Quantum electromagnetic oscillator in the field of a gravitational wave and the problem of nondemolition measurements

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A theory of the interaction of a quantum electromagnetic field in a cavity with a classical gravitational wave is developed. It is shown that the interaction of the electromagnetic oscillator with the gravitational radiation is described by a Hamiltonian which contains not only an external force but also a parametric variation of the mass and frequency of the oscillator. The explicit form of the evolution operator of such a system is found. The probabilities of various processes that take place due to the parametric action of the gravitational signal are found. The concept of a parametric quantum nondemolition operator is introduced. A general algorithm for constructing such operators is indicated and their explicit form is found.

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§1. INTRODUCTION

Weak gravitational waves from cosmic or laboratory sources can be detected only be means of instruments with very high sensitivity. In all probability, feasible methods of raising the sensitivity of gravitational antennas will require the creation of physical conditions under which it is important to take into allowance the quantum nature of the sensitive elements of the antenna. A quantum oscillator may be an adequate model of a (macroscopic) gravitational antenna-mechanical or electromagnetic-and, possibly, a sensor of small displacements, for which a parametrically excited electric circuit is usually employed.

The first studies in this direction led to the concept of the so-called quantum sensitivity limit, and then to the concept of *quantum nondemolition measurements*, which are designed to overcome this limit.[1] Many papers on this subject[2–6] then followed. In particular, the actual concept of quantum nondemolition measurements was made more precise.

From the formal point of view, the crux of the problem of nondemolition measurements consists of finding operators (observables) with the following property. If the system at the initial time is in one of the eigenstates of this operator, then it will remain in an eigenstate of this operator at subsequent times. Operators for which this is true at all times are particularly convenient, and they are called *continuous quantum nondemolition operators* (the abbreviation QND operators is used). Integrals of the motion are examples of such operators but not the only ones. A necessary and sufficient condition for an operator $Z(t)$ in the Heisenberg representation to be a continuous QND operator is taken[3,4] to be

$$[Z(t), Z(t')]=0, \tag{1}$$

where t and t' in the commutator are arbitrary times. If an external force acts on the system, the eigenvalue of a QND operator must vary continuously and thus permit precise measurement of the acting force without the introduction of disturbances into the system by the measurement process. Such operators are called continuous force QND operators (abbreviated QNDF, where F stands for force).

The actual sensitivity of the force detection will be determined by the properties of the materials, the numerical values of the coupling constants, etc., but not by limitations imposed by the quantum-mechanical uncertainty principle.

In all the cited studies, a harmonic oscillator excited by an external classical force was considered. The equations for an oscillator with a Hamiltonian containing an external force arise naturally when one considers the simplest mechanical antennas. It has been shown in a number of papers (see, for example, Refs. 7 and 8) that electromagnetic systems can also be used as gravitational-wave detectors. They have some distinctive features and potential advantages compared with mechanical detectors. In the present paper, we develop a quantum theory of the interaction of the electromagnetic field in a cavity with a classical gravitational wave. In §§2 and 3, we show that the interaction of the electromagnetic oscillator with the gravitational wave is described by a Hamiltonian containing not only an external force but also a parametric variation of the mass and frequency of the oscillator. (See Refs. 9 and 10 for other approaches to describing the interaction of a quantum oscillator with a gravitational field.) Thus, the quantum description of the interaction of the oscillator with the wave becomes much more informative. We find the probabilities of various processes induced by the parametric action of the gravitational signal. The different nature of the interaction Hamiltonian makes it necessary to extend the analysis of quantum nondemolition measurements (§4) to the case of an oscillator excited not only by a force but also parametrically. In this paper, we introduce the concept of a parameteric quantum nondemolition operator (abbreviated QNDP operator) as an Hermitian operator which satisfies Eq. (1) in the presence of parametric excitation of the oscillator. Prescriptions for the construction of such operators are given.

§2. THE CLASSICAL MAXWELL EQUATIONS IN HAMILTONIAN FORM

We write the Maxwell equations in an external gravitational field in the form

$$((-g)^{1/2}F^{\alpha\beta})_{,\beta}=-4\pi c^{-1}(-g)^{1/2}j^{\alpha}, \tag{2}$$

$$F_{\alpha\beta,\gamma}+F_{\gamma\alpha,\beta}+F_{\beta\gamma,\alpha}=0. \tag{3}$$

As the gravitational field, we take

$$g_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta};$$

$$|h_{\alpha\beta}|\ll 1,\quad h_{0\alpha}=0,\quad h^{\beta}_{\alpha,\beta}=0,\quad \eta_{\alpha\beta}h^{\alpha\beta}=0,\quad \Box h_{\alpha\beta}=0.$$

