Theory of nuclear level shifts in hadronic atoms

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Because of the strong interaction at short distances, the Coulomb levels of hadronic atoms are shifted and acquire a width. Because the condition $r_e \ll a_B$ is satisfied, one can construct an analytic theory of this phenomenon (r_e is the effective range of the strong potential, and a_B is the Bohr radius). The equation that describes the level shifts takes into account terms of order r_e, the proton form factor, and the influence of a second channel (with nearly the same energy). A study is made of the level shifts, the level widths, and the rearrangement of the atomic spectrum in the case of a complex scattering length, i.e., in the presence of absorption. The limits of applicability of perturbation theory with respect to the scattering length are determined. The results are generalized to states with nonzero orbital angular momentum.

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§1. INTRODUCTION

In the theory of hadronic atoms, it is necessary to take into account the strong interaction which leads to a shift and broadening of the levels. For the lightest hadronic atoms ($p\bar{p}$, $\Sigma^- p$, and others) the condition $r_e \ll a_B$ is well satisfied; here, r_e is the effective range of the strong potential $V_s(r)$, and $a_B = \hbar^2/me^2$ is the Bohr radius. This makes it possible to develop an analytic theory of nuclear level shifts and relate the shifts to the parameters of low-energy scattering, i.e., the scattering length a_s and the effective range r_e. As long as the level shift is small, it can be calculated by perturbation theory.[1-4] For example, for s levels[1)]

$$\Delta E_{ns} = 2a_s/n^3, \tag{1.1}$$

where a_s is the scattering length on the strong potential V_s.

However, (1.1) has a fairly narrow range of applicability (see Ref. 5 and in more detail §4 below) and does not describe the phenomenon of rearrangement of the atomic spectrum. Expressions going beyond the framework of perturbation theory were obtained earlier by the method of matching[5] and in the previous paper[6] by analytic continuation of the effective-range expansion.[7-10] We emphasize that these expressions, in contrast to (1.1), are valid for an arbitrary shift ΔE_{nl} and permit a model-free description of the rearrangement of the atomic spectrum[2,11] when a bound state appears in the strong potential V_s.

An interesting situation arises when there is a sufficiently large positive shift of the atomic s levels. This case may be realized in the $p\bar{p}$ atom; for in accordance with Ref. 12, for the ground 1s level

$$\Delta E_{1s} = 3.04 \pm 0.06 \text{ keV}, \quad \Gamma_{1s} \approx 300 \text{ eV} \tag{1.2}$$

(the distance between the undisplaced 1s and 2s levels is 9.37 keV). It is clear from physical considerations that in this case the $p\bar{p}$ system must have a nuclear level (Qs) with low binding energy, which pushes the Coulomb levels upward. The equations obtained earlier in Refs. 5 and 6 enable one, from known shift of one of the atomic s levels, to predict the position and width of the Qs level. Using the experimental data (1.2), we find that its binding energy ε_{Qs} is of order 1 MeV.

In view of the importance of this conclusion for following experiments, it is necessary to consider the theory of nuclear level shifts in more detail. This is the aim of the present paper. In §2, we discuss an equation in which allowance is made for the terms $\sim r_e/a_B$ and r_e/a_e, and also a correction for the electromagnetic form factor of the proton. We have made numerical calculations of the effective range and other parameters for the one-boson exchange potential (OBEP). For the majority of hadronic atoms, one must calculate the spectrum with allowance for the multichannel nature of the problem (in connection with the $p\bar{p}$ atom, this was pointed out by Kerbikov[13]; see also Ref. 14). We shall show that allowance for the second channel $n\bar{n}$ (which has nearly the same energy) reduces in the majority of cases to a renormalization of the effective range r_e for the $p\bar{p}$ atom. For sufficiently strong coupling of the channels, r_e may become negative [see Eq. (2.20) below], with the consequence that the binding energy ε_{Qs} decreases.

In §3, we investigate the behavior of the levels in the presence of absorption [associated, for example, with annihilation of p and $\bar{p}$ at distances $r \sim r_e \sim (2m_N)^{-1}$]. In the case of weak absorption, increasing depth of $\mathrm{Re}\, V_s(r)$ leads to a rearrangement of the atomic spectrum, as was found for purely real potentials.[5,11] There is however a certain critical value of the absorption. If the absorption exceeds the critical value, then there is no rearrangement of the spectrum, and the levels fluctuate around the unperturbed positions when there is a monotonic variation in $\mathrm{Re}\, V_s(r)$. In the same section, we calculate the shift and width for highly excited ($n \gg 1$) levels. In §4, we discuss the region of applicability of the formula (1.1) obtained in perturbation theory with respect to the scattering length. In §5, we investigate the equation for the shift of levels with nonvanishing angular momentum l and draw attention to an important difference between the cases $l = 0$ and $l \geq 1$.

The experimental investigation of the lightest hadronic atoms has only just begun.[2)] In particular, it is not known what spin ($S = 0$ or 1) corresponds to the measured value (1.2) of the shift for the $p\bar{p}$ atom. The situation will evidently be rapidly clarified once the slow antiproton storage ring (LEAR) at CERN has been commissioned.[15]

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We note finally that a Coulomb potential distorted at short distances ($r \ll a_B$) is frequently encountered in solid-state physics. Indeed, the very phenomenon of rearrangement of the atomic spectrum was discovered by Zel'dovich[2] when considering the energy spectrum of a valence electron bound to an impurity atom in a semiconductor. Therefore, it appears to us that the method presented below and some of the results may also be helpful in a number of problems in solid-state theory.

§2. EQUATION FOR THE LEVEL SPECTRUM ($l=0$)

In Ref. 6, an equation was obtained [Eq. (2.4)] for the nuclear level shift, the following approximations being made: 1) it was assumed that the depth of the strong potential V_s corresponds exactly to the value at which a bound state arises; 2) the Coulomb interaction in the region $r \leq r_e$ was ignored. For $a_s \gg r_e$, these approximations are rather accurate. However, for the $p\bar{p}$ system[3)] $a_s \approx 7-8$ F, i.e., the parameter r_e/a_B is not very small. Therefore, in the above equation it is necessary to take into account the corrections of first order in the range of the strong interaction.

The single-channel case. Corrections of order r_e. These corrections can be calculated by means of the method developed in Ref. 16 to obtain an "effective range" expansion near the boundary of the lower continuum. Referring the reader to Ref. 17 for details of the derivation, we give the final expression:

$$\psi(1-\lambda^{-1})+\frac{\lambda}{2}+\ln\lambda r_C+c_0+c_1 r_e+\frac{c_2}{(\Lambda r_e)^2}=(b_0+b_1 r_e+\ldots)\frac{1}{a_s}+\frac{1}{4}r_e\lambda^2+\ldots \tag{2.1}$$

