Electron localization in the random-phase model and in a magnetic field

M. V. Sadovskiĭ

Institute of Metal Physics, Ural Scientific Center, USSR Academy of Sciences
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Electron localization with disorder in the phases of the transport integrals is considered in the Anderson model. It is shown with the aid of Anderson's method that a disorder of this type leads in the general case to an effective decrease of the lattice connectivity constant, and contributes to the localization. Total localization of the band on account of the phase disorder alone, however, is impossible. The influence of an external magnetic field and the positions of the mobility edge is considered (neglecting the spin effects). It is shown that the result of the action of the magnetic field is determined by the distribution function of the areas of the self-avoiding walks on the lattice. In the genreal case, the magnetic field contributes to the localization, and its action is similar to the effect of random phases of the transport integrals. The results are valid in the region of sufficiently strong fields, in which the effects connected with the Langer-Neal diagrams are suppressed.

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INTRODUCTION

Interest in the localization of electrons in disordered systems has increased lately.[1-3] This is due both to the importance of this phenomenon to the theory of disordered systems, and to the reports of new experiments in which the localization manifests itself in unusual manner.[3] At the same time, the level of the theoretical understanding of the localization is still too low; this is manifest, in particular, in the fact that the roles of different external fields (primarily magnetic) and of different types of disorder have not yet been investigated. Until recently most papers were devoted to the study of localization in the Anderson model[4,5] with diagonal (site) disorder of the electron in the lattice. What was mainly discussed was the critical disorder that leads to complete localization of all states in a band. Only recently has serious interest been evinced in the role of off-diagonal disorder (transport integrals), and this led immediately to conclusions concerning the unusual role of this disorder in the phenomenon considered, especially to the conclusion that complete localization in a band on account of only a disorder of this type is impossible.[6] Finally, a paper by Abrahams *et al.*[7] increased sharply the interest in the critical behavior at the mobility edge and in the interesting predictions made concerning localization in two-dimensional systems[7-10] (see the review[3]).

In this paper we consider the localization phenomenon in a specific model of off-diagonal disorder (the random-phase model), whose interesting distinguishing feature is the presence of local gauge invariance. Generalization of the results obtained with this model make it possible to examine the effect of an external magnetic field in the positions of the mobility edges in terms of its influence on the orbital motion (neglecting spin effects). The analysis is carried out within the framework of Anderson's standard approach.[4-6] The relation between our results and those obtained within the framework of another approach[9,10] will be discussed in the Conclusion.

1. LOCAL GAUGE INVARIANCE IN THE ANDERSON MODEL

We consider the Hamiltonian of the Anderson model

$$H=\sum_{\langle ij\rangle} J_{ij}a_i^+a_j+\sum_i E_ia_i^+a_i, \tag{1}$$

where a_i^+ and a_j are the electron creation and annihilation operators on the i-th and j-th lattice sites. The energies E_i at the sites are assumed to be random, and their distribution is specified in the usual form

$$P\{E_i\}=\prod_i P(E_i),\qquad P(E_i)=\begin{cases} W^{-1}, & |E_i|<W \\ 0, & |E_i|>W\end{cases}. \tag{2}$$

The transport integrals J_{ij}, which are assumed do differ from zero only between nearest neighbors, also take on random values.

We consider a specific disorder model, in which the random quantity is the phase rather than the modulus of the transport integral, as considered by Antoniou and Economou.[6] We thus assume (the random-phase model)

$$J_{ij}=J\exp(i\Phi_{ij})=JU_{ij},\qquad J_{ij}=J_{ji}^*,\quad \Phi_{ij}=-\Phi_{ji}, \tag{3}$$

where Φ_{ij} is a random quantity whose distribution in the lattice is assumed to be factorizable in the bonds:

$$P\{\Phi_{ij}\}=\prod_{\langle ij\rangle} P(\Phi_{ij}), \tag{4}$$

and we consider for $P(\Phi_{ij})$ different cases:

$$P(\Phi_{ij})=\frac{1}{(2\pi)^{1/2}\Phi}\exp\left\{-\frac{\Phi_{ij}^2}{2\Phi^2}\right\}, \tag{5}$$

$$P(\Phi_{ij})=\begin{cases}\Phi^{-1}, & |\Phi_{ij}|<\Phi\\ 0, & |\Phi_{ij}|>\Phi'\end{cases} \tag{6}$$

$$P(\Phi_{ij})=c\delta(\Phi_{ij}-\pi)+(1-c)\delta(\Phi_{ij}),\quad 0<c<1, \tag{7}$$

etc. Case (7) corresponds to random introduction (with density c) of "antiferromagnetic" bonds:

$$J_{ij}=JA_{ij}, \tag{8}$$

$$P(A_{ij})=\begin{cases} c & A_{ij}=-1\\ 1-c, & A_{ij}=+1\end{cases}. \tag{9}$$

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It is easily seen that the Anderson Hamiltonian (1) has local gauge symmetry. It is invariant to a transformation of the type

$$\{G\}: a_i^+ \to \exp(i\Phi_i)a_i^+,\ a_j \to \exp(-i\Phi_j)a_j, \quad J_{ij} \to \exp(-i\Phi_i)J_{ij}\exp(i\Phi_j). \tag{10}$$