In the considered linear approximation $(-g)^{1/2}\approx 1$, and Eq. (2) simplifies

$$F^{\alpha\beta}{}_{,\beta}=-4\pi c^{-1}j^{\alpha}. \tag{4}$$

Maxwell's equations in the form (3) and (4) are convenient in that they can be interpreted as differential equations for the functions $F^{\alpha\beta}$ and $F_{\alpha\beta}$ in flat space-time and in Cartesian coordinates. The metric of the curved space-time (the gravitational field) occurs in the expressions that establish the connection between these functions:

$$F^{\alpha\beta}=F_{\mu\nu}(\eta^{\alpha\mu}\eta^{\beta\nu}-\eta^{\alpha\mu}h^{\beta\nu}-\eta^{\beta\nu}h^{\alpha\mu}). \tag{5}$$

We assume that the electromagnetic field is confined to a cavity with perfectly conducting walls. In the field of the gravitational wave, the elements of the walls behave as test particles.[11] Boundary conditions must be formulated on their worldlines. We consider first a cavity in flat space-time, i.e., outside the gravitational field. We expand the field in the cavity in the cavity wave functions $u_i(x^1, x^2, x^3)$:

$$F^{0i}=\frac{1}{c}\sum_m p_{(m)}u^i_{(m)},\quad F_{ik}=\sum_m q_{(m)}(u_{(m)i,k}-u_{(m)k,i}), \tag{6}$$

$$u_{(m)i,k,k}=-k^2_{(m)}u_{(m)i},\quad u_{(m)i,i}=0,$$
$$\int u_{(m)i}u_{(n)i}d^3x=\delta_{mn},\quad u^i_{(m)}=\delta^{ik}u_{(m)k}, \tag{7}$$

where the subscript in the brackets denotes the set of numbers labeling the cavity modes; in what follows, if no confusion is possible, we shall omit the brackets. On the boundaries of the cavity the following boundary conditions are satisfied for each mode:

$$u_i|_{tang}=0,\quad (u_{i,k}-u_{k,i})|_{norm}=0. \tag{8}$$

The Maxwell equations (without sources) reduce to the equations

$$\dot q_m=p_m,\quad \dot p_m=-c^2k_m^2q_m. \tag{9}$$

In a gravitational field, we seek a solution for F^{0i} and F_{ik} as before in the form (6), retaining the conditions (7) and (8). Equations (3) with index $\alpha=i$ and Eq. (4) with index $\alpha=0$ are satisfied identically. The remaining equations, in which (5) and (6) are used, contain the functions h_{ik}. To eliminate the spatial dependence in these equations, we multiply by $u_{(m)i}$ and sum over i the left- and right-hand sides of Eqs. (4) with index $\alpha=i$, and we then integrate over the volume of the cavity. We obtain the equation

$$-\dot p_m=c^2k_m^2q_m+c^2\sum_n k_n^2q_nA_{mn}, \tag{10}$$

$$-A_{mn}(t)=\frac{1}{k_m^2}\int d^3x\, u^i_{(m)}[(u^{j,k}_{(n)i}-u^{j,k}_{(n),i})h_{jk}+(u^j_{(n),k}-u^{,j}_{(n)k})h_{ij}{}^{,k}-k_n^2u^j_{(n)}h_{ij}].$$

If besides the variable electromagnetic field the cavity contains a constant magnetic field H_l, $F^{(0)}_{ik}\equiv e_{ikl}H^l$, then F_{ik} must be sought in the form

$$F_{ik}=\sum_m q_m(u_{(m)i,k}-u_{(m)k,i})+e_{ikl}H^l,$$

and then Eq. (10) is augmented by a further term:

$$-\dot p_m=c^2k_m^2q_m+c^2\sum_n k_n^2q_nA_{mn}+B_m, \tag{11}$$

$$-B_m(t)=c^2H^l\int d^3x u_{(m)i}e_{jkl}h^{ij,k}.$$

Equations (3) with the indices $\alpha=0$, $\beta=i$, $\gamma=k$ must first be differentiated and summed over the index k, then multiplied by $u^i_{(m)}$ and summed over i, and, finally, integrated over the cavity volume. This yields the equation

$$\dot q_m=p_m+\sum_n{}' p_nC_{mn}, \tag{12}$$

$$C_{mn}(t)=\frac{1}{k_m^2}\int d^3x\, u^i_{(m)}[u^{j,k}_{(n)}(2h_{ij,k}-h_{kj,i})+u^j_{(n)}h^{,l}_{ij,l}-k_n^2u^j_{(n)}h_{ij}-u^{j,k}_{(n),i}h_{kj}].$$

We make two comments. First, Eqs. (11) and (12) are integral consequences of the Maxwell equations. They are sufficient for our purposes. However, to show that the Maxwell equations are satisfied at each point within the cavity, we should also have to expand the components of the gravitional field in eigenfunctions. Second, the employed representation (6), in which all the components F^{0i} (and all the components F^{ik}) contain the same function of the time, is not the most general. In general, one should introduce a corresponding function for each component F^{0i}, i.e., one should introduce $p^i_{(m)}(t)$. For sufficiently arbitrary h_{ik}, there would be a coupling between not only p_m and q_m (as in the present equations) but also between the different components p^i and q^i of the same mode. For simplicity, we assume that the configuration of the electromagnetic field in the resonator and the structure of h_{ik} are such that this type of coupling is absent, i.e., as in the absence of a gravitational field, we regard each mode as an oscillator with a single degree of freedom.