($a_B=\hbar^2/me^2=1$). If the value of the shift is given, this equation makes it possible to find the "purely nuclear" scattering length a_s (for scattering on the potential V_s with the Coulomb interaction switched off). The notation is as follows: $\lambda=(-2E)^{1/2}$, where E is the energy of the level in atomic units; $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function; the effective range r_e and the Coulomb radius r_C are determined by the expressions

$$r_e=2\int_0^\infty(1-\chi_0^2)\,dr,$$
$$r_C=\lim_{\rho\to 0}\exp\left\{\ln\rho+\int_\rho^\infty[1-\chi_0^2(r)]\frac{dr}{r}\right\} \tag{2.2}$$

where $\chi_0(r)$ is the wave function in the strong potential at the moment when the bound state arises ($E=l=0$, $g=g_0$):

$$\chi_0''+g_0 v\chi_0=0.$$

At the same time, we have taken

$$2mV_s/\hbar^2=-gv(r),$$

and $\chi_0(r)$ must satisfy the boundary conditions

$$\chi_0(0)=0;\quad \chi_0(r)\to 1,\quad r\to\infty. \tag{2.3}$$

In (2.1), we have taken into account the first two terms of the expansion in the small parameter r/a_B. In the zeroth approximation,

$$b_0=1/2,\quad c_0=2C+\ln 2=1.848 \tag{2.4}$$

[see Ref. 6; here, $C=-\psi(1)=0.5772\ldots$]. The following coefficients can be found by expanding g, $1/a_s$, and the wave function with zero energy in a series in $g-g_0$ (which gives b_1) and e^2 (which gives c_1). The result can be obtained in a closed form[17]:

$$b_1=I_1/I_0,\quad c_1=I_2/I_0, \tag{2.5}$$

$$I_0=\frac{1}{2}r_e=\int_0^\infty(1-\chi_0^2)\,dr,\quad I_1=\int_0^\infty\left(1+\frac{\chi_0\chi_1}{\alpha_1 r}\right)dr,$$
$$I_2=\int_0^\infty\left(1-\ln\frac{r}{r_C}-\frac{\chi_0\varphi_1}{r}\right)dr,\quad \alpha_1=-\lim_{r\to\infty}\frac{\chi_1(r)}{r},\quad r\to\infty, \tag{2.6}$$

where χ_1 and φ_1 are solutions of the inhomogeneous equations and are uniquely determined by the following boundary conditions:

$$\chi_1''+g_0 v\chi_1=-v\chi_0; \tag{2.7}$$
$$\chi_1(0)=0;\quad \chi_1(r)=-\alpha_1 r+o(1),\quad r\to\infty.$$

$$\varphi_1''+g_0 v\varphi_1=-\frac{\chi_0}{r}, \tag{2.8}$$
$$\varphi_1(0)=0;\quad \varphi_1(r)/r=-[\ln(r/r_C)-1]+o(1),\quad r\to\infty.$$

In the simplest cases (rectangular well, Hulthen and Yamaguchi potentials), the functions χ_1 and φ_1 and the constants b_1 and c_1 can be found analytically.[17] The term $c_2/(\Lambda r_e)^2$ in (2.1) takes into account the finite sizes of the p and $\bar{p}$ and is determined below [see Eq. (2.13)]. Note that in Eq. (2.1) we have omitted the terms $\propto r_e^2$, whose contribution is small.

If the strong potential is local, then the coefficients b_1 and c_1 can be transformed to a form containing only the wave function χ_0. Indeed, one of the solutions of the Schrödinger equation $\chi''+g_0 v\chi=0$ is $\chi=\chi_0(r)$, and the second solution (linearly independent of χ_0) can be determined by quadrature:

$$\chi(r)=\chi_0(r)\int^r[\chi_0(x)]^{-2}dx.$$

Using the method of variation of constants, we find χ_1 and φ_1 explicitly:

$$\chi_1(r)=-\chi_0(r)\left\{A+\int_0^r\frac{dr'}{\chi_0^2(r')}\int_0^{r'}v(\rho)\chi_0^2(\rho)\,d\rho\right\},$$
$$A=\int_0^\infty dr\,v(r)\chi_0^2(r)\left[r-\int_r^\infty dr'(\chi_0^{-2}(r')-1)\right],\quad \alpha_1=\int_0^\infty v\chi_0^2 dr; \tag{2.9}$$

$$\varphi_1(r)=\chi_0(r)\left\{B-\int_0^r\frac{dr'}{r'}\chi_0^2(r')\int_{r'}^r d\rho[\chi_0^{-2}(\rho)-1]+\int_r^\infty dr'[1-\chi_0^2(r')]\right.$$
$$\left.+r\left[1-\int_0^r\frac{dr'}{r'}\chi_0^2(r')\right]\right\}, \tag{2.10}$$
$$B=\int_0^\infty\frac{dr}{r}\chi_0^2(r)\int_r^\infty dr'(\chi_0^{-2}(r')-1).$$

For an arbitrary local potential $V_s(r)$ the determination of the constants b_1 and c_1 reduces to the finding of the wave function $\chi_0(r)$ and the calculation of triple integrals.

Allowance for the proton form factor. In determining r_C in (2.2), we took the Coulomb interaction between

the p and $\bar{p}$ in the form $1/r$, which corresponds to point particles. At short distances, this expression is modified by the electromagnetic form factor of the proton. Using for it the double dipole representation,[18] we must make the substitution

$$\frac{1}{q^2}\to\frac{1}{q^2}\left(\frac{\Lambda^2}{\Lambda^2+q^2}\right)^4,\ \Lambda^2=0.71\ \text{GeV}^2;$$
$$\frac{1}{r}\to\frac{1}{2\pi^2}\int e^{iqr}\left(\frac{\Lambda^2}{\Lambda^2+q^2}\right)^4\frac{d^3q}{q^2} \qquad (2.11)$$
$$=\frac{1}{r}\left[1-e^{-\mu}\left(1+\frac{11}{16}\mu+\frac{3}{16}\mu^2+\frac{1}{48}\mu^3\right)\right]$$

($\mu=\Lambda r$). We rewrite (2.2) in the form

$$\ln r_C=\ln R+\int_R^\infty\frac{1-\chi_0^2}{r}dr-\int_0^R\frac{\chi_0^2(r)}{r}dr, \qquad (2.12)$$

which, in contrast to (2.2), does not contain the limiting operation and is more convenient for numerical calculations. Here, R is an arbitrary parameter whose value does not affect r_C. Bearing in mind that $\Lambda^{-1}\ll r_e$, we choose it to make $\Lambda^{-1}\ll R\ll r_e$. Then the modification of the Coulomb potential (2.11) is important only in the last integral of (2.12), where, by virtue of $r\ll r_e$, we can set $\chi_0(r)\approx\chi_0'(0)r$. Denoting $A=r_e[d\chi_0/dr]_{r=0}$, we obtain $r_C\to\tilde{r}_C$, where

$$\ln\tilde{r}_C=\ln r_C+c_2/(\Lambda r_e)^2,\quad c_2=4A^2. \qquad (2.13)$$

Numerical calculations. We have considered some model potentials of the strong interaction:

$$V_s(r)=-gv(x),\quad x=\beta r, \qquad (2.14)$$

and also the "realistic" OBEP potential[19,20]

$$V_s(r)=-\sum_{i=1}^{N}\frac{g_i}{r}e^{-\beta_i r}. \qquad (2.15)$$

The parameter β^{-1} in (2.14) determines the range of the nuclear forces, and the function $v(x)$ the shape of the potential. The results of the calculations are given in Table I for the 1s (upper row) and 2s levels. As a rule, r_e increases by 1.8–2 times on the transition from the ground to the first excited state. The value of r_C increases by about the same amount. Note that the ratio of the effective range to the Coulomb radius and also the coefficient b_1 depend weakly on the shape of the nuclear potential and the level number. This fact was verified[17] for the first ten s levels in the Hulthen potential.