This the analog of the local gauge transformation in the Yang-Mills theory on a lattice, a transformation actively used of late in the theory of random spin systems (spin glasses).[11,12] This invariance is known to lead to a number of nontrivial conclusions for magnetic systems,[11,12] some of which can be directly crossed over also to the model considered. In particular, if in (3)

$$\Phi_{ij} = \alpha_i + \alpha_j, \tag{11}$$

where α_i and α_j are random quantities, then this disorder is trivial and can be eliminated by a suitable local gauge transformation. This crosses over to the case (8) if $A_{ij} = c_i c_j$, where $c_i = \pm 1$ in random fashion (the analog of the Mattis model in spin-glass theory). Interest attaches to the nontrivial (gauge-invarient) disorder determined[11,12] by distribution of the frustrations on the considered lattice. The definition of the frustration[11] (or of the frustration angle[12]) can be formulated in the considered electronic model in complete analogy with the definitions in the theory of random spin systems. The frustration distributions investigated in spin lattices[12,13] can turn out to be useful also in localization theory.

Proceeding to consideration of the electron Green's function in the Anderson lattice, we note that the single-electron Green's function

$$G_{ij}(E) = \left\langle i \left| \frac{1}{E-H} \right| j \right\rangle = \left\langle 0 \left| a_i \frac{1}{E-H} a_j^+ \right| 0 \right\rangle \tag{12}$$

is not gauge-invariant:

$$\{G\}: G_{ij}(E) \to G_{ij}(E)\exp[i(\Phi_i - \Phi_j)]. \tag{13}$$

The only gauge invariant element in this function is $G_{ii}(E)$, which is diagonal in the sites and is customarily used in the study of localization in the standard Anderson approach.[4-6] It is obvious from the foregoing that in the random-phase model the averaged single-electron Green's function is diagonal in the site indices:

$$\langle G_{ij}(E) \rangle = G(E)\delta_{ij}, \tag{14}$$

a reflection of the vanishing of the gauge-noninvariant off-diagonal elements upon averaging over the gauge-invariant distribution of the frustrations. It is therefore meaningless to use (14) for the investigation of the localization. For the averaged two-particle Green's function we have

$$\langle G_{ij}(E) G_{kl}(E') \rangle \sim \delta_{ij}\delta_{kl};\ \delta_{ik}\delta_{jl};\ \delta_{il}\delta_{jk}. \tag{15}$$

A similar situation (in another model) was dealt with in Refs. 14 and 15.

We can introduce the gauge-invariant electron Green's functions

$$\mathcal{G}_{ij}^{\Gamma}(E) = \langle 0 | a_i \frac{1}{E-H} \prod_{\Gamma} U_{kl} a_j^+ | 0 \rangle, \tag{16}$$

where $\Pi_\Gamma U_{kl}$ determines the product of the elements U_{kl} from (3) [or A_{kl} from (9)] along an arbitrary walk Γ that connects the sites i and j of the lattice. Expression (16) is obviously gauge invariant, and (14) does not hold for it. Correlators of the type (16) are therefore capable of containing definite information on the localization, but they have an explicit dependence on the walk Γ, and their behavior after the averaging has not been investigated.

At the same time, as noted above, Anderson's standard approach[4-6] is perfectly suited for the analysis of problems of this type, in view of the local gauge invariance of $G_{ii}(E)$.

2. LOCALIZATION IN THE RANDOM PHASE MODEL

Following Anderson's method, we investigate the convergence of the renormalized perturbation-theory series for the self-energy part $\Delta_i(E)$ that enters in the matrix element, diagonal in the sites, of the non-averaged Green's function:

$$G_{ii}(E) = \frac{1}{E - E_i - \Delta_i(E)}; \tag{17}$$

$$\Delta_i(E) = \sum_{k \neq i} J_{ik} \frac{1}{E - E_k - \Delta_k^i(E)} J_{ki} + \sum_{k \neq i, l \neq ik} J_{il} \frac{1}{E - E_l - \Delta_l^{ik}(E)} J_{lk} \frac{1}{E - E_k - \Delta_k^i(E)} J_{ki} + \dots, \tag{18}$$

