar partial elimination of the causal horizon occurs for a number of anisotropic singularities of special form [and very different from (1)–(3)] in homogeneous cosmological models of Bianchi type VI with a common "diagonal" metric[12,13,15b]

$$-ds^2=-d\tau^2+X^2(\tau)dx_1^2+Y^2(\tau)\exp\left[-2(a_0+k_0)x_1\right]dx_2^2+$$
$$+Z^2(\tau)\exp\left[-2(a_0-k_0)x_1\right]dx_3^2. \tag{4}$$

This metric belongs to class B, has negative anisotropic curvature of a V_3 space of "open" type,

$$K_1^1=-2(a_0^2+k_0^2)/X^2,\ K_2^2=-2(a_0^2+a_0k_0)/X^2,\ K_3^3=-2(a_0^2-a_0k_0)/X^2,$$
$$K=-2(3a_0^2+k_0^2)/X^2<0;\ a_0,\ k_0=\mathrm{const}\ , \tag{5}$$

and includes the axisymmetric VI_0 type of class A as a limiting case with $a_0 = 0, Y(\tau) = Z(\tau)$.[12]

We consider first the special solution of the Einstein vacuum equations $R_{ik} = 0$ for the metric (4) found in the analytic form[15b]

$$X=(1+\delta^2)\tau,\ Y=\tau^{(1+\delta)/(1+\delta^2)},\ Z=\tau^{(1-\delta)/(1+\delta^2)},\ \delta=k_0/a_0=\mathrm{const}, \tag{6}$$

which augments the general vacuum solution for VI_h type.[12] The special metric (6) has an anomalous vacuum singularity at $\tau = 0$ with infinite components of the Riemann curvature tensor, which is of algebraic type II(N) in the Petrov classification[2,3] and has vanishing values for all its invariants (such as $I_1 = R_{ijkl}R^{ijkl} = 0$):

$$R_{02}{}^{02}=R_{13}{}^{13}=-R_{12}{}^{12}=-R_{03}{}^{03}=\delta(1-\delta^2)/(1+\delta^2)^2\tau^2=\mu;$$
$$R_{12}{}^{02}=-R_{13}{}^{03}=\mu\tau e^{-x_1}. \tag{7}$$

This singularity of (6) is qualitatively different from the linear Kasner singularity[12,15] characteristic of the VI_h type models and corresponds to a singular degenerate regime of anisotropic collapse of V_3 into a line ($|\delta| > 1$) or a point ($|\delta| < 1$), the spatial anisotropic curvature of (5) being dynamically important in this case:

$$|K|\propto X^{-2}\propto\tau^{-2}\to\infty.$$

It was found that the special vacuum solution (6) for the Bianchi type VI_h (4) is actually equivalent to the homogeneous variant of the anomalous asymptotic behavior near the singularity, being a degenerate special case of the Kasner singularity in null coordinates. This special case was noted and investigated earlier by Lifshitz and Khalatnikov (Ref. 7, Appendices B and F) on the basis of the following self-similar representation of its metric in a synchronous system[3]:

$$-ds^2=-d\tau^2+(\tau/x)^2dx^2+(\tau/x)^{2s_2}dy^2+(\tau/x)^{2s_3}dz^2,$$
$$s_2+s_3=s_2^2+s_3^2=\mathrm{const}\leqslant 2,\quad s_2=(1+\delta)/(1+\delta^2),\quad s_3=(1-\delta)(1+\delta^2). \tag{8}$$

Here, the dependence on the spatial coordinate x is unimportant, as in (4), since the components of the curvature tensor of V_4 are functions of the time τ alone and are equal to (7), while the metric (8) itself can be readily transformed to the homogeneous canonical form (4), (6) of the Bianchi type VI_h. The solution (8), (6) can also be written down in the null coordinates $\eta=\tau/\sqrt{2}x$, $\xi=-x\tau/\sqrt{2}$, which are more natural for it[7]:

$$-ds^2=2d\eta d\xi+\eta^{2s_2}dy^2+\eta^{2s_3}dz^2;\quad s_2+s_3=s_2^2+s_3^2, \tag{9}$$

and it then depends explicitly on only a single null coordinate.

But this form of the solution (9) in the null coordinates is more general than the original VI_h metric (6), (8), since it contains not only this homogeneous T region but also a static inhomogeneous L region of V_4 in vacuum with a new metric related to (8):

$$-ds^2=d\zeta^2-(\zeta/t)^2dt^2+(\zeta/t)^{2s_2}dy^2+(\zeta/t)^{2s_3}dz^2. \tag{10}$$

The transition from the extended metric in the null coordinates (9) to this static inhomogeneous solution of the vacuum Einstein equations $R_{ik}=0$ is realized by means of the substitution $\eta=\zeta/\sqrt{2}t$, $\xi=-\zeta t/\sqrt{2}$ or by the formal substitution $\tau\to i\zeta$, $x\to it$ in (8) and, accordingly, in (7). The resulting metric of its curvature tensor (7), depends essentially on only the single spatial coordinate ζ. Therefore, the spatial singularity at $\zeta=0$ can be interpreted as a localized δ-functional source of gravitating mass, which follows from qualitative analysis of the motion of test particles and light rays in such a field (10), (9) (like the static variant of the Kasner–Weyl metric).[18]