TABLE I.

$v(x)$	βr_e	r_C/r_e	A	b_1	c_1	Remarks
$(e^x-1)^{-1}$	3.000	0.374	3.00	1.000	0.983	Hulthen potential
	5.500	0.397	−11.00	0,905	1.453	
e^{-x}/x	2.120	0.364	3.27	1,056	0.842	Yukawa potential
	3.962	0.400	−11.44	0,962	1.428	
e^{-x}	3.541	0.395	2.21	0.903	1.215	–
	6.597	0.398	−6.20	0.852	1.495	
$(\text{ch}\,x)^{-2}$	2.000	0.406	2.00	0.863	1.323	–
	3.667	0.393	−5.50	0.815	1.491	
e^{-x^2}	1.435	0.418	1.83	0.814	1.457	Gaussian potential
$\theta(1-x)$	1.000	0.439	1.57	0.757	1.634	Rectangular well
см. [31]	3.000	0.374	3.00	0.667	1.333	Yamaguchi potential

Note. In the first and second rows, we give the parameters βr_e, etc., for levels 1s and 2s, respectively. For the potentials to which only one row corresponds in the table, these values refer to the ground 1s level.

The individual terms in (2.15) correspond to the exchange of scalar, pseudoscalar, and vector mesons ($N=6$). However, the OBEP potential is not well known at short distances. Because of this, we varied its shape in the region $r<r_0=0.6$ F, considering three variants: (I) $V(r)=0$ for $r<r_0$; (II) $V(r)=V(r_0)=\text{const}$, $r<r_0$; and (III), in which the expression (2.15) was used for all r. The scale of the changes is characterized by the following numbers: $r_e=1.43$, 1.14, and 0.68 F; then $r_C/r_e=0.417$, 0.401, and 0.350; $b_1=0.86$, 0.90, and 1.14; and $c_1=1.35$, 1.20, and 0.57 for the variants I, II, and III (spin $S=0$, ground-state level). For the states with spin $S=1$, the picture is similar.

For the final calculations, we chose[4)] variant I (see Table II). On the transition from the 1s to 2s level, the values of r_e and r_C increase appreciably; the remaining parameters change very little. Since the OBEP potential for the $N\bar{N}$ system is much deeper than for NN, the nuclear level with low binding energy is here evidently the state 2s. Since r_C occurs in Eq. (2.1) under the logarithm sign, the existing uncertainty in the value of r_C is unimportant.

As one would expect, when allowance is made for the proton form factor the Coulomb radius of the $p\bar{p}$ system increases. The numerical values of the parameters r_C and $\tilde{r}_C$ for the OBEP potential are given in Table II.

Multichannel case. For the $p\bar{p}$ system, there is a neutral ($n\bar{n}$) channel with nearly the same energy [the difference between the thresholds is $\Delta=2(m_n-m_p)=2.59$ MeV]. The same is true for the other hadronic atoms: $\Delta=5.29$ MeV for K^0n and K^-p, etc. The effective-range approximation for the multichannel problem has been considered by various people.[22,23,10] Generalizing Bethe's method[7] to the two-channel case, we arrive at an equation for the energies of s levels[24]:

$$\lambda+2\zeta[\psi(1-\zeta\lambda^{-1})-\ln(\zeta/\lambda)]=1/a_{cs}+R\lambda^2/2+O(\lambda^4),$$
$$a_{cs}=(-\alpha_{11}+\rho\alpha_{12})^{-1},\quad \rho=\alpha_{12}/[\alpha_{22}+(2\Delta)^{1/2}], \qquad (2.16)$$
$$R=r_{11}-\alpha_{12}\frac{2r_{12}+\rho\{2[(2\Delta)^{1/2}+(2\Delta+\lambda^2)^{1/2}]^{-1}-r_{22}\}}{\alpha_{22}+(2\Delta+\lambda^2)^{1/2}-r_{22}\lambda^2/2},$$

where $\zeta=-e_1e_2$ (for the $p\bar{p}$ atom, $\zeta=1$); $E=-\lambda^2/2$ is the level energy, measured from the threshold of the $p\bar{p}$ channel; a_{cs} is the nuclear–Coulomb $p\bar{p}$ scattering length;

$$\alpha_{ij}=\lim_{R\to0}\{C_{ii}(\eta)M_{ij}(E)C_{jj}(\eta)\}\quad(i,j=1,2), \qquad (2.17)$$
$$C(\eta)=\text{diag}\{[2\pi\eta/(1-e^{-2\pi\eta})]^{1/2},1\},$$

TABLE II. Values of the parameters for the OBEP potential.

S	r_e	r_C	r_C/r_e	A	b_1	c_1	$\tilde{r}_C$
0	1.43	0.60	0.417	1.62	0.86	1.35	0.79
	3.34	1.46	0.436	−1.72	1.06	1.45	1.55
1	1.23	0.54	0.440	1.47	0.77	1.65	0.64
	2.24	1.06	0.475	−1.44	0.89	1.61	1.16

Note. The first and second rows correspond to the levels 1s and 2s for the given S. The values of r_e, r_C, and $\tilde{r}$ are given in fermis; $S=0$ or 1 is the spin of the state.

where the matrix $M(E)$ is related to the matrix T by[5)] $T=(M-ip)^{-1}$, r_{ij} is the matrix of effective ranges, and $\eta=\zeta/k$ is the Coulomb parameter.

Let us elucidate the physical meaning of ρ, the channel coupling parameter. Outside the range of the nuclear forces, the wave function of the scattering problem with incident wave in the first ($p\bar{p}$) channel has the form

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_{(r>r_e)}=\begin{pmatrix}k^{-1}F_0(k,r)+T_{11}\exp(-\pi\eta/2+i\sigma_0)W_{i\eta,1/2}(-2ikr)\\ T_{21}e^{iqr}\end{pmatrix}, \quad (2.18)$$

where $F_0(k,r)$ is a regular Coulomb function; $W_{i\eta,1/2}$ is a Whittaker function; $\sigma_0=\arg\Gamma(1-i\eta)$, $k=(2E)^{1/2}$, $q=[2(E-\Delta)]^{1/2}$; the indices 1 and 2 correspond to the $p\bar{p}$ and $n\bar{n}$ channels. For bound states,

$$\det(M-ip)=0, \quad k=i\lambda, \quad q=i(2\Delta+\lambda^2)^{1/2},$$

and T_{11} and T_{21} become infinite. For $r<r_e$, the Coulomb interaction can be ignored, by virtue of which $\chi_1(r)$ and $\chi_2(r)$ have the same r dependence and differ from the function $\chi_0(r)$ introduced above [see (2.2) and (2.3)] only by a constant factor. From (2.18), we find that in the region $r\lesssim r_e$

$$\frac{\chi_2(r)}{\chi_1(r)}=\frac{C_0(\eta)\exp(iqr_e)T_{21}(E)}{T_{11}(E)}\Big|_{E=0}=-\exp[-(2\Delta)^{1/2}r_e]\rho \quad (2.19)$$

(we have here expressed T_{ij} in terms of the elements of the matrix M and taken into account the definition of α_{ij}). Thus, $\rho^2\exp[-2(2\Delta)^{1/2}r_e]$ gives the relative probability of finding the system in the state $n\bar{n}$ (within the range of the nuclear forces). It can be seen from this that if the second channel is eliminated for kinematic ($\Delta\to\infty$) or dynamical ($\alpha_{12}\to 0$) reasons, the system is in the pure $p\bar{p}$ state.