where $\Delta_k^{ijl\cdots}(E)$ is determined by a series such as (18), but corresponding to the Hamiltonian (1), in which we put $E_i = E_j = E_l = \dots = \infty$.[5] We have excluded from (18) the repeated indices of the sites, i.e., in $(N+1)$-st order in J_{ij} the summation proceeds along a self-avoiding walk Γ_N consisting of N steps on the lattice, starting with the i-th site and returning to the i-th site [Fig. 1(a)]. Multiple scattering processes [Fig. 1(b)] with return are implicitly taken into account here by introducing $\Delta_k^{ijk\cdots}(E)$ in the denominators of (18),[4,5] and it is this which allows us to consider self-avoiding walks on the lattice. The representation (8) is exact. An electron of energy E is localized if the series $\Delta_i(E)$ converges in the sense of convergence with respect to probability.[4,5]

To investigate the convergence of the series (18), we consider the modulus of the term of $(N+1)$-st order in J_{ij}:

$$\left| \Delta_i^{(N)}(E) \right| = \left| \sum_{\Gamma_N} T_{\Gamma_N}(E) \right|, \tag{19}$$

where $\sum$ denotes summation over self-avoiding walks consisting of N steps starting and ending at the site i, and $T_{\Gamma_N}(E)$ is the contribution of one such walk. According to Economou and Cohen,[5] it can be shown that

$$|\Delta_i^{(N)}(E)| \approx L^N(E) = \left| \sum_{\Gamma_N} J\exp(i\Phi_{ij})\bar{G}_j^i(E) J\exp(i\Phi_{jk})\bar{G}_k^{ij}(E) \dots J\exp(i\Phi_{li}) \right|, \tag{20}$$

$$\ln \bar{G}_k^{ij\dots l}(E) = \left\langle \ln \left| \frac{1}{E - E_k - \Delta_k^{ij\dots l}(E)} \right| \right\rangle, \tag{21}$$

where the angle brackets denote averaging over the diagonal disorder (2). The quantity $L^N(E)$ is obviously gauge-invariant, since the walks Γ_N on the lattice are closed. Then $L(E) < 1$ is the condition for the convergence of the series (18) (Ref. 5) and can be regarded as

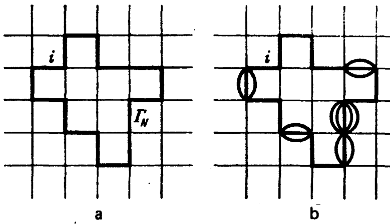

FIG. 1.

a criterion of the localization. The delocalized states correspond to the condition $L(E) > 1$.

Expression (21) is too complicated for actual calculations, owing to the need for taking the contribution $\Delta_k^{ij\cdots l}(E)$ into account. There are several ways of getting around this difficulty,[4-6] but we shall use the simplest one—we neglect completely the contributions of these quantities[5,16]:

$$G_k^{ij\ldots l}(E) \approx \exp\{-\langle \ln|E-E_k|\rangle\}. \tag{22}$$

This approximation facilitates all the calculations, and its result for the positions of the mobility edges and for the critical disorder do not differ greatly from those of the more accurate analysis.[4,5,17] It is therefore usually assumed that a more consistent account of $\Delta_k^{ij\cdots l}(E)$ in (21) leads simply to a quantitative refinement of the localization condition[5] (see, however, the discussion in the Conclusion).

We then have

$$L^N(E) \approx J^{N+1}\left|\sum_{\Gamma_N}\exp(i\Phi_{\Gamma_N})\right|\exp\{-N\langle\ln|E-E_k|\rangle\}, \tag{23}$$

where

$$\Phi_{\Gamma_N}=\Phi_{ij}+\Phi_{jk}+\ldots+\Phi_{li} \tag{24}$$

is the phase advance along the walk Γ_N. Equation (24) has only N terms, $\sum_{\Gamma_N}$ contains $\sim K^N$ terms, where K is the so-called lattice connectivity constant 18:

$$\ln K=\lim_{N\to\infty}\frac{1}{N}\ln Z_N,$$

where Z_N is the total number of self-avoiding walks of N steps.