However, the timelike singularity ($\tau=0$) [for the homogeneous vacuum solution (6), (8)] and the spacelike singularity ($\zeta=0$) [for the static variant (10)] correspond to a physical singularity $\eta=0$ of the metric in the null coordinates (9), at which the components of the Riemann curvature tensor are infinite (although $I_1=R_{ijkl}R^{ijkl}=0$):

$$\mu=s_2(s_2-1)/2\eta^2\to\infty,\quad \eta\to 0. \tag{11}$$

Therefore, the singularities at $\tau=0$ and $\zeta=0$ of the solutions (6), (8), and (10) are in reality null and form a common boundary null surface $\eta=0$ in the global coordinates (9). Such a general metric (9) gives a simple analytic continuation between the causally connected homogeneous T region of (6), (8) and the static L region of (10), these forming together a single space-time V_4 (see Fig. 1). As is shown by the analysis below of the behavior of the geodesics for the metric (9), this metric gives a maximally extended V_4, since it possesses geodesic completeness in the sense that all possible

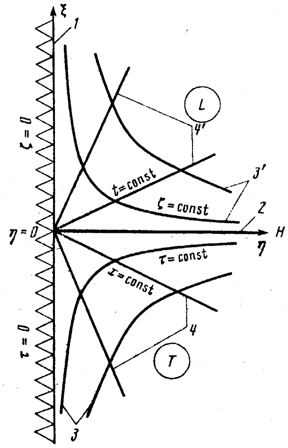

FIG. 1. Global structure of the complete space-time V_4 with Lifshitz–Khalatnikov vacuum metric (9). Here, 1 is the radiative null singularity $\eta=0$; 2 is the regular null Killing–Cauchy horizon separating the stationary L region and the homogeneous T region of Bianchi VI_h type (6); 3 and 3′ are the homogeneous spacelike V_3 and invariant timelike V_3^{ξ} sections, the transitivity hypersurfaces of the VI_h group G_3; 4 and 4′ are the geodesic rays which are orthogonal to them and in the T region coincide with the world lines of the fluid particles for the cosmological models of Bianchi type VI_h.

trajectories of test particles and light rays can be continued without limit with respect to the affine parameter or terminate on the true singular boundary $\eta=0$.

The special vacuum solution (6), (8) for the Bianchi type VI_h has a specific feature—the elimination of the homogeneity of the original V_4 metric due to the existence of the locally regular null Cauchy event horizon ($\xi=0$ in Fig. 1), beyond which there is the additional static L region (10). Through this invariant null boundary the world lines of particles and light rays (i.e, all causal influences) can pass only in one direction: from the T to the L region or vice versa, as for a semipermeable causal membrane. Such intermediate singularities in the form of regular Killing–Cauchy null horizons[19] are inherent in T–NUT universes,[14,15] and also for the axisymmetric Bianchi type V model with moving fluid.[20] They are, apparently, an exceptional phenomenon in general relativity.

In the T region (6), (8), the homogeneous spatial sections V_3 ($\tau=$const) degenerate in finite proper time into isotropic but different boundaries—the singularity $\eta=0$ and the regular Cauchy horizon $\xi=0$; when they pass through this horizon, they change their orientation to timelike and become transitivity hypersurfaces $\tilde{V}_3$ ($x=$const) of a group of motions G_3 of the same algebraic type VI_h in the static inhomogeneous L region (10).

In what follows, we shall be interested in only the homogeneous T region (6)–(9), which becomes an ordinary cosmological model of Bianchi type VI_h when it is filled by a gravitating fluid at rest with, for example, the equation of state $P=n\varepsilon$ ($0\leqslant n<1$). For this special case, the vacuum asymptotic behavior (6)–(8) of the collapse of V_3 into a line is preserved near the initial singularity if

$$n<\frac{(\delta^2-1)}{(3+\delta^2)}>0,\quad \delta^2>\frac{1+3n}{1+n}\geqslant 1,$$

when the dynamical influence of the gravitating fluid is unimportant, although it always eliminates the vacuum regime of point V_3 collapse (6)–(8) with $|\delta| < 1$.[15b] If matter is present in the T region (6)–(8), there is a radical change in the global structure of the V_4 (9), since the null Cauchy horizon is transformed into the physical singularity $\tau=0$ ($\varepsilon, P\to\infty$), and one can no longer have causal connection with the L region, which is eliminated from the cosmological models.

It is natural to expect that in these special type VI_h models near the initial singularity with vacuum (different from Kasner) asymptotic behavior (6)–(8) there will, because of the null orientation of the singularity, be no causal horizon in at least one of the directions of V_3 ($\tau=\mathrm{const}$). But to prove this, we must investigate the null geodesics in the T region (6)–(9), since even in "diagonal" models of class B the direction of propagation of the light rays in V_3 is not preserved,[12] so that the time dependence $X_a(\tau)\propto\tau^{s_a}$ of the scale factors along the principal axes near the singularity $\tau=0$ does not by itself indicate elimination of the causal horizon even if some exponents satisfy $s_a\geqslant 1$, as, for example, for the asymptotic behavior (6)–(8).