Except for the special case[6)] when $(\alpha_{22}+(2\Delta)^{1/2})\to 0$ and in (2.16) there is a pole associated with the $n\bar{n}$ channel, the function $R(\lambda^2)$ can be expanded in powers of λ^2. Further, bearing in mind that the Coulomb correction to r_e is small, and that the nondiagonal elements r_{ij} are much smaller than r_{ii} and r_{jj} (see Ref. 23), we set $r_{ij}\approx r_e\delta_{ij}$. Then

$$R(\lambda^2)\approx r_{cs}=(1+\rho^2)r_e-\rho^2(2\Delta)^{-1/2}. \quad (2.20)$$

For the $p\bar{p}$ atom, $(2\Delta)^{-1/2}=4.0$ F. Using the values of r_e in Table II, we see that with increasing ρ^2 (i.e., with increasing coupling of the channels $p\bar{p}$ and $n\bar{n}$) the effective range r_{cs} decreases, and for $\rho^2>r_e/[(2\Delta)^{-1/2}-r_e]\sim 1$ becomes negative. This is the main difference between the two- and single-channel problems.

When r_{cs} decreases, the binding energy of the quasinuclear level ε_{Qs} also decreases. However, for the physically reasonable values $\rho^2\sim 1$ it remains in the region of 1 MeV (see Fig. 1 in Ref. 24), which corresponds to the predictions of Ref. 5.

Physical consequences. The large shift of the 1s level of the $p\bar{p}$ atom indicates the existence of a level Qs of quasinuclear type[25] with low binding energy ($\varepsilon_{Qs}\sim 1$ MeV). The position of this level is fairly sensitive to the value of the effective range r and the channel coupling parameter ρ^2. In Fig. 1, we have plotted the binding energy of the level Qs as a function of the shift

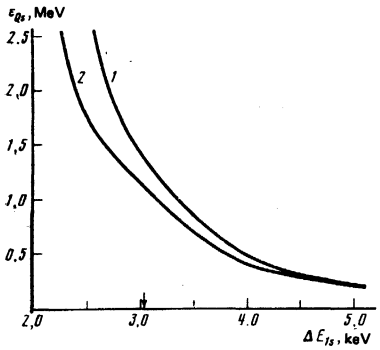

FIG. 1. Binding energy ϵ_{Q_s} of the quasinuclear meson as a function of the shift of the ground level of the $p\bar{p}$ atom. Curves *1* and *2* correspond to the values $r_{cs}=1.5$ F and 0. The arrow indicates the shift Δ_{1s} in accordance with Ref. 12.

of the atomic level 1s calculated using Eq. (2.16) in the approximation (2.20). Curves *1* and *2* correspond to $r_{cs}=1.5$ F and $r_{cs}=0$.

To estimate the magnitude of the corrections $\propto r_e$ in Eq. (2.1), we use this equation to calculate the scattering length a_s on the basis of the experimental level shift (1.2). We assume that the quasinuclear level is a 2s state in the OBEP potential with spin $S=0$. Then $a_s=6.75$ F for $b_1=c_1=c_2=0$ and $a_s=7.34$ F with allowance for all corrections $\propto r_e$. The inclusion in (2.1) of the terms containing b_1, c_1, and c_2 changes the calculated value of a_s by ≈10, 2, and 1%, respectively. For $S=1$, the situation is analogous.

Coulomb corrections to the scattering lengths. From (2.1), we obtain the formula

$$(1+2\zeta b_1r_e)/a_s=1/a_{cs}+2\zeta[\ln|\zeta|\,r_C+c_0+\zeta c_1r_e], \quad (2.21)$$

which gives the connection between the "purely nuclear" scattering length a_s and the nuclear–Coulomb length a_{cs}, which is measured directly in the experiments:

$$a_{cs}=-(1/2\pi)\lim_{k\to 0}\tan\delta_{cs}(k).$$

Here $\zeta=-e_1e_2$ ($\zeta=1$ for $p\bar{p}$), and $a_B=1$. Comparison of (2.21) with the exact calculations (numerical solution of the Schrödinger equation) for some model potentials V_s shows[17] that the error in extracting the value of a_s does not exceed 1.5%. On the other hand, if we assume $b_1=c_1=0$ in (2.21), then the accuracy of this expression deteriorates by an order of magnitude. Therefore, the corrections $\propto r_e$ play a fairly important part.

A generalization of the relation (2.21) to the case $l\geq 1$ was obtained earlier.[17] In this case, the Coulomb interaction renormalizes $a_l^{(cs)}$ by an amount of order $r_0/a_B\ll 1$ [a large logarithm of the type $\ln(r_C/a_B)$ is not present in the difference $(a_l^{(s)-1})-(a_l^{(cs)})^{-1}$].

§3. ATOMIC SPECTRUM WITH ALLOWANCE FOR ABSORPTION

Introducing appropriate notation, we write (2.1) in the form

$$2[\psi(1-\lambda^{-1})+\ln\lambda+\lambda/2]=1/a_{cs}+{}^1/_2r_{cs}\lambda^2. \quad (3.1)$$

This equation also remains valid in the case of a complex scattering length (which takes into account absorption phenomenologically, i.e., the presence in the problem of open channels). At the same time, the atomic levels not only undergo a shift but also acquire a width, i.e., the parameter $\nu=(-2E)^{-1/2}$ becomes complex.

We study the properties of the solutions of Eq. (3.1) in the complex plane of ν. We set

$$a_{cs}(\nu)=[2(\psi(1-\nu)-\ln\nu+1/2\nu)-r_{cs}/2\nu^2]^{-1}. \qquad (3.2)$$

Figures 2(a) and 2(b) show curves of constant values of $\operatorname{Re} a_{cs}(\nu)$ and $\operatorname{Im} a_{cs}(\nu)$ in the region $0.5<\operatorname{Re}\nu<2$, $-1<\operatorname{Im}\nu<0$. The calculations were made for $r_{cs}=1/30$ (the influence of the term $r_{cs}/2\nu^2$ is here small). Note that the curves in Fig. 2(b) have different behaviors depending on the value of $\operatorname{Im} a_{cs}$. The nature of the curves changes significantly at $\operatorname{Im} a_{cs}=-1/2\pi$. The curve corresponding to this value of $\operatorname{Im} a_{cs}$ goes to infinity and divides two types of curves.

To elucidate the question of the shifts of the Coulomb levels in the presence of absorption, it is necessary to specify the scattering length as a function of the depth of the real part of the potential V_s. For this, we use the model of a complex rectangular well (range r_0, depth V_0), when

$$a_s=r_0(1-z^{-1}\operatorname{tg} z),\quad z=r_0(2mV_0)^{1/2}=(U+iW)^{1/2}, \qquad (3.3)$$

where U and W are dimensionless variables. Assuming $W\ll U$, we have approximately

$$z=U^{1/2}+iWU^{-1/2}/2,$$
$$\operatorname{Re} a_s=r_0\left\{1+\frac{2(U-U_0)}{[(U-U_0)^2+\gamma^2/4]\operatorname{ch}^2\alpha}\right\} \qquad (3.4)$$
$$\operatorname{Im} a_s=-r_0\frac{\gamma}{(U-U_0)^2+\gamma^2/4},$$
$$\gamma=4U_0^{1/2}\operatorname{th}\alpha=\begin{cases}2W, & W\ll U^{1/2},\\ 4U_0^{1/2}, & W\gg U^{1/2},\end{cases} \qquad (3.5)$$

where $\alpha=W/2U^{1/2}$, $U_0=(n-\frac{1}{2})^2\pi^2$, $n=1,2,\ldots$. If $W=0$, then a_s becomes infinite at $U=U_0$. In the presence of absorption, the maximum and minimum of $\operatorname{Re} a_s$ are $r_0(1\pm 2\gamma^{-1})$ and are attained at $U=U_0\pm\gamma/2$, and $\operatorname{Im} a_s$ is described by the Breit–Wigner formula with width γ.