We consider now

$$X_N=\sum_{\Gamma_N}\exp(i\Phi_{\Gamma_N}). \tag{25}$$

If all the phase shifts in (24) are zero (or fixed), then obviously $|X_N|\sim K^N$ and we obtain correspondingly the usual answer[5,16]

$$\begin{gathered}L^N(E)\approx J^{N+1}K^N\exp\{-N\langle\ln|E-E_k|\rangle\},\\ L(E)\approx\alpha K\exp\mathcal{F}(E,W/J),\end{gathered} \tag{26}$$

$$\begin{aligned}\mathcal{F}\left(E,\frac{W}{J}\right)&=\frac{1}{W}\int_{-W/2}^{W/2}dE_k\ln\left|\frac{J}{E-E_k}\right|\\ &=1-\frac{1}{2}\left\{\left(1+\frac{2E}{W}\right)\ln\left|\frac{W}{2J}+\frac{E}{J}\right|+\left(1-\frac{2E}{W}\right)\ln\left|\frac{W}{2J}-\frac{E}{J}\right|\right\}.\end{aligned} \tag{27}$$

Here $\alpha = Z/K$ (Z is the number of nearest neighbors) is a correction factor[5] that makes it possible to describe correctly the limit of the regular lattice. The condition $L(0) = 1$ then yields for the critical disorder need for complete localization

$$(W/J)_c=2eZ. \tag{28}$$

In the random phase model, $|X_N|$ is the length of the walk on a plane as a result of random K^N steps of unit length. The most substantial effect is reached for fully random phases, when Φ_{ij} have a distribution (6), with $\Phi = 2\pi n$ (n = 1, 2, ...). We are then dealing with Brownian motion on a plane, and $|X_N|$ has the Rayleigh distribution[19,20]

$$P(|X_N|)=\frac{2|X_N|}{K^N}\exp\left\{-\frac{|X_N|^2}{K^N}\right\}, \tag{29}$$

$$\langle|X_N|^2\rangle=K^N. \tag{30}$$

The most probable value is $|X_N|\sim\langle|X_N|^2\rangle^{1/2}\sim K^{N/2}$. Then

$$\begin{gathered}L^N(E)\approx J^{N+1}K^{N/2}\exp\{-N\langle\ln|E-E_k|\rangle\},\\ L(E)\approx\alpha K^{1/2}\exp\mathcal{F}(E,W/J).\end{gathered} \tag{31}$$

The stochastization of the phases leads thus to a decrease of the effective connectivity constant of the lattice The condition for complete localization when the phases are completely random takes then the form

$$(W/J)_c^{RPM}=2e\alpha K^{1/2}=2eZK^{-1/2} \tag{32}$$

It is obvious that

$$(W/J)_c^{RPM}<(W/J)_c,$$

i.e., the localization condition is less stringent.

In the absence of diagonal disorder, $E_k = E_0$ for all k. Then, if nondiagonal disorder is also absent, we have[5]

$$L(E)=\alpha K\frac{J}{|E-E_0|}=Z\frac{J}{|E-E_0|}, \tag{33}$$

and when the phases are completely randomized

$$L(E)=\alpha K^{1/2}\frac{J}{|E-E_0|}=\frac{Z}{K^{1/2}}\frac{J}{|E-E_0|}. \tag{34}$$

Then at $L(E) > 1$ we obtain the width of the band of extended states in the model of completely random phases:

$$B_{ext}^{RPM}=K^{-1/2}B, \tag{35}$$

where $B = 2ZJ$ is the usual width of the band in the regular case.

Thus, in the absence of diagonal disorder complete localization in the enitre band is impossible, and a region of extended states, of widths B_{ext}^{RPM}, always remains around the center of the band. Table I shows the values of K and $K^{-1/2}$ for different lattices.[18] It is seen that the phase disorder can localize in all cases approximately $\frac{1}{3}$ to $\frac{2}{3}$ of the initial band.

In the general case, obviously, $K^{N/2}\lesssim|X_N|\lesssim K^N$. The problem of calculating the statistical distribution of sums of the type (25) was investigated in detail in connection with various problems of statistical radio engineering.[19-22] This distribution is relatively easy to obtain when the distribution of the Φ_{Γ_N} is such that the central limit theorem is satisfied.[21,22] In particular,

$$\begin{aligned}P(|X_N|)=\frac{|X_N|e^{-S}}{(s_1s_2)^{1/2}}\sum_{m=0}^{\infty}(-1)^m\varepsilon_m I_m(P)I_{2m}((Q^2+R^2)^{1/2})\\ \times\cos\left[2m\,\mathrm{arctg}\frac{R}{Q}\right]\end{aligned} \tag{36}$$

TABLE I.