3. The equations of geodesics for the V_4 metrics (6)–(10) are obtained from the variational principle

$$\delta\int_A^B ds=0$$

and can be expressed in the Lagrangian or Hamiltonian forms

$$L=\frac{1}{2}g_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda},\quad H\frac{1}{2}g^{ik}p_ip_k,\quad p_i=g_{ik}\frac{dx^k}{d\lambda}=g_{ik}u^k. \tag{12}$$

Here, λ is an affine parameter determined by the normalization condition

$$2L=2H=g_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda}=g^{ik}p_ip_k=e=\pm1,0 \tag{13}$$

respectively, for timelike, spacelike, and null geodesics, respectively, i.e., it is equal to the proper time of test particles when $e=1$.

Because of the high symmetry of V_4—the explicit homogeneity of the metric (9) in the chosen null coordinates with no dependence at all on the spatial variables z, y, and ξ—the equations of the geodesics have three obvious integrals of the motion:

$$p_\xi=\frac{d\eta}{d\lambda}=\mathrm{const},\quad p_y=\eta^{2s_2}\frac{dy}{d\lambda}=\mathrm{const},\quad p_z=\eta^{2s_3}\frac{dz}{d\lambda}=\mathrm{const}, \tag{14}$$

and these, in conjunction with the normalization condition (13) in, for example, the case of null geodesics ($e=0$) of the form

$$-\frac{d\xi}{d\lambda}=\frac{1}{2p_\xi}\left(\frac{p_y^2}{\eta^{2s_2}}+\frac{p_z^2}{\eta^{2s_3}}\right);\quad \eta=\lambda p_\xi+\eta_0, \tag{15}$$

permit a complete investigation of the behavior of light rays in the V_4 vacuum field (9). They can be integrated in a closed analytic form and give the following expressions for $p_\xi\neq0, s_2, s_3\neq\frac{1}{2}$:

$$y-y_0=\frac{p_y}{p_\xi}\frac{\eta^{1-2s_2}}{1-2s_2},\quad z-z_0=\frac{p_z}{p_\xi}\frac{\eta^{1-2s_3}}{1-2s_3}, \tag{16}$$

$$-\xi+\xi_0=\frac{1}{2}\left(\frac{p_y}{p_\xi}\right)^2\frac{\eta^{1-2s_2}}{1-2s_2}+\frac{1}{2}\left(\frac{p_z}{p_\xi}\right)^2\frac{\eta^{1-2s_3}}{1-2s_3},$$

where $x_0, y_0, \xi_0=\mathrm{const}$. If $s_2=(1+\delta)/(1+\delta^2)>0$ or $s_3=(1-\delta)/(1+\delta^2)=\frac{1}{2}$ ($\delta=\sqrt{2}\pm1$), when the corresponding divergent factor $\eta^{1-2s_a}/(1-2s_a)$ must be replaced by $\ln\eta$.

In the degenerate case when $p_\xi=0$, we have one-dimensional motion: $p_y=0, p_z=0$, and the rays $\eta=\mathrm{const}$ parallel to the singular null boundary of V_4 form together with the family $\xi=\mathrm{const}$ the coordinate mesh of the null system (9) in Fig. 1.

It is readily seen from the relations (16) that for s_2, $s_3<\frac{1}{2}$ the light rays reach the singularity $\eta=0$ at finite values of all the coordinates $\xi, x_a=\mathrm{const}$, so that it has a causal horizon in the null system (9). However, if at least one of the exponents satisfies $s_a\geqslant\frac{1}{2}$, the rays reach the singularity $\eta=0$ only asymptotically with

$$\xi,\ x_a\propto\eta^{1-2s_a}\to\infty,\quad a=2,3 \tag{17}$$

and necessarily in the static L region, some of them (or even all for $s_a=\frac{1}{2}$) passing through the null Cauchy horizon $\xi=0$ from the homogeneous T region of V_4. Therefore, around the null singularity $\eta=0$ of the vacuum metric (9) it is only for $s_a\geqslant\frac{1}{2}$ that the causal horizon is eliminated along the two different directions (17) or even along all three when $s_2>1, s_3>\frac{1}{2}$ ($\delta<\sqrt{2}-1$).

Note that the special vacuum solution (6)–(10) has a higher symmetry and admits the complete group G_4, since besides the group G_3 of the Bianchi type VI_h, which acts transitively on the homogeneous spatial sections V_3 ($\tau=\mathrm{const}$), there is a further group operator $\hat{X}_4=\partial_\xi$ with Killing null vector corresponding to a shift along the null coordinate ξ in (9). This additional symmetry agrees with the interpretation of the homogeneous vacuum metric (4)–(10) as a purely radiative V_4 field with curvature tensor of wave type II(N), for which all the scalar invariants are zero (as for a plane gravitational wave[2]), although its physical components are nonzero, $R_{ijkl}\neq0$, and may even become infinite, which leads to a completely new type of singularity of V_4 (see Refs. 2 and 19).