Equations (11) and (12) generalize Eq. (9) and have the form of Hamiltonian equations of motion. They are obtained in accordance with the usual rules

$$-dp_m/dt=\partial H/\partial q_m,\quad dq_m/dt=\partial H/\partial p_m \tag{13}$$

from the Hamiltonian

$$H={}^1/_2\sum_m(p_m^2+\omega_m^2q_m^2)+{}^1/_2\sum_{m,n}(p_mp_nC_{mn}+\omega_m\omega_nq_mq_nA_{mn})+\sum_m q_mB_m \tag{14}$$

the coefficients A_{mn} and C_{mn} in (11) and (12) must be redenoted as follows:

$$A_{mn}\to(A_{mn}+A_{nm})/2,\quad C_{mn}\to(C_{mn}+C_{nm})/2).$$

In what follows, we assume that q_m and p_m are generalized coordinates and momenta, and we establish canonical commutation relations between them. The choice of quantities taken to be the coordinate and momentum operators and subject to the usual commutation relations is not unique. The solution of the Maxwell equations could also be sought in the form

$$F_{0i}=-c^{-1}\sum_m \pi_m(t)u_{(m)i},\quad F^{ik}=\sum_m \varkappa_m(t)\left(u_{(m)}^{i,k}-u_{(m)}^{k,i}\right),$$

and the remaining components obtained for the pairs of variables π_m, q_m and p_m, $\varkappa_m$; one would then obtain equations different from (10) and (12).

In principle, one could call π_m and q_m, or p_m and $\varkappa_m$, or, finally, π_m and $\varkappa_m$ canonical variables. We choose p_m and q_m for the following reasons. It is possible to choose the Hamiltonian in such a way that Eqs. (11) and (12) follow from it in accordance with the rules (13), whereas it is impossible to choose a Hamiltonian from which the equations for the other pairs of variables follow in accordance with the rules (13).

§3. QUANTIZATION OF THE ELECTROMAGNETIC FIELD AND TRANSITIONS UNDER THE INFLUENCE OF A GRAVITATIONAL WAVE

We take p_m and q_m to be operators and make them satisfy the commutation relations

$$[p_m,p_n]=[q_m,q_n]=0,\qquad [q_m,p_n]=i\hbar\delta_{mn}.$$

In accordance with the usual rules, we introduce creation and annihilation operators:

$$a_m=(2\hbar\omega_m)^{-1/2}(\omega_m q_m+ip_m),$$

$$a_m^+=(2\hbar\omega_m)^{-1/2}(\omega_m q_m-ip_m),\qquad [a_m,a_n^+]=\delta_{mn}.$$

In terms of a and a^+, the Hamiltonian (14) takes the form

$$H=\hbar\sum_n(a_n^+a_n+{}^1/_2)\omega_n+{}^1/_4\hbar\sum_{m,n}(\omega_m\omega_n)^{1/2}[Y_{mn}(t)(a_m^+a_n^++a_ma_n)$$

$$+X_{mn}(t)(a_ma_n^++a_m^+a_n)]+\hbar\sum_n B_n(t)(a_n^++a_n),\tag{15}$$

$$Y_{mn}(t)=A_{mn}(t)-C_{mn}(t),\qquad X_{mn}(t)=A_{mn}(t)+C_{mn}(t).$$

For a single-mode system, the Hamiltonian, expressed by means of the operators p and q, has the form

$$H=\{1+C(t)\}p^2/2+\omega^2\{1+A(t)\}q^2/2+B(t)q.$$

It can be seen from this expression that the action of the gravitational wave on a constant electric field in the cavity has the nature of an external force [the term $B(t)q$], while the effect of the wave on an eigenmode reduces to the appearance of a variable frequency and variable mass of the oscillator. In a many-mode system, the gravitational wave also establishes a coupling between the different modes.