Lattice	z	K	$K^{-1/2}$	ln K
Triangular	6	4.1515	0.4908	1.4235
Quadratic	4	2.6390	0.6156	0.9704
Diamond	4	2.878	0.5896	1.0571
PC	6	4.6826	0.4621	1.5438
BCC	8	6.5288	0.3914	1.8762
FCC	12	10.035	0.3157	2.3061

the so-called Nakagami distribution.[22] Here $I_m(x)$ is a modified Bessel function of order m, $\varepsilon_0=1$, and $\varepsilon_m=2$ at $m\neq 0$,

$$S=\frac{s_1+s_2}{4s_1s_2}|X_N|^2+\frac{\alpha^2}{2s_1}+\frac{\beta^2}{2s_2},$$

$$P=\frac{s_1-s_2}{4s_1s_2}|X_N|^2,\quad Q=|X_N|\frac{\alpha}{s_1},\quad R=|X_N|\frac{\beta}{s_2};$$

$$\alpha=\sum_{\Gamma_N}\int d\Phi_{\Gamma_N}P(\Phi_{\Gamma_N})\cos\Phi_{\Gamma_N}=\sum_{\Gamma_N}\alpha_{\Gamma_N},\tag{37}$$

$$\beta=\sum_{\Gamma_N}\int d\Phi_{\Gamma_N}P(\Phi_{\Gamma_N})\sin\Phi_{\Gamma_N}=\sum_{\Gamma_N}\beta_{\Gamma_N},$$

$$s_1=\sum_{\Gamma_N}\int d\Phi_{\Gamma_N}P(\Phi_{\Gamma_N})\cos^2\Phi_{\Gamma_N}-\sum_{\Gamma_N}\alpha^2_{\Gamma_N},$$

$$s_2=\sum_{\Gamma_N}\int d\Phi_{\Gamma_N}P(\Phi_{\Gamma_N})\sin^2\Phi_{\Gamma_N}-\sum_{\Gamma_N}\beta^2_{\Gamma_N},\tag{38}$$

where $P(\Phi_{\Gamma_N})$ is the distribution function of Φ_{Γ_N}. It is easily seen here that[21,22]

$$\langle|X_N|^2\rangle=s_1+s_2+\alpha^2+\beta^2,\tag{39}$$

i.e., it is determined completely by the mean values α and β and by the variances s_1 and s_2 from (38). The Rayleigh distribution (29) is obtained from (36) at $\alpha=\beta=0$ and $s_1=s_2=K^N/2$.

We consider now several examples. We begin with the Gaussian case (5). It is then easily seen that

$$P(\Phi_{\Gamma_N})=\frac{1}{(2\pi)^{1/2}\Phi_N}\exp\left\{-\frac{\Phi^2_{\Gamma_N}}{2\Phi_N^2}\right\},\tag{40}$$

$$\Phi_N^2=\langle\Phi_{ij}^2\rangle+\langle\Phi_{jk}^2\rangle+\ldots+\langle\Phi_{li}^2\rangle=N\Phi^2.\tag{41}$$

From (38) and (39) we easily obtain

$$\langle|X_N|^2\rangle=K^{2N}\exp(-N\Phi^2)+K^N(1-\exp(-N\Phi^2)).\tag{42}$$

In the general case we get from (23)

$$L(E)\approx a\mathscr{K}\exp\mathscr{F}(E,W/J),\tag{43}$$

where the effective connectivity constant $\mathscr{K}$ is defined as

$$\mathscr{K}=\lim_{N\to\infty}\langle|X_N|^2\rangle^{1/2N}.\tag{44}$$

For the case (40)–(42) it is correspondingly easy to show that

$$\mathscr{K}=\begin{cases}K\exp(-\Phi^2/2), & \Phi^2<\ln K\\ K^{1/2}, & \Phi^2>\ln K\end{cases}.\tag{45}$$

The "effective connectivity" of the lattice as a function of the phase disorder is shown in Fig. 2. $\mathscr{K}$ first increases with increasing Φ^2, and at $\Phi^2>\ln K$ the phases become completely randomized and $\mathscr{K}$ takes the asymptotic form $K^{1/2}$.

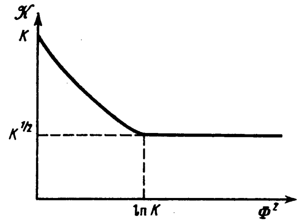

FIG. 2.

We consider now the case of the distribution (6) with $\Phi\neq 2\pi n$ ($n=1,2,\ldots$). Then, obviously,

$$P(\Phi_{\Gamma_N})=\begin{cases}1/N\Phi, & |\Phi_{\Gamma_N}|<N\Phi\\ 0, & |\Phi_{\Gamma_N}|>N\Phi\end{cases}.\tag{46}$$

We then obtain

$$\mathscr{K}=\lim_{N\to\infty}\left\{K^{2N}\frac{\sin^2N\Phi}{(N\Phi)^2}+K^N\left(1-\frac{\sin^2N\Phi}{(N\Phi)^2}\right)\right\}^{1/2N}=K.\tag{47}$$

Thus, in this case there is not even partial randomization of the phases. A nondiagonal disorder of this type does not influence the localization. It is seen from (47) that at $\Phi=\pi n$ ($N=1,2,\ldots$) we return to the full stochastization considered above.