By direct transformation from the null coordinates (9) to the homogeneous canonical form (4), (6) of the Bianchi type VI_h, using transition to the metric (8),

$$\tau=(-2\eta\xi)^{1/2},\quad x=e^{x_1}=(-\xi/\eta)^{1/2}, \tag{18}$$

we obtain from (16) the corresponding equations of null geodesics in the parametric form ($p_\xi\neq0, s_2, s_3\neq\frac{1}{2}$)

$$\begin{gathered} x_1=\frac{1}{2}\ln\left\{\frac{1}{\eta}\left[\frac{1}{2}\left(\frac{p_y}{p_\xi}\right)^2\frac{\eta^{1-2s_2}}{1-2s_2}+\frac{1}{2}\left(\frac{p_z}{p_\xi}\right)^2\frac{\eta^{1-2s_3}}{1-2s_3}-\xi_0\right]\right\}\\ \tau=\left\{\frac{1}{2}\left(\frac{p_y}{p_\xi}\right)^2\frac{\eta^{2(1-s_2)}}{1-2s_2}+\frac{1}{2}\left(\frac{p_z}{p_\xi}\right)^2\frac{\eta^{2(1-s_3)}}{1-2s_3}-\xi_0\eta\right\}^{1/2},\\ x_a-x_a^{(0)}=\frac{p_a}{p_\xi}\frac{\eta^{1-2s_a}}{1-2s_a},\quad a=2,3. \end{gathered} \tag{19}$$

These expressions describe the propagation of light rays in the comoving, synchronous coordinate frame with the metric (4), (6) and hold even during the vacuum stage of the expansion of the special cosmological model of type VI_h with gravitating matter near the initial singularity

$\tau=0$.

In the case of one-dimensional degenerate motion in (14), (15), when $p_\xi=0$ and $p_y=p_z=0$, the light rays (η = const) transverse an infinite distance $|x_1| \propto |\ln\tau| \to \infty$ in approaching the singularity $\tau=0$ independently of the choice of the exponents s_a [and even for the exotic variant of point collapse of V_3 with $s_2>1, 0<s_3<1$ in the purely vacuum metric (6)–(10) when $\delta<1$].

For the vacuum linear collapse of V_3 remaining in the type VI_h model with matter and asymptotic behavior (4), (6) the exponents are

$$s_2=(1+\delta)/(1+\delta^2)<1,\quad s_3=(1-\delta)/(1+\delta^2)<0,\quad \delta>1,$$

so that we obtain from (19) in the neighborhood of the singularity $\tau=0$ a simple picture of the behavior of the null geodesics. As the singularity $\tau=0$ is approached, the velocities along the y and z axes die away, $x_a \propto \tau \to 0$, so that all light rays tend asymptotically to the distinguished x_1 direction in V_3 (τ = const), along which a causal particle horizon is always eliminated:

$$|x_1| \propto |\ln\tau| \to \infty, \text{ when } \tau\to 0. \tag{20}$$

Note that the fraction of rays propagating in the positive direction of the x_1 axis decreases with increasing exponent $s_2<\frac{1}{2}$, and for $s_2 \geqslant \frac{1}{2}$ there remains a unique ray moving in the positive x_1 direction strictly along this axis.

Thus, in this model of VI_h type near the initial singularity $\tau=0$ of vacuum nature (4), (6) there is indeed no causal horizon along the one direction of V_3, so that all events within the infinite cylinder along the axis x_1 with radius $c\tau \to 0$ can be causally collected.

The cosmological type VI_h model (4) with anomalous vacuum singularity $\tau=0$ (6), (8) admits approximate isotropization during the intermediate expansion stage, when the gravitating fluid with $P<\varepsilon$ predominates in the dynamics and the spatial anisotropic curvature (5) is negligible (although its influence then increases and again gives like (6) an asymptotic behavior of unlimited expansion of V_3 as $\tau\to\infty$; see Ref. 15). Therefore, such an anisotropic model of VI_h type, having near the initial singularity no causal horizon along one direction of V_3 (τ = const) in a frame comoving with the matter, can, like the axisymmetric T and Taub–NUT models (1), also pretend to a description of the Universe in the quasi-Euclidean variant if the energy density is close to the critical value $\varepsilon_0 \approx \varepsilon_{cr} \approx 0.5\times 10^{-29}$ g/cm^3.

4. We give three further new examples of anisotropic material singularities (for which the gravitation of the matter is important and determines their nature) in special Bianchi type VI cosmological models (4) and show that they too have no causal horizon along one distinguished direction of V_3 (τ = const) in a coordinate system comoving with the matter.