If $W=0$, then to the unshifted level ns there corresponds $\nu=n$. As U increases, this point moves to the left, remaining on the real axis the whole time. To the points $\nu=\tilde\nu_n$ there correspond poles of $a_{cs}(\nu)$ (see Table III). When ν passes through $\tilde\nu_n$, there is a rearrangement of the atomic spectrum.[11,6] In the case of weak absorption ($W\ll 1$, $\cosh\alpha\approx 1$) the trajectory of the level $\nu=\nu(U)$ is shifted into the complex plane, though its nature does not change qualitatively. This can be seen from the fact that at $U=U_0$ the condition $\operatorname{Re} a_s=r_0\ll 1$ holds, and $|\operatorname{Im} a_s|=r_0/W$ is not small. As can be seen from Fig. 2(b), the point ν is then near $\tilde\nu_L$. In the case of strong absorption ($W\gtrsim 1$), the picture is quite different, and with increasing U the level is first shifted to the left and then has a retrograde motion. At the same time, $\operatorname{Re}\nu$ fluctuates around the original value $\nu=n$ (see Fig. 2, in which the broken curves show the level trajectories in the two cases $W=0.1$ and $W=0.5$). The transition from the one regime to the other occurs at $W\approx 0.4$ (which happends to correspond to $\operatorname{Im} a_{cs}=1/2\pi$ for $\operatorname{Re} a_{cs}=0$). Such a value of $\operatorname{Im} a_{cs}$ corresponds to the curve that divides the two types of contour in Fig. 2(b). Thus, the nature of the trajectories $\nu=\nu(U)$ is determined by the value $\operatorname{Im} a_{cs}$ at $\operatorname{Re} a_{cs}=0$, and if $|\operatorname{Im} a_{cs}|>1/2\pi$ then the levels move downward monotonically and there is a rearrangement of the spectrum, but if $|\operatorname{Im} a_{cs}|<1/2\pi$ the motion of the levels is oscillatory in nature.

Note that an oscillatory variation of the shifts and widths of levels for hadronic atoms was discovered earlier by Erickson.[26] However, he did not note that for weak absorption the oscillatory regime is replaced by the regime of rearrangement of the spectrum [this is evidently due to the circumstance that in Ref. 26 the level shifts were studied in the framework of the perturbation theory (1.1), which does not enable one to study correctly the case of large scattering lengths]. The replacement of the rearrangement regime by the oscillatory regime when the magnitude of the absorption changes was considered qualitatively by Markushin.[27]

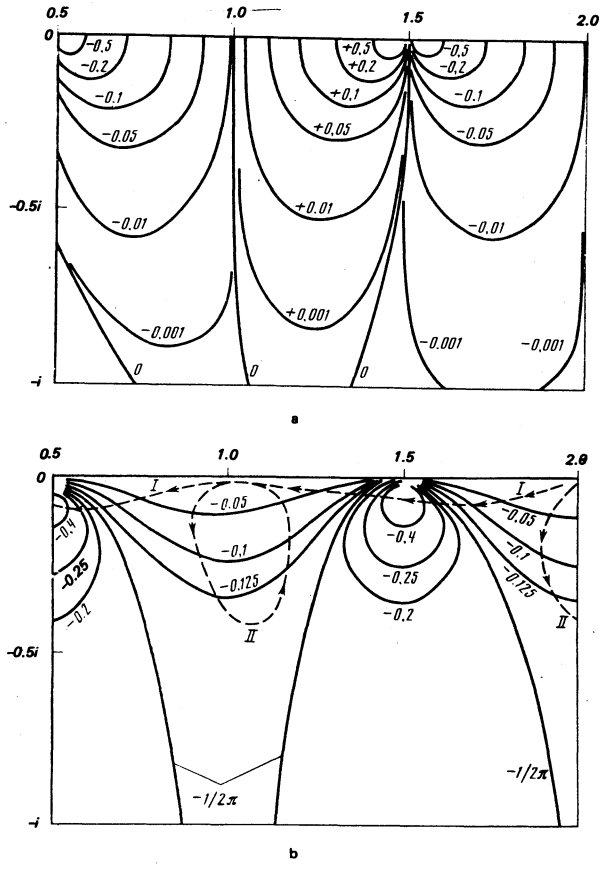

FIG. 2. Contours for the real (a) and imaginary (b) parts of the function $a_{cs}(\nu)$ on the complex plane of $\nu=\nu_1+i\nu_2$. The numbers next to the curves give the values of $\operatorname{Re} a_{cs}(\nu)$ and $\operatorname{Im} a_{cs}(\nu)$. The broken curves are the trajectories of the levels in the model of a complex rectangular well for $W=0.1$ (curve I) and $W=0.5$ (curve II). The arrows indicate the direction of motion of these levels with increasing U.

TABLE III. Roots of Eq. (3.1) for $a_{cs}=\infty$, $r_{cs}=0$.

n	$\tilde{\nu}_n$	n	$\tilde{\nu}_n$
–	0.4696	4	4.4996
1	1.4964	5	5.4997
2	2.4987	10	10.49992
3	3.4993	20	20.49998

We now turn to the case $n \gg 1$ (highly excited states). In this case, it is convenient to rewrite Eq. (3.1) in the form

$$\operatorname{ctg}\pi\nu - f(\nu) = \frac{1}{2\pi}\left(\frac{1}{a_{cs}} + \frac{r_{cs}}{2\nu^2} + \ldots\right),$$

$$f(\nu) = \frac{1}{\pi}\int_0^\infty e^{-2\nu x}\left(\operatorname{cth} x - \frac{1}{x}\right)dx, \tag{3.6}$$

the function $f(\nu)$ being numerically small in the region of the atomic spectrum. Therefore, with increasing principal quantum number n the differences $\sigma_n = \nu_{ns} - n$ rapidly reach the limiting value σ_∞, and the dependence of ν on n becomes periodic. This is illustrated by Table III, in which we give the results of a numerical calculation in accordance with Eq. (3.1) for $1/a_{cs} = r_{cs} = 0$. We obtain[7] analytically from (3.1) in the limit $n \to \infty$ the result $\tilde{\nu}_n = n + \frac{1}{2} - (12\pi^2 n^2)^{-1} + \ldots$,

$$\sigma_n = \sigma_\infty(1 - \mu/n^2 + \ldots), \tag{3.7}$$

$$\sigma_\infty = \frac{1}{\pi}\operatorname{arctg} 2\pi a_{cs},\quad \mu = \frac{\sin^2\pi\sigma_\infty}{12\pi^2\sigma_\infty}(1+3r_{cs}) \tag{3.8}$$

(we recall that in our case $a_B = 1$).