The cases (7)–(9) can be treated similarly. Let n be the number of negative bonds along a given walk Γ_N. The probability of realization of such a bond distribution is given by the binomial distribution

$$P_N(n_{\Gamma_N})=C_N^{n_{\Gamma_N}}c^{n_{\Gamma_N}}(1-c)^{N-n_{\Gamma_N}}.\tag{48}$$

Using the limiting behavior of (48) as $N\to\infty$ and the fact that in this case $\Phi_{\Gamma_N}=\pm n_{\Gamma_N}\pi$, we obtain directly

$$P(\Phi_{\Gamma_N})=\frac{1}{2}[2\pi^3Nc(1-c)]^{-1/2}\{\exp[-(\Phi_{\Gamma_N}-c\pi N)^2/2Nc(1-c)\pi^2]+\exp[-(\Phi_{\Gamma_N}+c\pi N)^2/2Nc(1-c)\pi^2]\}.\tag{49}$$

The appearance of two terms in (49) is due to the fact that the walk Γ_N can follow two circuit directions, and the phase advance is $\pm n_{\Gamma_N}\pi$ by virtue of $\Phi_{ij}=-\Phi_{ji}$ (3). From (49) we get

$$K=\begin{cases}K\exp[-c(1-c)\pi^2/2], & c<c_1^*,\quad c>c_2^*=(1-c_1^*),\\ K^{1/2}, & c_1^*<c<c_2^*,\end{cases}\tag{50}$$

where $c^*_{1,2}$ is obtained by solving the equation $c(1-c)\pi^2=\ln K$.

Thus, the inclusion of a sufficient number of antiferromagnetic bonds (with $c>c_1^*$) leads to complete stochastization of the phases and brings $\mathscr{K}$ to the level $K^{1/2}$. The values of $c^*_{1,2}$ for different lattices are given in Table II. We note that the critical concentrations obtained in this manner are very close to the critical antiferromagnetic-bond concentrations at which ferromagnetism vanishes in the corresponding Ising lattices.[23,24]

TABLE II.

Lattice	Triangular	Quadratic	Diamond	PC	BCC	FCC
c_1^*	0.1748	0.1106	0.1220	0.1941	0.2445	0.3722
c_2^*	0.8252	0.8894	0.8780	0.8059	0.7455	0.6278

3. LOCALIZATION IN A MAGNETIC FIELD

Application of a constant external magnetic field $\mathbf{H}$ adds, as is well known, an additional phase factor in the transport integral J_{ij} (the Peierls factor)[25,26]

$$J_{ij}^{H}=\exp(-i\Phi_{ij})J_{ij},$$

$$\Phi_{ij}=\frac{1}{2}\frac{e}{\hbar c}\mathbf{H}[\mathbf{R}_i\times\mathbf{R}_j], \tag{51}$$

where $\mathbf{R}_i$ is the radius vector that determines the position of the i-th site in the lattice. The main property of these factors is[26] that the sum along a closed walk Γ_N on the lattice is gauge-invariant and is equal to the flux of the magnetic field $\mathbf{H}$ through the area S_{Γ_N} enclosed by the contour Γ_N measured in units of the magnetic-flux quantum $\Phi_0=\hbar c/e$:

$$\Phi_{\Gamma_N}=\Phi_{ij}+\Phi_{jk}+\ldots+\Phi_{li}=\Phi_0^{-1}\mathbf{H}\mathbf{S}_{\Gamma_N}. \tag{52}$$

The result (51) is valid in not too strong fields, for which one can neglect the deformation of the atomic wave functions in the magnetic field (this changes also the modulus J_{ij}).

We see now that the influence of the magnetic field on the localization is similar to the influence, considered above, of the random phases J_{ij}. It is determined completely by the statistics of the area $\mathbf{S}_{\Gamma_N}$ of the self-avoiding walks on the lattice. To my knowledge, the problem of the distribution of the areas of the self-avoiding paths has not been considered before. It appears that reliable results can be obtained here only by computer simulation. Nonetheless, regardless of the statistics of $\mathbf{S}_{\Gamma_N}$, it is clear from the foregoing that the appearance of the phase factors Φ_{Γ_N} (52) in (23) can only improve the convergence of the Anderson series (or at least have no effect on it), and decrease effectively the connectivity constant of the lattice. Therefore, neglecting spin effects (their influence on the hopping conductivity was considered recently by Fukuyama and Yosida[27]), the magnetic field can only promote localization in this approximation. We present below a simple qualitative analysis aimed at revealing the principal relations and estimating the scale of the phenomena.