For the "diagonal" triaxial type VI_h model (4) filled with gravitating fluid with $P=n\varepsilon$ $(0\leqslant n<1)$ when $\delta^2>(1+3n)/(1-n)$ or $n<(\delta^2-1)/(3+\delta^2)$, $\delta>1$, a special solution of exact power-law form is possible[13,15b]:

$$X=x_0\tau,\quad Y=\tau^r,\quad Z=\tau^{-q},\quad \varkappa\varepsilon=\mu^2/\tau^2,$$

$$\mu^2=\frac{\delta^2-1-n(3+\delta^2)}{(1+n)^2\delta^2},\quad r=\frac{1}{2\delta}\left[2+\frac{(1-n)}{(1+n)}(\delta-1)\right],$$
$$q=\frac{1}{2\delta}\left[2-\frac{(1-n)}{(1+n)}(1+\delta)\right];\quad x_0^2=\frac{2(1+n)^2\delta^2}{(1-n)(1+3n)}. \tag{21}$$

It has an initial singularity $\tau=0$ with anomalous asymptotic behavior of anisotropic collapse of V_3 into a line for

$$[(1+3n)/(1+n)]^{1/2}<\delta<(1+3n)/(1+n)\quad (q>0,\ Z(\tau)\to\infty)$$

or into a point if $\delta>(1+3n)/(1-n)$, and is characterized by the fact that the dynamical influence of the gravitating matter and the anisotropic spatial curvature (5) are always of the same order: ε, $|K| \propto \tau^{-2}\to\infty$.

A similar special solution exists for the axisymmetric model of Bianchi type VI_0 of the form[14b,15]

$$-ds^2=-d\tau^2+X^2(\tau)dx_1^2+Y^2(\tau)[\exp(-2k_0x_1)dx_2^2+\exp(2k_0x_1)dx_3^2],$$
$$X=x_0\tau,\quad Y=\tau^{(1-n)/2(1+n)},\quad \varkappa\varepsilon=\frac{(1-n)}{(1+n)^2\tau^2};\quad x_0=\frac{2k_0(1+n)}{[(1-n)(1+3n)]^{1/2}}, \tag{22}$$

when the gravitating fluid with $P=n\varepsilon$ $(0\leqslant n<1)$ and the anisotropic V_3 curvature,

$$-K_1{}^1=-K=2k_0^2/X^2\propto\tau^{-2}\to\infty,$$

jointly determine the dynamics and at the singularity $\tau=0$ lead to an anisotropic point collapse.

We also draw attention to an unusual general singular asymptotic behavior in the axisymmetric Bianchi type VI_0 in the presence of a free electromagnetic field $\mathrm{H}(\tau)\|\mathrm{E}(\tau)\|x_1$ in vacuum. The field prevents a Kasner linear collapse along lines of force along the symmetry axis and, in view of the impossibility of the kinematic collapse (2) of V_3 into a pancake for type VI_0, it leads in conjunction with the spatial curvature to a new form of anisotropic point collapse[15b]:

$$X\approx x_0\tau\to 0,\quad Y\propto\tau^{1/2}\to 0,\quad \varkappa w\approx\frac{q^2}{\tau^2}\to\infty,\quad |K|\approx\frac{2\bar{k}_0^2}{\tau^2}\to\infty, \tag{23}$$

this asymptotic behavior being identical to that of the exact special solution of the Einstein–Maxwell equations in vacuum:

$$q^2={}^1/_2,\quad \bar{k}_0^2=k_0^2/x_0^2={}^3/_4.$$

The physical singularity, at which the intensity and energy density $w=(\mathrm{E}^2+\mathrm{H}^2)/8\pi$ of the electromagnetic field (together with the spatial curvature) are infinite. is eliminated by a gravitating fluid with $P\neq 0$:

$$\varepsilon\sim V^{-(1+n)}\propto\tau^{-2(1+n)}\to\infty,\quad \varepsilon/w\sim\tau^{-2n}\to\infty\quad (n\neq 0).$$

Therefore, in cosmological "magnetic" models of axisymmetric VI_0 type with matter the singularity is always replaced by the special regime of anisotropic V_3 collapse to a point (22), at which the influence of the magnetic field is negligible: $w/\varepsilon\sim\tau^{4n/(1+n)}\to 0$ (see Ref. 15b).

We now consider the equations of null geodesics in the general "diagonal" type VI metric (4) and show that for all these specific singularities (6), (21)–(23) there is a partial elimination of the causal horizon along the distinguished direction $|x_1|\propto|\ln\tau|\to\infty$, as for the kinematic Kasner collapse (2) of V_3 into a pancake.

The isometries of the V_4 field lead to corresponding conservation laws in (12), and with each independent Killing vector $\xi_a^i(x^k)$ there is associated a first integral of the dynamical equations of the geodesics[2,18]:

$$C_a = g_{ik}\xi_a{}^i \frac{dx^k}{d\lambda} = \xi_a{}^i p_i = \text{const.} \tag{24}$$

Using the canonical form of the generators and the Killing vectors of the Bianchi type VI group G_3,[12] we obtain for the homogeneous "diagonal" metric (4) the following expressions for the conserved characteristics of test particles and light rays:

$$C_2 = p_2 = Y^2(\tau)\frac{dx_2}{d\lambda} = \text{const}, \quad C_3 = p_3 = Z^2(\tau)\frac{dx_3}{d\lambda} = \text{const},$$
$$C_1 = X^2(\tau)\frac{dx_1}{d\lambda} + (a_0+k_0)x_2C_2 + (a_0-k_0)x_3C_3 = \text{const}. \tag{25}$$