The periodicity property of ν_{ns} gives a convenient method for calculating the shifts and widths of excited levels. For this, determining $\nu_1 = (1 - 2\Delta E_1 + i\Gamma_1)^{-1/2}$ from the experimental data for the $1s$ level, we set

$$E_n - \frac{i}{2}\Gamma_n = -\frac{1}{2}(n+\sigma_\infty)^{-2},\ n \geqslant 2,$$
$$\sigma_\infty = \frac{1}{\pi}\operatorname{arc\,tg}\left[\frac{\operatorname{tg}\pi\nu_1}{(1 - f(\nu_1)\operatorname{tg}\pi\nu_1)}\right]. \tag{3.9}$$

Because of absorption due to annihilation of $\bar{p}$ and p at short [$r \sim r_a = (2m_N)^{-1}$, see Ref. 25] distances, the parameter σ_∞ becomes complex. The width Γ_n obtained from (3.9) is the annihilation width of the level ns.

We note finally that in the case $\Gamma_n \ll \Delta E_n$ formula (3.9) admits further simplification (see Appendix A).

§4. THE REGION OF APPLICABILITY OF PERTURBATION THEORY

It has already been noted[5] that perturbation theory (PT) for $\Delta E_{1s} \approx 3$ keV gives a scattering length a_s with error ~100%. Let us consider the accuracy of (1.1) in more detail.

Setting $\nu = n + \sigma_n$, we obtain in the limit $\sigma_n \to 0$

$$\psi(1-\nu) = \sigma^{-1} + \psi(n) + \ldots,\quad \Delta E_n = \frac{\sigma_n}{n^3}\left[1 - \frac{3}{2n}\sigma_n + \ldots\right]$$

and using Eq. (2.4) from the previous paper Ref. 6, we find

$$a_s = \frac{\sigma_n}{2(1 - A_n\sigma_n)},\quad A_n = \ln\frac{n}{r_c} - \psi(n) - \frac{1}{2n} - c_0. \tag{4.1}$$

Denoting $\Delta E_n^{PT} = 2a_s/n^3$, we have

$$\Delta E_n/\Delta E_n^{PT} = 1 - \alpha_1 a_s + \alpha_2 a_s^2 + \ldots, \tag{4.2}$$

and because of the large logarithm ($\ln(a_B/r_C) \gg 1$) the coefficient α_1 is anomalously large:

$$\alpha_1 = 2[\ln(n/r_c) - \psi(n) - 1/n - c_0],\quad c_0 = 1.848.$$

By virtue of this, the region of applicability of perturbation theory with respect to the scattering length is rather narrow:

$$|a_s| \ll a_B/[2\ln(a_B/r_c)], \tag{4.3}$$

which is also confirmed by the numerical calculation (Fig. 3). Already at a shift $\Delta E_{1s} = 0.6$ keV ($\nu_1 = 1.025$) the error of (1.1) for a_s reaches 10%.

The question of the accuracy of perturbation theory can also be approached from a different side. In accordance with (1.1), $n^3\Delta E_n$ must not depend on n. The curves in Fig. 4, obtained by means of Eq. (3.1), clearly demonstrate the extent to which perturbation theory breaks down.

§5. STATES WITH NONZERO ANGULAR MOMENTUM

The case $l \geqslant 1$ has a specific feature due to the circumstance that the behavior of the Coulomb wave functions in the limit $r \to 0$ is different for $l = 0$ and $l \geqslant 1$.

Equation for the level spectrum. The energy dependence of the phase shifts for small k is determined by the equation[3,10]

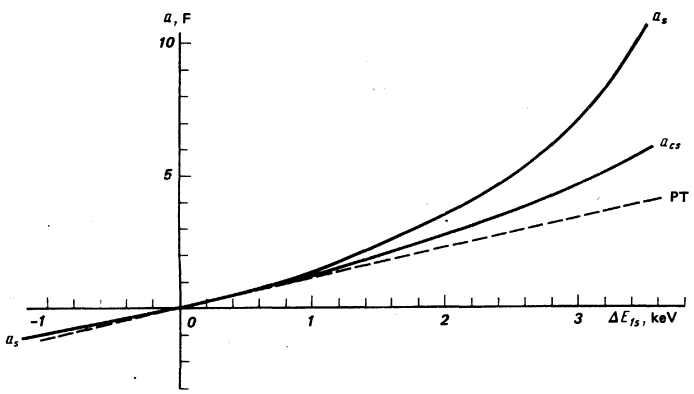

FIG. 3. Dependence of the scattering lengths a_s and a_{cs} on the shift of the ground level of the $p\bar{p}$ atom. The broken straight line corresponds to perturbation theory [formula (1.1)].

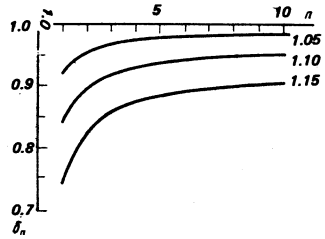

FIG. 4. The values of $\delta_n = n^3 \Delta E_n / 2a_{cs}$ for ns levels. The numbers next to the curves are the values of ν_1 (the corresponding shifts of the 1s level are $\Delta E_{1s} = 1.16$, 2.17, and 3.05 keV).

$$\prod_{s=1}^{l}\left(1+\frac{\eta^2}{s^2}\right) k^{2l+1}[C_0^2(\eta)\operatorname{ctg}\delta_l^{cs} - 2\eta h(\eta)] = -\frac{1}{a_l^{cs}} + \frac{1}{2} r_l^{cs} k^2 + \ldots,$$
$$\eta = \zeta/k, \quad C_0(\eta) = [2\pi\eta/(1-e^{-2\pi\eta})]^{1/2}, \qquad (5.1)$$
$$h(\eta) = \operatorname{Re}\psi(i\eta) - \ln\eta = \psi(1-i\eta) - \ln\eta + {}^1/_2 i(\pi\operatorname{ctg}\pi\eta - \eta^{-1});$$

a_l^{cs} and a_l^s have the dimensions of (length)$^{2l+1}$. The poles of the scattering amplitude are determined by the condition $\cot\delta_l^{cs}(k) = i$. To go over from (5.1) to the discrete spectrum, it is necessary to make the substitution $k = i\lambda$, $\eta = -i\zeta/\lambda$, $\cot\delta_l^{cs} \to i$, which gives

$$\Phi_l(\lambda)[\psi(1-\lambda^{-1}) + \ln\lambda + \lambda/2] = 1/a_l^{cs} + {}^1/_2 r_l^{cs}\lambda^2 + \ldots,$$
$$\Phi_l(\lambda) = \frac{2}{(l!)^2}\prod_{s=1}^{l}(1-s^2\lambda^2) = \begin{cases} \dfrac{2}{(l!)^2}\left[1 - \dfrac{1(l+{}^1/_2)(l+1)}{3\nu^2} + \ldots\right], & \nu\to\infty \\ \dfrac{2^{l+1}(2l+1)!!}{l!(l+1)^{2l+1}}, & \nu = l+1 \end{cases}$$
$$(\nu = \lambda^{-1},\ \zeta = 1; \quad \text{for} \quad l=0 \quad \Phi_0(\lambda) = 2). \qquad (5.2)$$