The solutions of the Heisenberg equations of motion for a_n and a_n^+ can be written in the form

$$a_n(t)=U^+(t)a_nU(t),\quad a_n^+(t)=U^+(t)a_n^+U(t),\tag{16}$$

where $U(t)$, the evolution operator of the system, satisfies the equation

$$i\hbar dU/dt=HU.$$

We shall solve this equation by the usual methods of perturbation theory, making a series expansion in the small quantities X_{mn}, Y_{mn}, B_n. Then the evolution operator corresponding to the Hamiltonian (15) has the form

$$U(t)=U^{(0)}(t)\{I-iW(t)\}.$$

Here, $U^{(0)}(t)$ is the operator which describes the evolution of the free oscillator, and $W(t)$ is a small correction due to the wave:

$$W(t)=\frac{1}{4}\sum_{m,n}(\omega_m\omega_n)^{1/2}\Big\{a_m^+a_n^+\int_0^t Y_{mn}(\tau)\exp[i(\omega_m+\omega_n)\tau]d\tau$$

$$+a_m^+a_n\int_0^t X_{mn}(\tau)\exp[i(\omega_m-\omega_n)\tau]d\tau+\text{h.c.}\Big\}$$

$$+\sum_n\Big\{a_n^+\int_0^t B_n(\tau)\exp(i\omega_n\tau)d\tau+\text{h.c.}\Big\}$$

(the Hermitian-conjugate terms are denoted by the letters h. c.).

Knowing the evolution operator, we can find the change in the state vector $|\psi(t)\rangle=U(t)|\psi(0)\rangle$ of the system and the probabilities of transitions between the different states. The time dependence of the operator $U(t)$ is determined by the functions $A_{mn}(t), C_{mn}(t), B_n(t)$. In their turn, they repeat the time dependence of the components h_{ik}. We write

$$X_{mn}(t)=x_{mn}G(t),\quad Y_{mn}(t)=y_{mn}G(t),\quad B_n(t)=b_nG(t),$$

where x_{mn}, y_{mn}, b_n do not depend on the time, and $G(t)$ is determined by the time dependence of the gravitational signal. We shall be interested in the effect of a monochromatic wave on the oscillator, and we retain only the resonance terms. Analysis of the effect of a short pulse also presents no difficulties.

It is convenient to distinguish two different situations, which differ in the relationship between the frequencies Ω of the wave and of the oscillator modes:

1) $\Omega=\omega_m+\omega_n$. Using the terminology adopted in quantum electronics, we shall call this the case of a parametric amplifier.[12] Then

$$W(t)=(\omega_m\omega_n)^{1/2}ty_{mn}[\exp(i\varphi_{mn})a_ma_n-\exp(-i\varphi_{mn})a_m^+a_n^+]/8i.$$

Suppose that a single-mode ($\omega_m=\omega_n=\omega$) oscillator is initially in a state with definite number of quanta: $|\psi(0)\rangle=|n\rangle$. The selection rules are $n\to n\pm2$. The probabilities of remaining in the level n or going over to the levels $n\pm2$ are

$$P_{n,n}=1-\omega^2t^2y^2(n^2+n+1)/32,\quad P_{n,n-2}=-\omega^2t^2y^2n(n-1)/64,$$
$$P_{n,n+2}=-\omega^2t^2y^2(n+1)(n+2)/64.$$

These formulas illustrate the advantage of electromagnetic detectors, which interact parametrically with the gravitational wave, over mechanical detectors, on which the wave acts as a classical force.[8] In the first case, the probability of changing level at large n is proportional to n^2, whereas in the second case it is proportional to only the first power of n. The mean number of quanta at the time t is $\bar{n}=n+\Delta n$, where $\Delta n=1/8\omega^2t^2y^2(n+1/2)$, so that at large n we have $\Delta n/n\sim(\omega t)^2y^2$ in accordance with the equations of the classical theory for the change in the energy of an oscillator with undetermined phase of the oscillations. The variance of the number of quanta is

$$\delta(n^2)=(\omega t)^2 y^2 n(n^3+6n-1)/16.$$

For nonequal frequencies, $\omega_m \neq \omega_n$, the selection rules are modified: $n, m \to n\pm 1, m\pm 1$, so that $n+m \to n+m \pm 2$. The mean number of quanta in the modes ω_m and ω_n is increased.

We now assume that the initial state is coherent. The state vector is determined by the formula

$$|\psi\rangle=\exp(\beta_m a_m^+ - \beta_m^* a)\exp(\beta_n a_n^+ - \beta_n^* a_n)|0_m\rangle|0_n\rangle.$$

The mean number of quanta in the initial state is

$$\bar{n}_m+\bar{n}_n=|\beta_m|^2+|\beta_n|^2.$$

In the final state, the mean number of quanta is

$$\bar{n}_m+\bar{n}_n=|\beta_m|^2+|\beta_n|^2+{}^1/_4(\omega_m\omega_n)^{1/2} t y_{mn}|\beta_m||\beta_n|\cos(\varphi_{mn}+\varphi_m+\varphi_n), \qquad (17)$$

where we have used the notation

$$\beta_m=|\beta_m|\exp i\varphi_m, \qquad \beta_n=|\beta_n|\exp i\varphi_n.$$

Equation (17) agrees with the classical formula $\Delta n/n \sim \omega t y \cos\varphi$ for the increment in the energy of an oscillator with definite and constant phase of the oscillations.