These integrals of the motion together with the normalization condition (13) for $e=0$ of the form

$$\left(\frac{d\tau}{d\lambda}\right)^2 = \frac{1}{X^2(\tau)}[C_1-(a_0+k_0)x_2C_2-(a_0-k_0)x_3C_3]^2 + \frac{C_2^2}{Y^2(\tau)}\exp[-2(a_0+k_0)x_1] + \frac{C_3^2}{Z^2(\tau)}\exp[-2(a_0-k_0)x_1] \tag{26}$$

for the system of coupled equations

$$\frac{dx_1}{d\lambda} = \frac{1}{X^2(\tau)}[C_1-(a_0+k_0)x_2C_2-(a_0-k_0)x_3C_3], \tag{27}$$
$$\frac{dx_2}{d\lambda} = \frac{C_2}{Y^2(\tau)}\exp[2(a_0+k_0)x_1], \quad \frac{dx_3}{d\lambda} = \frac{C_3}{Z^2(\tau)}\exp[2(a_0-k_0)x_1],$$

which makes it possible to investigate qualitatively the behavior of light rays in the cosmological models of Bianchi type VI.

When light propagates along the distinguished axis x_1, then $x_2, x_3 = \text{const}, C_2 = C_3 = 0$, and in accordance with (26) and (27)

$$dx_1/d\tau = 1/X(\tau). \tag{28}$$

Therefore, for all the singularities (6), (21)–(23) there is, because of the same dependence $X(\tau) \propto \tau \to 0$ of the scale factor, a characteristic divergence of the distance traversed by light along this distinguished axis:

$$|x_1| \propto |\ln \tau| \to \infty, \quad \tau \to 0. \tag{29}$$

By an analysis of Eqs. (26) and (27) near material singularities with the power-law asymptotic behaviors (21)–(23) one can show (for example, when $C_1 = 0, C_2 = 0, x_2 = \text{const}$) that all light rays must tend in the limit $\tau \to 0$, as for the vacuum singularity (6), to the distinguished direction of V_3, $|x_1| \propto |\ln\tau| \to \infty$, so that along it there is no causal horizon.

But such anisotropic type VI models with material singularities do not admit even intermediate isotropization, and, like the "magnetic" vacuum metric (23), they can hardly be used to describe the Universe.

5. Thus, it may be concluded that even partial removal of the horizon for causal connectedness near the initial cosmological singularity is a rather rare and, apparently, exceptional phenomenon in classical general relativity. In anisotropic homogeneous cosmology, the spatial direction and only near certain special singularities with null orientation, namely, for one vacuum (of Lifshitz–Khalatnikov type[7]) and three material null singularities with special asymptotic behavior of anisotropic linear and point collapse of the homogeneous spatial sections V_3 ($\tau = \text{const}$) in special Bianchi type VI models. A causal horizon is absent similarly along one distinguished direction of space for kinematic Kasner collapse of V_3 into a pancake ($p_1 = 1, p_2 = p_3 = 0$) on null caustics, which replace the vacuum null Cauchy-Killing horizons, in the axisymmetric T and Taub–NUT models, and also for the purely vacuum "quasi-Milne" asymptotic behavior of point collapse of V_3 in the "nondiagonal" wave metric of VII_h type.[16]

One can hardly expect complete elimination of the causual particle horizon in more general inhomogeneous cosmological models, since their typical Kasner and oscillator singularities are locally the same as for the homogeneous case.[3,7,9] However, this question requires further study, relating, in particular, to a number of degenerate singularities[7,13-16] and locally regular cosmological metrics.[14] Although the Penrose–Hawking theorems[2] predict that in general relativity a cosmological singularity, manifested in the form of the impossibility of extending the world lines of particles and light rays, is unavoidable for normally gravitating material sources, the physical origin of this causal incompleteness of V_4 has not been fully elucidated even in homogeneous models with moving matter.[19,20]

The special vacuum asymptotic behavior (6) of linear V_3 collapse for type VI_h models, which is identical with the Lifshitz–Khalatnikov solution (8), (9), demonstrates a qualitatively new type of null singularity. It is characterized by the circumstance that all the physical components of the Riemann tensor is a homogeneous coordinate frame tend to infinity (together with the energy density and pressure of the fluid, whose gravitation is negligible), although the scalar invariants of the conformal Weyl curvature of V_4 are zero and always regular.

The Lifshitz–Khalatnikov metric (8)–(10) is a very special wave field with homogeneous T region of VI_h type (which can be interpreted by analogy with a "converging" gravitational wave of Bianchi type VII_h),[16] since the curvature tensor of V_4 has the purely radiative algebraic structure II(N) with zero invariants. This homogeneous metric (6), (8), as can be seen from its analytic extension in null coordinates (9), contains a regular vacuum null Killing–Cauchy horizon. This is a semipermeable causal membrane, beyond which there is a stationary inhomogeneous L region of the geodesically complete space-time V_4 (Fig. 1). This special example of Bianchi type VI_h complements the very small set of known spatially homogeneous solutions of the Einstein equations that possess a null Killing–Cauchy horizon and include in addition to homogeneous T regions stationary inhomogeneous L regions of V_4 as well. The set consists of: a) the axisymmetric family of vacuum T and Taub–NUT universes V_4 ($\Lambda \neq 0$), including the case when a free electromagnetic field is present[14,15]; b) the axisymmetric type V models with moving fluid.[20]