Since the function $\Phi_l(\lambda)$ is positive for $\nu > l$ and increases monotonically with increasing ν, the graph of the left-hand side of Eq. (5.2) in the interval $l < \nu < \infty$ has qualitatively the same form as in the case $l = 0$ (see Fig. 1 in Ref. 5). Therefore, the conclusions drawn earlier for the s states[5,6] about the dependence of the level shifts on the scattering length remain valid. In particular, for $|a_l^{cs}| \ll a_B$, the function $\psi(1-\lambda^{-1})$ in (5.2) is near a pole: $\nu = n + \sigma_{nl}$, $|\sigma_{nl}| \ll 1$, whence

$$a_l^{cs} = \frac{(l!)^2(n-l-1)!\,n^{2l+1}}{2[(n+l)!]}\frac{\sigma_{nl}}{1-A_n'\sigma_{nl}}, \qquad (5.3)$$
$$A_n' = \ln n - \psi(n) - 1/2n.$$

Numerically, the coefficient A_n' is small: $A_1' = 0.0772$, $A_n' \approx (12n^2)^{-1}$ for $n \gg 1$. Therefore, (5.3) goes over into the formula of perturbation theory with respect to the scattering length[3,4]:

$$\Delta E_{nl} = \frac{2(n+l)!}{(l!)^2(n-l-1)!\,n^{2l+4}} a_l^{cs}. \qquad (5.4)$$

Collapse of bound state with $l \neq 0$. The solution of the Schrödinger equation with Coulomb potential decreasing at infinity can be expressed in terms of a Whittaker function:

$$\varphi_l(r, \lambda) \propto W_{\nu, l+1/2}(2\lambda r).$$

In particular,

$$W_{l+1, l+1/2}(x) = e^{-x/2}x^{l+1}, \quad W_{l, l+1/2}(x) = e^{-x/2}\sum_{m=-l}^{l}\frac{(2l)!}{(l+m)!}x^m. \qquad (5.5)$$

When $\nu = l+1$, the function $\varphi_l(r, \lambda)$ is regular at the origin. As $\nu \to l$, a difference arises between the cases $l = 0$ and $l \geq 1$. For $l = 0$, $\varphi_l(r, \lambda)$ preserves a physical meaning and remains normalized, going over in the limit $\nu \to 0$ into the wave function for a δ potential:

$$\varphi_0(r) \approx (2\lambda)^{1/2} e^{-\lambda r}.$$

But if $l \neq 0$, then $\varphi_l(r, \lambda)$ has an inadmissible singularity $\sim r^{-l}$ at the origin. This means that the idealization of a point charge ($V_C(r) = -\zeta/r$, $0 < r < \infty$) is here invalid, and when a cutoff of the Coulomb potential V_C in the region $0 < r < r_0$ is introduced the system "sits" at distances $\sim r_0$. Thus, in the interval $l < \nu < l+1$ there is a collapse of the bound state, and its mean radius $\langle r\rangle$ decreases from values of order a_B to $\langle r\rangle \sim r_e$. The calculations shows that this occurs in a fairly narrow range of energies near $\nu = l+1$:

$$l+1-\nu_{cr} \sim \begin{cases} (r_0/a_B)^{l-1}, & l \geq 2 \\ [\ln(a_B/r_0)]^{-1}, & l = 1 \end{cases} \qquad (5.6)$$

(see Appendix B).

The inescapability of collapse at $\nu = \nu_{cr}$ near $l+1$ also follows from examination of the curves of the effective potential energy

$$U(r) = -\frac{1}{r} + \frac{l(l+1)}{2r^2}.$$

For $l = 0$, the level sinks, remaining in the Coulomb field until the mean radius $\langle r\rangle$ of the state is comparable with r_e. Using the expressions of §5 in Ref. 6, we obtain

$$\langle r\rangle = \begin{cases} \dfrac{3}{2}\nu^2 + \dfrac{\sin^2\pi\nu}{4\pi^2\nu}(1+3r_e) + \ldots, & \nu \gg 1 \\ \dfrac{1}{2}\nu + \dfrac{1}{2}r_e + \ldots, & \nu \ll 1 \end{cases} \qquad (5.7)$$

For $\nu = 1, 2, 3, \ldots$, Eq. (5.7) goes over into the well-known[9] expression $\langle r\rangle = 1.5\nu^2$ for s levels in the field of a point charge. For nonintegral ν there is a correction $\propto \sin^2\pi\nu$, which, however, rapidly vanishes with increasing ν and for $\nu > 0.8$ does not exceed 1%. The condition $\langle r\rangle \gg r$ is satisfied to $\nu \sim r_e \ll 1$, i.e., to $\lambda r_e \sim 1$.

On the other hand, in the case $l \geq 1$ a level in a Coulomb potential and having radius $\langle r\rangle \geq 1$ cannot have energy less than $U_{min} = -1/2l(l+1)$. In fact (with allowance for the zero-point energy) the minimal energy is $-1/2(l+1)^2$, i.e., $\nu_{min} = l+1$. For $\nu < l+1$, the level can no longer remain in the Coulomb well and collapses into the region of the nuclear forces. This explains why ν_{cr} is close to $l+1$.

Note that Eq. (5.2), as in the case $l = 0$, remains valid to $\lambda r_e \sim 1$. Using it, one can calculate the position of the quasinuclear level Ql with angular momentum l from the known shifts of the atomic nl levels if the binding energy satisfies $\varepsilon_{Ql} \ll 1/2r_0^2 \sim 10$ MeV (for the $p\bar{p}$ atom).

The collapse of states with orbital angular momentum $l \neq 0$ when the potential becomes deeper in the region of small r was recently discovered[29] in numerical calculations of the 4f shell of rare earth elements made in the Fock–Dirac approximation. The rearrangement of the atomic spectrum in a narrow range $\Delta Z \sim 0.2$ was also

demonstrated (see Fig. 2 in Ref. 29). These numerical calculations are in qualitative agreement with the above analytical theory.

We thank B.O. Kerbikov, L.A. Sliv, Yu.A. Simonov, and I.S. Shapiro for discussing the results of this work, and also I.M. Band and V.I. Fomichev for information about Ref. 29.

APPENDIX A

In the case $\Gamma_n \ll \Delta E_n$ (weak absorption), Eq. (3.1) can be solved as follows. First, ignoring the width Γ_n, we find real roots $\nu = \nu_n$ from the shift of the ground level. After this, the annihilation widths of the s levels are calculated in accordance with the formula

$$\Gamma_{ns} = \frac{g(\nu_i)}{g(\nu_n)} \Gamma_{1s}, \tag{A.1}$$

$$g(\nu) = \nu^3 \left[\psi'(1-\nu) + \frac{2\nu+1}{2\nu^2}\right] = \begin{cases} \nu^2 + \frac{1}{2}\left(1+\frac{\pi^2}{3}\right)\nu^3 + \ldots, & \nu \to 0 \\ \nu\left(\frac{\pi\nu}{\sin \pi\nu}\right)^2 - \frac{1}{6} + \ldots, & \nu \to \infty \end{cases}. \tag{A.2}$$

The widths Γ_{ns} decrease $\propto 1/n^3$:

$$\Gamma_{ns} \approx \frac{\sin^2 \pi\sigma_\infty}{\pi^2 n^3} g(\nu_i)\Gamma_{1s}, \quad n \gg 1. \tag{A.3}$$

It is also easy to explain the increase in the width on the transition from the $1s$ level to the quasinuclear Qs level. Indeed, for $\nu_{1s} \sim 1$ and $\nu_{Qs} \ll 1$, we obtain

$$\frac{\Gamma_{Qs}}{\Gamma_{1s}} \sim \left(\frac{\pi}{\nu_{Qs} \sin \pi\nu_{1s}}\right)^2.$$

For $\nu_{Qs} \approx 0.1$ and $\nu_{1s} = 1.15$, this simple estimate gives $\Gamma_{Qs}/\Gamma_{1s} \sim 10^3$, which agrees qualitatively with the numerical calculation of Ref. 5 for the $p\bar{p}$ atom.