2) $\Omega=\omega_m-\omega_n$. This case corresponds to a parametric frequency converter. The operator $W(t)$ has the form

$$W(t)=(\omega_m\omega_n)^{1/2} t x_{mn}[\exp(i\varphi_{mn})a_m a_n^+ - \exp(-i\varphi_{mn})a_m^+ a_n]/8i.$$

Suppose the initial state is a $|n, m\rangle$-quantum state. The transitions satisfy the selection rules $n, m \to n\pm 1, m\mp 1$. The probabilities for remaining in the n, m level and of transitions to the levels $n+1, m-1$ or $n-1, m+1$ are

$$P_{nm}=1-\omega_m\omega_n t^2 x_{mn}^2(2mn+m+n)/64,$$
$$P_{n+1,\,m-1}=\omega_m\omega_n t^2 x_{mn}^2 m(n+1)/64,$$
$$P_{n-1,\,m+1}=\omega_m\omega_n t^2 x_{mn}^2 n(m+1)/64.$$

At the time t, the mean numbers of quanta in the modes ω_m and ω_n are, respectively,

$$\bar{n}=n+\omega_m\omega_n t^2 x_{mn}^2(m-n)/64,$$
$$\bar{m}=m+\omega_m\omega_n t^2 x_{mn}^2(n-m)/64.$$

Thus, there is an increase in the population of one mode and a decrease in the other. The number of quanta in the system remains the same, but the total energy of the system changes.

§4. QUANTUM NONDEMOLITION MEASUREMENTS OF A PARAMETRICALLY EXCITED OSCILLATOR

We consider a single-mode oscillator with Hamiltonian that is a special case of the Hamiltonian (15):

$$H=\hbar\omega(a^+a+{}^1/_2)+\hbar\omega[y(t)(a^{+2}+a^2)+x(t)(aa^++a^+a)]/4+\hbar f(t)(a^++a).$$

The solution of the Heisenberg equations of motion (16) in the linear approximation in the small functions $x(t)$ and $y(t)$ has the form

$$a(t)=e^{-i\omega t}[a_0(1+\xi)+a_0^+\eta+\chi],$$
$$a^+(t)=e^{i\omega t}[a_0^+(1+\xi^*)+a_0\eta^*+\chi^*]; \qquad (18)$$

$$\xi(t)=-\frac{i\omega}{2}\int_0^t x(t')\,dt'=-\xi^*(t),$$

$$\eta(t)=-\frac{i\omega}{2}\int_0^t y(t')e^{2i\omega t'}dt',$$

$$\chi(t)=-i\int_0^t f(t')e^{i\omega t'}dt'.$$

We find an Hermitian operator $Z(t)$ composed of linear and quadratic combinations of the operators $a(t)$ and $a^+(t)$ and satisfying Eq. (1). For $\xi=0$, $\eta=0$, $\chi=0$ such an operator will be a QND operator of the free oscillator; for $\chi\neq 0$, $\xi=0$, $\eta=0$ it will be a QNDF operator; and for $\chi=0$ and at least one of the functions ξ or η nonvanishing it will be a QNDP operator. If the functions that describe both the force and the parametric action are nonzero, the operator will be simultaneously a QNDF operator and QNDP operator. We stipulate that in the construction of $Z(t)$ at a given instant of time we use only a and a^+ taken at the same time. Such an operator may be called instantaneous, in contrast to the shift operator considered below, in which we permit the use of a and a^+ taken at preceding instants of time.

The general form of the required operator is

$$Z(t)=R_1 I+R_2 e^{i\omega t}a(t)+R_2^* e^{-i\omega t}a^+(t)+R_3 e^{2i\omega t}a(t)a(t)$$
$$+R_3^* e^{-2i\omega t}a^+(t)a^+(t)+R_4(t)a^+(t)a(t),$$

where I is the identity operator, R_1, R_2, R_3, R_4 are as yet arbitrary c-number functions of the time, and R_1 and R_4 are real functions. Using (18) and taking into account only the terms linear in ξ, η, χ, we rewrite $Z(t)$ in the form

$$Z(t)=IZ_1+a_0Z_2+a_0^+Z_2^*+a_0a_0Z_3+a_0^+a_0^+Z_3^*+a_0^+a_0Z_4, \qquad (19)$$

where Z_1, Z_2, Z_3, Z_4 can be readily expressed in terms of R_1, R_2, R_3, R_4 and ξ, η, χ.