In all the above cases, a removable intermediate singularity[19] is realized in the form of the disappearance of the original homogeneity on the null Cauchy–Killing horizons and the appearance of additional stationary L regions of V_4, since the transitivity hypersurfaces V_3 (τ=const) and one of the Killing vectors of the corresponding groups of motion G_4 change their orientation from spacelike to timelike. It is probable that these exceptional examples of various form do not exhaust all possibilities with a removable intermediate singularity in the class of spatially homogeneous Einstein–Maxwell vacuum fields and cosmological models.[19,15] However, they must be a degenerate set of measure zero, since all such known solutions have higher G_4 mobility, although the requirement of axial V_3 symmetry is not necessary, as can be seen from the example of the special wave metric of VI_h type (cf. Ref. 15). In addition, for the Lifshitz–Khalantnikov vacuum solution (6)–(9) the regular null Killing–Cauchy horizon is unstable against perturbations of the free field, which destroy the higher symmetry of V_4, as for T and Taub-NUT universes.[15] Similarly, if the T region is filled with a gravitating fluid at rest, it is also transformed into a local vacuum singularity of the curvature (and not into a material singularity such as Kasner V_3 collapse into a pancake, as for the axisymmetric T and Taub-NUT models).

It is known[7] that the Lifshitz–Khalatnikov asymptotic behavior adjoins the Taub–Kasner behavior in a special null coordinate system of the form

$$-ds^2=2d\eta d\xi+\lambda\eta^{2s_1}d\xi^2+\eta^{2s_2}dy^2+\eta^{2s_3}dz^2,$$
$$s_2+s_3=s_2^2+s_3^2,\quad s_1=1/2\,(1-s_2-s_3),\quad \lambda=\text{const},\tag{30}$$

which can be interpreted as a comoving coordinate frame attached to relativistically moving test matter ($P=0$) on a vacuum background near the Kasner singularity $\eta=0$. Such a metric form is more general than the canonical form in a homogeneous synchronous system for Bianchi type I, and it includes as limiting degenerate case for $\lambda=0$ the special vacuum type VI_h solution (6)–(9), which now has null singularity $\eta=0$. This singularity does not have a causal horizon along two or even along all three spatial directions of the metric (9) when one or both exponents satisfy $s_a>\frac{1}{2}$.

We emphasize that the existence and nature of the removal of the causal horizon may, as can be seen from the example of the Lifshitz–Khalatnikov null singularity (17),(20), depend on the choice of the various coordinate systems (6) or (9). However, it follows from our investigation of the geodesics for the metric in the null coordinates (30) that near the Kasner singularity $\eta=0$ there always remains (except for the kinematic regime with $s_2=s_3=0, s_1=1$) a causal horizon with respect to relativistically moving test fluid of the frame, i.e., essentially in the most general regimes of initial expansion of an anisotropic inhomogeneous Universe.[3,7,9]

We note that the degenerate vacuum asymptotic behavior with the null Lifshitz–Khalatnikov singularity (6)–(9) certainly admits inhomogeneous generalizations,[7] but the extent of its generality is not yet known.

The problems of the initial cosmological singularity and its causal horizon are intimately related, and it is natural to assume that they will be resolved only when allowance is made for quantum effects in general relativity,[4)] which must eliminate the classical singularity and ensure a regular transition at

$$t\sim t_P\sim(\hbar G/c^5)^{1/2}\sim10^{-43}\text{ sec}$$

from the preceding contraction phase to the present expansion of the Universe.[21]

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1) Note that for the "nondiagonal" Bianchi type VII_h an exact vacuum solution is known in the form of a "converging" gravitational wave with anisotropic "quasi-Milne" asymptotic behavior of V_3 collapse into a point at $\tau=0$, for which there is also no causal horizon along one direction. But this degenerate singularity is possible only in vacuum, and it is eliminated in the presence of matter, being transformed into a linear Kasner singularity with causal particle horizon.[16]

2) Note that an improbable "long" era of oscillations of the oscillator asymptotic approach to the singularity[8,9] can be interpreted as a slightly perturbed kinematic regime of V_3 collapse into a pancake with approximate Kasner exponents $p_1=1, p_2=p_3=0$. It describes the initial stage of development of instability on the null Killing–Cauchy horizons of the original axisymmetric variants of Bianchi types VIII and IX under the influence of small perturbations of the free gravitational field, when, as a result of deviations from axial symmetry of V_3 (τ=const), the null horizons regular in the vacuum are eliminated and transformed into an oscillator singularity.[14,15] It is precisely because of the proximity to the kinematic Kasner asymptotic behavior (2) in the oscillatory regime of the long era that light can traverse long spatial distances along the direction of monotonic expansion of V_3 (and even go several times round the closed 3-spherical space in the Bianchi type IX).[10]