It is known from the scaling theory of self-avoiding walks[28,29] that the mean squared dimension of Φ_{Γ_N} is

$$\langle R^2\rangle\sim N^{2\nu}a^2, \tag{53}$$

where a is the lattice constant and ν is the critical exponent of the correlation length. For a qualitative treatment we can therefore assume

$$\langle|\mathbf{S}_{\Gamma_N}|\rangle\sim\pi\langle R^2\rangle\sim\pi a^2N^{2\nu} \tag{54}$$

as an estimate of the average area of Γ_N.

In the two-dimensional case, with the magnetic field perpendicular to the plane of the system, it is clear that the values of Φ_{Γ_N} (52) are distributed about

$$\pm\mathcal{F}_0=\pm H\langle|\mathbf{S}_{\Gamma_N}|\rangle\sim\pi N^{2\nu}Ha^2.$$

(The two signs are again connected with the two possible circuit directions of Φ_{Γ_N}). The distribution function Φ_{Γ_N} can then be simulated by two Gaussian peaks:

$$P(\Phi_{\Gamma_N})=\frac{1}{2\sqrt{2\pi}\sigma_N}\left\{\exp\left[-\frac{(\Phi_{\Gamma_N}-\mathcal{F}_0)^2}{2\sigma_N^2}\right]+\exp\left[-\frac{(\Phi_{\Gamma_N}+\mathcal{F}_0)^2}{2\sigma_N^2}\right]\right\}, \tag{55}$$

where

$$\sigma_N^2\sim f(N)H^2a^4/\Phi_0^2 \tag{56}$$

is the variance of this distribution. It is difficult at present to draw any definite conclusions concerning the behavior of $f(N)$, other than it apparently increases like a certain power of N. In addition, we assume that the distributions of the areas (i.e., and of Φ_{Γ_N}) of the different Γ_N are independent, an assumption that is of course rather doubtful when the statistics of self-avoiding walks is considered.

From the foregoing analysis of the random-phase model we then obtain directly for the effective connectivity constant in a magnetic field the expression

$$\mathcal{K}=\lim_{N\to\infty}\{K^{2N}e^{-\sigma_N^2}\cos^2\mathcal{F}_0+K^N[1-e^{-\sigma_N^2}\cos^2\mathcal{F}_0]\}^{1/2N}$$

$$=\begin{cases}K; & \sigma_N^2\sim N^{1-\delta}H^2a^4/\Phi_0^2,\\ K^{1/2}; & \sigma_N^2\sim N^{1+\delta}H^2a^4/\Phi_0^2\quad\delta>0\,.\end{cases} \tag{57}$$

Only in the case $\sigma_N^2\sim NH^2a^4/\Phi_0^2$ do we obtain

$$\mathcal{K}=\begin{cases}K\exp(-\mathrm{const}\,H^2a^4/\Phi_0^2); & \mathrm{const}\,H^2a^4/\Phi_0^2<\ln K\\ K^{1/2}; & \mathrm{const}\,H^2a^4/\Phi_0^2>\ln K\end{cases}, \tag{58}$$

i.e., a behavior of the type shown in Fig. 2. At $Ha^2\sim\Phi_0$, the phases are thus completely randomized. The behavior (57a), i.e., the absence of an influence of the field on the localization, is also perfectly feasible. The case (57b) has low probability.

We note that in the case (58) the effect saturates in fields $Ha^2\sim\Phi_0$, i.e., $H\sim10^6$ G at $a\sim3$ A. In the limit $Ha^2\ll\Phi_0$ it follows from (58) that

$$L(E)\approx\alpha K\left\{1-\mathrm{const}\frac{H^2a^4}{\Phi_0^2}\right\}\exp\mathcal{F}\left(E,\frac{W}{J}\right), \tag{59}$$

i.e., the mobility thresholds are shifted inside the band in proportion to the square of the field.

In the three-dimensional case we again assume factorization of the distribution function $\mathbf{S}_{\Gamma_N}$ with respect to various Γ_N. In addition we assume also complete randomization of the orientations of $\mathbf{S}_{\Gamma_N}$ in space, so that

$$P(\mathbf{S}_{\Gamma_N})=P(S_{\Gamma_N}^x)P(S_{\Gamma_N}^y)P(S_{\Gamma_N}^z). \tag{60}$$

Simulating each of the factors in (60) by a simple Gaussian distribution (with zero mean value), we obtain for the distribution function of the flux through the contour Γ_N

$$P(\Phi_{\Gamma_N})=\frac{1}{(2\pi)^{1/2}\sigma_N}\exp\left(-\frac{\Phi_{\Gamma_N}^2}{2\sigma_N^2}\right), \tag{61}$$

where for σ_N we again assume a behavior of the type (56). In the three-dimensional case we then obtain the results (57)–(59).