APPENDIX B

We calculate the mean radius of the state with angular momentum l for $\nu \approx l+1$. The wave function is

$$\varphi_l(r, \lambda) = \text{const} \cdot \begin{cases} \varphi_l(r), & 0 < r < r_0 \\ W_{\nu, l+1/2}(2\lambda r), & r > r_0 \end{cases} \tag{B.1}$$

[we assume for simplicity that $V_s(r) \equiv 0$ for $r > r_0$], where $W_{\nu, l+1/2}$ is a Whittaker function[30]:

$$W_{\nu,\mu}(x) = \frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\nu)} M_{\nu,-\mu}(x) + \frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\nu)} M_{\nu,\mu}(x)$$

(in the limit $r \to 0$, the first term is $\propto r^{-l}$, and the second $\propto r^{l+1}$). For $\nu = l+1$, the regular solution in the Coulomb field is

$$M_{\nu, l+1/2}(2\lambda r) = e^{-\lambda r}(2\lambda r)^{l+1},$$

and

$$\Gamma(-2\mu)/\Gamma(1/2-\mu-\nu) = \Gamma(l+1+\nu)/\Gamma(2l+2) \approx 1.$$

Hence

$$W_{\nu, l+1/2}(2\lambda r) \approx \left\{\frac{\Gamma(2l+1)}{\Gamma(l+1-\nu)}(2\lambda r)^{-l} + (2\lambda r)^{l+1}\right\} e^{-\lambda r}$$

($0 < l+1-\nu \ll 1$). Using (5.3), we express $1/\Gamma(l+1-\nu) \approx l+1-\nu$ in terms of the scattering length, and we finally obtain

$$W_{\nu, l+1/2}(2\lambda r) \approx (2\lambda)^{l+1} e^{-\lambda r}[r^{l+1} - \eta_l r_0^{2l+1} r^{-l}], \tag{B.2}$$

$$\eta_l = \frac{\xi - (l+1)}{\xi + l}, \quad a_l = \frac{2^{2l}(l!)^2}{(2l)!(2l+1)!}\eta_l r_0^{2l+1} \tag{B.3}$$

[see Eq. (3) in Ref. 5].

Thus, for $\eta_l \neq 0$ the solution which is regular at the origin has the singular solution mixed to it with a coefficient proportional to the level shift. Using the approximation (B.2) for the wave function and making some elementary calculations, we obtain ($l \geq 2$)

$$\langle r \rangle \approx (l+1)\left(l+\frac{3}{2}\right)\frac{1+\alpha_l(1+A_l)\eta_l^2 r_0^{2l+4}}{1+\beta_l(1+B_l)\eta_l^2 r_0^{2l+3}}, \tag{B.4}$$

$$\alpha_l = \frac{1}{(l^2-1)(2l+3)!}\left(\frac{2}{l+1}\right)^{2l+3}, \quad \beta_l = \frac{1}{(2l-1)(2l+2)!}\left(\frac{2}{l+1}\right)^{2l+3},$$

in which A_l and B_l also depend on the wave function in the region $0 < r < r_0$:

$$A_l = \frac{2(l-1)}{r_0^2}\int_0^{r_0} \varphi_l^2(r) r dr, \quad B_l = \frac{2l-1}{r_0}\int_0^{r_0} \varphi_l^2(r)\, dr.$$

Here, we assume $\eta_l \gg 1$ and we have adopted the normalization condition $\varphi_l(r_0) = 1$. Therefore, A_l and B_l are of the order unity.

As long as $\eta_l \ll r_0^{-(l+3/2)}$, the expression (B.4) agrees with the usual formula (Ref. 9) $\langle r \rangle = (l+1)(l+\frac{3}{2})$ for a level with $n = l+1$ (hydrogenlike atom with point nucleus). For $\eta_l \sim r_0^{-(l+3/2)}$, the radius $\langle r \rangle$ begins to decrease with increasing η_l. Finally, at $\eta_l \sim r_0^{-(l+2)}$ the value of $\langle r \rangle$ ceases to depend on η_l and becomes of the same order as the range of the nuclear forces:

$$\langle r \rangle \approx r_0 \frac{(2l-1)(A_l+1)}{(2l-2)(B_l+1)} \tag{B.5}$$

(the scattering length $a_l \sim r_0^{l-1}$ is here still small compared with a_B). Thus, the collapse of the bound state occurs before the rearrangement region (where $a_l \sim 1$) is entered.

Above, we assumed $l > 1$. The case of the p levels is singular and requires separate treatment. This is due to the circumstance that at the time when the level arises $\varphi_l(r) \sim r^{-l}$ for $r > r_0$, and therefore $\langle r \rangle$ becomes infinite for $l = 1$. But if $l \geq 2$, then $\langle r \rangle$ remains finite in the limit $\lambda \to 0$, which makes it possible to obtain expressions of the type (B.4) and (B.5).

[1)] We use atomic units: $\hbar = m = e = 1$, where m is the reduced mass of the system; the unit of length is the Bohr radius a_B. Thus, for the $p\bar{p}$ atom $a_B = 57.6$ F, and the binding energy of the unshifted Coulomb levels is $E_{nl}^C = 12.5/n^2$ keV. For the $\Sigma^- p$ atom, $a_B = 51.4$ F, $E_{nl}^C = 14.0/n^2$ keV, etc.

[2)] In this connection, we note that the authors of Ref. 12 themselves regard the result (1.2) for the shift of the $1s$ level of the proton–antiproton atom as preliminary.

[3)] Such a value of the s-wave scattering length in one of the spin states ($S = 0, 1$) follows from the experimental[12] shift of the atomic $1s$ level (see Fig. 2 in Ref. 6).

[4)] This variant of the cutoff of $V(r)$ for $r < r_0$ takes into account qualitatively the presence of a hard core in the nuclear forces and was already used in the calculations of Ref. 21. To calculate r_e, r_C, etc., we solved the single-channel Schrödinger equation with the potential $(V_0 + V_I)/2$, where $V_I(r)$ is the potential (2.15) for states with isotopic spin I.

[5)]In the single-channel case, $T=(e^{2i\delta}-1)/2ik$ coincides with the scattering amplitude, and $M=k\cot\delta$.

[6)]This case can occur only in a narrow range $a_s=-1/\alpha_{22}=3.5-4.0$ F of $n\bar{n}$ scattering lengths. In the remaining cases, the position of the Qs level is stable with respect to the opening of the second channel (for details, see Ref. 24, in which the correspondence with the results of Ref. 13 is also discussed).

[7)]The parameter σ_n differs only in sign from the quantum defect used in atomic physics.[10,28]

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