3) On the basis of a more general self-similar metric form of, for example, the power-law type

$$-ds^2=-(t/x)^{2\rho}dt^2+(t/x)^{2\alpha}dx^2+(t/x)^{2\beta}dy^2+(t/x)^{2\gamma}dz^2,$$

one can show that in general relativity there do not exist vacuum inhomogeneous solutions of the type of self-similar Einstein–Rosen gravitational waves with cylindrical symmetry. Indeed, the vacuum Einstein equations $R_{ik}=0$ impose stringent restrictions on the values of the exponents,

$$\alpha+\beta+\gamma=1+\rho,\quad \alpha^2+\beta^2+\gamma^2=(1+\rho)^2,\quad \beta^2+\gamma^2=(\alpha+\beta)(\beta+\gamma),$$

with which only the flat Minkowski metric is compatible for $\beta=\gamma=0, \alpha=1+\rho$. In addition, one can also have a special vacuum solution, which is actually homogeneous and of VI_h Bianchi type (8), when $\rho=0, \alpha=1, \beta^2+\gamma^2=\beta+\gamma$; it is a purely radiative field of type II(N) (cf. the wave interpretation[16] of the vacuum fields for the neighboring type VII).

4) The interesting question of the nature and back reaction of the quantum effects of particle pair production and vacuum polarization in the strong gravitational wave field near a null vacuum Lifshitz–Khalatnikov singularity requires a separate investigation.

[1] H. Bondi, Cosmology, Cambridge (1960).
[2] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time, CUP (1973) [Russian translation published by Mir, Moscow (1977)].
[3] L. D. Landau and E. M. Lifshitz, Teoriya polya, Nauka, Moscow (1973); English translation: The Classical Theory of Fields, 4th ed., Pergamon Press, Oxford (1975).
[4] Ya. B. Zel'dovich and I. D. Novikov, Stroenie i évolyutsiya

Vselennoĭ (Structure and Evolution of the Universe), Nauka, Moscow (1977).
[5]C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, W. H. Freeman, San Francisco (1973) [Russian translation published by Mir, Moscow (1977)].
[6]L. S. Shepley and M. P. Ryan, Homogeneous Relativistic Cosmology, Princeton (1974).
[7]E. M. Lifshitz and I. M. Khalatnikov, Usp. Fiz. Nauk **80**, 391 (1963) [Sov. Phys. Usp. **6**, 495 (1964)].
[8]C. W. Misner, Phys. Rev. Lett. 22, 1071 (1969); Phys. Rev. **186**, 1319 (1969).
[9]V. A. Belinskiĭ, E. M. Lifshitz, and I. M. Khalatnikov, Usp. Fiz. Nauk **102**, 463 (1970) [Sov. Phys. Usp. **13**, 745 (1971)].
[10]A. G. Doroshkevich, V. N. Lukash, and I. D. Novikov, Zh. Eksp. Teor. Fiz. **60**, 1201 (1971); **64**, 1457 (1973) [Sov. Phys. JETP **33**, 649 (1971); **37**, 739 (1973)].
[11]C. W. Misner, Ap. J. **151**, 431 (1968); J. D. Barrow, Nature **267**, 117 (1977).
[12]G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. **12**, 108 (1969); **19**, 31 (1970); **20**, 57 (1971); M. A. H. MacCallum, Proc. of 1971 Cargese Summer School, New York (1973), p. 64.
[13]C. B. Collins, Commun. Math. Phys. **23**, 137 (1971).
[14]V. A. Ruban, a) Dokl. Akad. Nauk. SSSR **204**, 1086 (1972) [Sov. Phys. Dokl. **17**, 992 (1973)]; b) Preprint No. 355 [in Russian], A. F. Ioffe Physicotechnical Institute, Leningrad (1971).
[15]V. A. Ruban, a) Zh. Eksp. Teor. Fiz. **72**, 1201 (1977) [Sov. Phys. JETP **45**, 629 (1977)]; b) Preprints Nos. 441 and 412 [in Russian], Leningrad Institute of Nuclear Physics (1978).
[16]V. N. Lukash, Zh. Eksp. Teor. Fiz. **67**, 1594 (1974) [Sov. Phys. JETP **40**, 792 (1975)].
[17]V. A. Ruban, a) Zh. Eksp. Teor. Fiz. **56**, 1915 (1969) [Sov. Phys. JETP **29**, 1027 (1969)]; b) in: Trudy IV sovetskoĭ gravitats. konf. (relyat. astrofizika. Gravitats. éksperiment. Kosmologiya) [Proc. Fourth Soviet Gravitational Conference (Relativistic Astrophysics Gravitational Experiments. Cosmology)], Minsk (1976), p. 31; Proc. Int. Conf. GR8, Canada (1977), p. 301.
[18]P. C. Aichelburg, J. Math. Phys. **11**, 1330 (1970); I. M. Khalatnikov (Chalatnikov) and S. L. Parnovsky, Phys. Lett. **66A**, 466 (1970).
[19]G. F. R. Ellis and A. R. King, Commun. Math. Phys. **38**, 119 (1974).
[20]C. B. Collins, Commun. Math. Phys. **39**, 131 (1974); I. S. Shikin, Zh. Eksp. Teor. Fiz. **68**, 1583 (1975) [Sov. Phys. JETP **41**, 794 (1975)].
[21]V. I. Gurovich and A. A. Starobinskiĭ, Zh. Eksp. Teor. Fiz. **77**, 1683 (1979) [Sov. Phys. JETP **50**, 844 (1979)].

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