Another possible approximation for $P(\Phi_{\Gamma_N})$ is obtained if the variance of $\mathbf{S}_{\Gamma_N}$ is neglected. It can then be assumed that all the Γ_N have a fixed area of the order of (54), but the directions of S_{Γ_N} are random in space. We obtain readily

$$P(\Phi_{\Gamma_N})=\begin{cases}\Phi_0/2H\langle|\mathbf{S}_{\Gamma_N}|\rangle; & |\Phi_{\Gamma_N}|<H\langle|\mathbf{S}_{\Gamma_N}|\rangle\\ 0; & |\Phi_{\Gamma_N}|>H\langle|\mathbf{S}_{\Gamma_N}|\rangle\end{cases}, \tag{62}$$

and for the effective connectivity we get a result of the type of (47), i.e., the magnetic field does not affect the localization. It seems to me that the most probable is a behavior of the type (58), (59), but the final solution of the problem depends on the behavior of the variance σ_N (56).

CONCLUSION

In conclusion, we discuss the relation between the results above and the deductions of the scaling theory of localization[3,7] and the predictions concerning the influence of the magnetic field.[9,10] It was shown in Ref. 7 that in two-dimensional systems an arbitrarily small disorder suffices for complete localization of all the states in a band. Although this conclusion met with certain objections (see the review[3]), it is confirmed by simple perturbation-theory calculations in the limit of weak disorder,[8,9] when $l \gg a$, where l is the mean free path due to elastic scattering. Analogous calculations[9,10] have demonstrated the strong influence of a magnetic field on two-dimensional localization, viz., a negative magnetoresistance sets in, i.e., the field destroys the localization. These results raise the question of the physical meaning of the two dimensional mobility edges obtained in Anderson's standard approach,[4,5] as well as of the meaning of the conclusion arrived at above, that the magnetic field can only promote localization (or, in the extreme case, have no influence on it).

We note first that despite the complete localization, two-dimensional thresholds retain according to Ref. 7 a certain definite physical meaning of the threshold energies that separate the quasimetallic energy region in two-dimensional systems from the dielectric region. When the Fermi energy passes through these threshold, a rather abrupt transition should take from quasimetallic to hopping conductivity.[3] The localization effects in the "quasimetallic" region are connected[7,8] with the singular behavior of a special class of perturbation-theory diagrams (the Langer-Neal graphs,[30]). In the standard Anderson approach[4,5] the analog of such processes are apparently multiple scattering with return [Fig. 1(b)], which contribute to the self-energy parts $\Delta_k^{ijl\cdots}(E)$. Neglect of such contribution or insufficient allowance for them in the usual approach[4,5] does not lead to weak (logarithmic) effects of complete localization in the quasimetallic region of a two-dimensional system.

At the same time, the contribution of the Langer-Neal graphs is quite sensitive to the magnetic field (and also to scattering by magnetic impurities).[9,10] A rather weak field suffices to exclude such scattering processes, i.e., to destroy the localization in the quasimetallic region, and it is this which leads[9,10] to the effect of negative magnetoresistance. However, even if the Langer-Neal processes are completely neglected, the ordinary Anderson localization, which sets in at $l \sim a$, is possible when the disorder increases in the system. In a two-dimensional zone of a system located in a magnetic field, at sufficiently strong disorder (and field), there exist ordinary mobility edges, whose behavior was in fact considered above.

It is clear from the foregoing that the results of Refs. 9 and 10 and of the present study pertain to different ranges of magnetic-field variation. In particular, if the critical-field estimates of Refs. 9–10 are rewritten in our notation, we find that the negative-magnetoresistance effect saturates ($\sim \ln H$) in fields $Hd^2/\Phi_0 \sim (a/l)^2$ at $T=0$, or in fields $Hd^2/\Phi_0 \sim d^2/ll_{in}$ at $T \neq 0$, where l_{in} is the mean free path for inelastic-scattering processes. By virtue of the condition $a \ll l \ll l_{in}$ $(T \to 0)$ used in Refs. 9 and 10, it is seen that $Hd^2 \ll \Phi_0$. Typical values of the critical field in Refs. 9 and 10 are of the order of 10–100 G. At the same time, the effects discussed above have a characteristic scale $Hd^2 \lesssim \Phi_0$ and saturate at $Hd^2 \sim \Phi_0$, i.e., they refer to fields $H \sim 10^4$–10^6 G, where they should lead to positive magnetoresistance [this can occur earlier in the case of the behavior (57b)]. We note that positive-magnetoresistance effects are implicitly contained in Refs. 9 and 10 via the magnetic-field dependence of the classical diffusion coefficient.

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