Substituting (19) in Eq. (1), we find that (1) will be satisfied if

$$Z_3(t)Z_3^*(t')-Z_3^*(t)Z_3(t')=0, \quad Z_3(t)Z_4(t')-Z_3(t')Z_4(t)=0, \qquad (20)$$

$$Z_2(t)Z_4(t')-Z_2(t')Z_4(t)+2Z_3(t)Z_2^*(t')-2Z_2^*(t)Z_3(t')=0, \qquad (21)$$

$$Z_2(t)Z_2^*(t')-Z_2^*(t)Z_2(t')=0. \qquad (22)$$

Since these equations must be satisfied for arbitrary t and t', they must, in particular, be satisfied at times separated by an infinitesimal amount: $t'=t+dt$. Then the functions $Z(t')$ in Eqs. (20)–(22) can be replaced by $\dot{Z}(t)$ and Eqs. (20)–(22) can be solved for $R_2(t)$, $R_3(t)$, $R_4(t)$.

We begin the investigation of Eqs. (20)–(22) with the case of the free oscillator: $\xi=0$, $\eta=0$, $\chi=0$. From Eqs. (20) and (22), we find

$$R_3(t)=|R_3(t)|e^{-i\varphi}, \quad C_1R_4(t)=|R_3(t)|, \quad R_2(t)=|R_2(t)|e^{-i\psi}. \qquad (23)$$

Here, $|R_2(t)|$ and $|R_3(t)|$ are arbitrary real functions of the time, and φ, ψ, C_1 are arbitrary real constants. It remains to satisfy Eq. (21). It decomposes into a product of two factors, and requires for its fulfillment either the equations $C_1=\frac{1}{2}$, $\varphi=2\psi$ (first variant) or the equation $R_2(t)=C_2R_3(t)e^{i(\varphi-\psi)}$, where C_2 is an arbitrary real constant (second variant).

In the first variant, the general form of the QND operator of the free oscillator is

$$Z(t)=R_1(t)I+|R_2(t)|\{e^{i(\omega t-\psi)}a(t)+e^{-i(\omega t-\psi)}a^+(t)\}$$
$$+|R_3(t)|\{e^{2i(\omega t-\psi)}a(t)a(t)+e^{-2i(\omega t-\psi)}a^+(t)a^+(t)+2a^+(t)a(t)\}. \quad (24)$$

In its terms linear in a and a^+ (the expression in the first curly brackets), it is, as noted in Ref. 4, a linear combination (with constant coefficients) of the operators $X_1(t)$ and $X_2(t)$, where

$$X_1(t)=(\hbar/2\omega)^{1/2}(e^{-i\omega t}a^+(t)+e^{i\omega t}a(t)),$$
$$X_2(t)=i(\hbar/2\omega)^{1/2}(e^{-i\omega t}a^+(t)-e^{i\omega t}a(t)). \quad (25)$$

In the terms quadratic in a and a^+ [the second curly brackets in (24)]. the operator $Z(t)$ contains the square of a linear combination of $X_1(t)$ and $X_2(t)$. It is easy to show that an arbitrary function of the expression

$$e^{i(\omega t-\psi)}a(t)+e^{-i(\omega t-\psi)}a^+(t)$$

can also serve as a QND operator of the free oscillator.

In the second variant, the general form of a QND operator of the free oscillator is

$$Z(t)=R_1(t)I+C_1C_2R_4(t)\{e^{i(\omega t-\psi)}a(t)+e^{-i(\omega t-\psi)}a^+(t)\}$$
$$+C_1R_4(t)\{e^{i(2\omega t-\varphi)}a(t)a(t)+e^{-i(2\omega t-\varphi)}a^+(t)a^+(t)\}+R_4(t)a^+(t)a(t). \quad (26)$$

The last term of this operator is the particle-number operator $N(t)$, which is obviously a QND operator of the free oscillator.

Thus, all (instantaneous) QND operators of the free oscillator in the class of operators containing $a(t)$ and $a^+(t)$ to not higher than the second power are exhausted by the expressions (24) and (26). We now turn to the construction of QNDF and QNDP operators, i.e., we assume $\chi\neq 0$, $\xi\neq 0$, $\eta\neq 0$. It is convenient to add to the functions $R_2(t)$, $R_3(t)$, $R_4(t)$ found above for the free oscillator small corrections of the same order as the functions χ, ξ, η. We write

$$R_2(t)=e^{-i\psi}(\beta+r_2(t)), \quad R_3(t)=e^{-i\varphi}(\gamma+r_3(t)), \quad R_4(t)=\gamma/C_1+r_4(t),$$

where β and γ are arbitrary real functions of the time. In the zeroth approximation, we use formulas (23). In the linear approximation, the system of equations (20)–(22) leads to the following restrictions. In the first variant (i.e., for $C_1=\frac{1}{2}$, $\varphi=2\psi$) the real part of the function $r_2(t)$ remains arbitrary, and the remaining functions are connected by the equations

$$r_3=r_4/2+\gamma(\xi^*-\xi-\eta^*e^{i\varphi}+\eta e^{-i\varphi}),$$
$$r_2-r_2^*=\beta(\xi^*-\xi-\eta^*e^{i\varphi}+\eta e^{-i\varphi}). \quad (27)$$

In the second variant (i.e., for $Z_2=C_2Z_3e^{i(\varphi-\psi)}$), we obtain from (20)–(22) the equations

$$r_3=C_1r_4+\gamma[\xi^*-\xi+2C_1(\eta e^{-i\varphi}+\eta^*e^{i\varphi})-\eta^*e^{i\varphi}/C_1],$$
$$r_2=C_1C_2r_4+\gamma[2C_1C_2(\eta e^{-i\varphi}+\eta^*e^{i\varphi})-C_2(\xi+\eta^*e^{2i\psi})-2\chi e^{-i(\varphi-\psi)}-\chi^*e^{i\psi}/C_1]. \quad (28)$$

The first of equations (28) is identical with the first of equations (27) if $C_1=\frac{1}{2}$. The second of equations (27) is a consequence of the second equation of the system (28) if we make the substitution $\beta=C_2\gamma$, $C_1=\frac{1}{2}$, $\varphi=2\psi$.

Equations (27) and (28) establish the required restrictions on the c-number functions $R_2(t)$, $R_3(t)$, $R_4(t)$. The operators $Z(t)$ constructed by means of them are QND operators for $\chi\neq 0$, $\xi\neq 0$, $\eta\neq 0$.

Suppose first $\chi\neq 0$, $\xi=\eta=0$. Since the function χ does not occur in Eq. (27), it follows that the operator (24) is not only a QND operator of the free oscillator but also a QNDF operator; moreover, it is obvious that its construction does not require knowledge of the function χ. In particular, $X_1(t)$ and $X_2(t)$ are operators of this kind.[4] In accordance with Eqs. (28), we can also construct a QNDF operator, this being the energy operator in the first approximation; however, in this case we need to know $\chi(t)$. Indeed, as follows from (28), for $\gamma/C_1=1$ ($\gamma\to 0$, $C_1\to 0$) the operator $N(t)$ is augmented in the linear approximation by the operators $a(t)$ and $a^+(t)$ with coefficient r_2 that depends on χ. For the QNDF operator

$$Z(t)=a^+(t)a(t)-\chi^*a(t)e^{i\omega t}-\chi a^+(t)e^{-i\omega t}$$

constructed in this manner, the eigenvalue n_0 does not change in the linear approximation, while in the quadratic approximation it is equal to $n_0-|\chi|^2$. Operators of this kind could be helpful in the detection of gravitational signals with shape known in advance, which could be the case in a laboratory experiment or in the observation of astronomical sources like binary stars.

Now suppose $\chi=0$ and at least one of the functions ξ and η is nonzero. The prescription for constructing QNDP operators using *a priori* knowledge of the functions ξ and η is indicated by Eqs. (27) and (28). Such operators may be helpful when one is detecting signals from the class of sources having a previously known signal shape. However, as can be seen from Eqs. (27) and (28), there are no instantaneous QNDP operators that do not depend on the signal functions ξ and η even if one of them is zero. *A fortiori* there are no operators of this type that are simultaneously QNDP and QNDF operators and do not depend on χ, ξ, η.

We now consider shift operators, i.e., in constructing $Z(t)$ we permit the use of a and a^+ taken at preceding instants. As an example of operators of this class, we can take the operator proposed in Ref. 5:

$$y(t)=q(t)-q(t-\tau)=(\hbar/2\omega)^{1/2}[a^+(t)+a(t)-a^+(t-\tau)-a(t-\tau)], \quad (29)$$

where $\tau=2\pi/\omega$ is the period of the oscillator. This operator is a QNDF operator, since

$$y(t)|\psi(t)\rangle=(\hbar/2\omega)^{1/2}\{e^{-i\omega t}(\chi(t)-\chi(t-\tau))$$
$$+e^{i\omega t}(\chi^*(t)-\chi^*(t-\tau))\}I|\psi(t)\rangle=R_1(t)|\psi\rangle,$$

and in the given case $|\psi(t)\rangle$ can be an arbitrary state vector. In other words, $y(t)$ has the form of the first term in the expansion (19), $Z(t)=R_1(t)I$, but R_1 is not determined from the condition (1) (this condition does not impose any restrictions on R_1) but arises automatically as a consequence of the time shift. It is clear that any other operator which is transformed after the shift into a constant operator (not necessarily I) multiplied by a c-number function will satisfy the condition (1) and thus be a QND operator.

Below, we shall not restrict ourselves to a shift τ

equal to an integral number of periods of the oscillator, and we shall regard τ as an arbitrary fixed parameter. In particular, we may have $\tau=2\pi/\omega$. Simple generalizations of the operator (29) are the following shift QNDF operators:

$$X_1(t)-X_1(t-\tau)=(\hbar/2\omega)^{1/2}[(\chi+\chi^*)(t)-(\chi+\chi^*)(t-\tau)]I,$$
$$X_2(t)-X_2(t-\tau)=-i(\hbar/2\omega)^{1/2}[(\chi-\chi^*)(t)-(\chi-\chi^*)(t-\tau)]I.$$

We now find shift QNDP operators. Their construction requires partial knowledge of the functions ξ and η, namely, the assumption that either $\xi=0$, $\eta\neq0$ or $\xi\neq0$, $\eta=0$. Such assumptions are not excessively restrictive. For example, if the components h_{ik} of the gravitational wave are concentrated near the frequency 2ω, then after Q periods have elapsed $|\eta|$ will be Q times greater than $|\xi|$ and, therefore, it may be assumed that $\xi=0$, $\eta\neq0$. On the other hand, if the functions h_{ik} vary slowly, then after a definite time $|\xi|\gg|\eta|$ and we can assume approximately $\xi\neq0$, $\eta=0$. Thus, we consider two cases: 1) $\xi=0$, $\eta\neq0$; 2) $\xi\neq0$, $\eta=0$.

We first introduce the notation

$$K(t)=X_1^2(t)-X_2^2(t),\quad L(t)=X_1(t)X_2(t)+X_2(t)X_1(t).$$

For $\xi=0$, $\eta\neq0$

$$K(t)-K(t-\tau)=\hbar\omega^{-1}[(\eta+\eta^*)(t)-(\eta+\eta^*)(t-\tau)](a_0^+a_0+a_0a_0^+),$$
$$L(t)-L(t-\tau)=i\hbar\omega^{-1}[(\eta^*-\eta)(t)-(\eta^*-\eta)(t-\tau)](a_0^+a_0+a_0a_0^+)$$

are shift QNDP operators.

We note a curious analogy between the operators $K(t)$ and $L(t)$ and the invariants of the electromagnetic field. Since the operator $q(t)$ is associated with the magnetic field, and $p(t)/\omega$ with the electric field, the invariants of the electromagnetic field, H^2-E^2 and $\mathbf{E}\cdot\mathbf{H}$, should be associated with the operators

$$k(t)=q^2(t)-p^2(t)/\omega^2,\quad l(t)=[q(t)p(t)+p(t)q(t)]/\omega.$$

Together with the operator $n(t)=q^2(t)+p^2(t)/\omega^2$, they satisfy the commutation relations

$$[k,l]=4i\hbar n/\omega,\quad [k,n]=4i\hbar l/\omega,\quad [n,l]=4i\hbar k/\omega.$$

The connection of the operators $k(t)$ and $l(t)$ to $K(t)$ and $L(t)$ is

$$k(t)=K(t)\cos2\omega t+L(t)\sin2\omega t,\quad l(t)=-K(t)\sin2\omega t+L(t)\cos2\omega t.$$

Thus, a shift through an arbitrary time interval τ in the operators $K(t)$ and $L(t)$ or a shift through $\tau=\pi/\omega$ in the operators $k(t)$ and $l(t)$ leads to QNDP operators for any function $\eta(t)$ and $\xi=0$.

If we also know that the function $\eta(t)$ can be represented in the form $\eta(t)=|\eta(t)|e^{i\alpha}$, where α does not depend on the time and $|\eta(t)|$ is an arbitrary function of the time, then we can find other QNDP operators. For example,

$$a^+(t)a(t)-a^+(t-\tau)a(t-\tau)=[|\eta(t)|-|\eta(t-\tau)|][a_0^2e^{-i\alpha}+(a_0^+)^2e^{i\alpha}],$$
$$X_1(t)-X_1(t-\tau)=(\hbar/2\omega)^{1/2}[|\eta(t)|-|\eta(t-\tau)|](a_0e^{-i\alpha}+a_0^+e^{i\alpha}),$$
$$X_2(t)-X_2(t-\tau)=i(\hbar/2\omega)^{1/2}[|\eta(t)|-|\eta(t-\tau)|](a_0e^{-i\alpha}-a_0^+e^{i\alpha}).$$

In the second case, i.e., for $\xi\neq0$, $\eta=0$,

$$K(t)-K(t-\tau)=2\hbar\omega^{-1}[\xi(t)-\xi(t-\tau)][a_0^2-(a_0^+)^2],$$
$$L(t)-L(t-\tau)=-i2\hbar\omega^{-1}[\xi(t)-\xi(t-\tau)][(a_0^+)^2+a_0^2],$$
$$X_1(t)-X_1(t-\tau)=(\hbar/2\omega)^{1/2}[\xi(t)-\xi(t-\tau)](a_0-a_0^+),$$
$$X_2(t)-X_2(t-\tau)=-i(\hbar/2\omega)^{1/2}[\xi(t)-\xi(t-\tau)](a_0+a_0^+)$$

are shift QNDP operators.

Thus, the QNDP operators (observables) found here can be used to avoid completely the quantum-mechanical limitations on the sensitivity of a detector whose model is a parametrically excited oscillator.

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