The mass shift of an accelerated charge

V. I. Ritus

P. N. Lebedev Physics Institute of the USSR Academy of Sciences, Moscow
(Submitted 20 September 1980)
Zh. Eksp. Teor. Fiz. **80**, 1288–1306 (April 1981)

The mass shift of an accelerated electron and of a scalar (field) charge (source) is investigated in the classical limit. The shift is produced by the inertia of the self-field of the particle, is analogous to reactive energy (in a circuit), and is accompanied by a logarithmic singularity in the emission probability and in the spectrum of radiation. The mass shift appears as a result of sufficiently long acceleration of the charge by a constant field which does work on the charge as a consequence of the 4-current structure of the interaction (the vector character of the self-field). For an accelerated scalar charge there is no mass shift, but the inertia of the scalar field as well as that of the electromagnetic field manifests itself in the aperiodic oscillation of the self-field mass, i. e., as a variable shift which is an odd function of the longitudinal speed of the charge. As the speed v tends to c, for the electron this function has the limit $-\alpha\hbar w_0/2c^3$, determined by the acceleration w_0, whereas for a scalar charge the limit is zero. The sign of the mass change as the particle traverses the turning point is related to the causality of the theory. The excitation spectra of the variable mass shifts of the electromagnetic and scalar charges coincide respectively with Bose and Fermi distributions corresponding to a temperature $kT=2\hbar w_0/\pi c$. It is shown that the limiting mass shift of an electron for arbitrary motion in an electric field does not depend on the transverse momentum, whereas the emission probability depends on it, and for $p_\perp \gg mc$ tends to the emission probability in a crossed field, retaining however its logarithmic singularity.

PACS numbers: 03.50. – z

1. INTRODUCTION

In a previous paper[1] the shift of the electron mass in a constant electromagnetic field was determined, a shift due to the change of the electron radiative self-interaction caused by the external field. This shift depends on the strengths of the electric and magnetic fields and on the quantum numbers which determine the state of the electron in the field. In the special case of a purely electric field ε and a state of the electron characterized by vanishing transverse momentum $p_\perp=0$ (so-called hyperbolic motion), the mass shift depends only on the dimensionless constant $\beta=e\varepsilon/m^2$ and for $\beta\ll 1$ is determined by the expansion[1)]

$$\Delta m=\frac{\alpha}{2\pi}m\left\{-\beta\pi+\beta^2\left(\frac{4}{3}\ln\frac{\gamma}{2\beta}+\frac{4}{9}\right)+\dots -i\left[\beta\left(2\ln\frac{2\beta m}{\gamma\mu}-1\right)-\beta^2\frac{2\pi}{3}+\dots\right]\right\}. \quad (1)$$

Here μ is the photon mass introduced to remove the infrared divergence caused by the infinite character of the motion, and $\gamma=1.781\dots$. The first terms of this expansion for the real and imaginary parts of the mass shift turn out to be purely classical—they do not depend on $\hbar$, if one takes into account the fact that in the classical theory one should use in place of the photon mass the smallest wave number $k_{\perp\min}=\mu c/\hbar$.

Thus, according to the classical term, $\mathrm{Re}\,\Delta m$ decreases linearly with the increase of the electric field strength, and for $\beta=2/\alpha$, i.e., $\varepsilon=2m^2c^4/e^3$ the electron mass should vanish. Thus the inapplicability of classical electrodynamics manifests itself visibly for fields of the order of m^2c^4/e^3 (the field at "the edge of the electron"), cf. Ref. 2, §75. However, as can be seen from Eq. (1), quantum effects become important already for $\beta\sim 1$, and these effects lead to a cessation of the decrease of the mass, so that in the region $\beta\sim 1$ the quantity $\mathrm{Re}\,\Delta m$ attains a minimum of the order of $-\alpha m$, and then the mass shift increases and for $\beta\gg 1$ is determined by the expression

$$\Delta m=\frac{\alpha}{2\pi}m\left\{\frac{1}{4}\left(\ln\frac{2\beta}{\gamma}\right)^2+\dots-i\left[2\beta\ln\frac{m}{\mu}+\frac{\pi}{4}\ln\frac{2\beta}{\gamma}+\dots\right]\right\}. \quad (2)$$

Although in principle an investigation of quantum electrodynamics in the region near the critical field $\varepsilon\sim m^2/e$ is of interest, the expression obtained for the mass shift exhibits quite intriguing properties of the electron even in relatively weak fields $\varepsilon\ll m^2/e$.

In this paper two methods are used to investigate the classical part of the mass shift. The first, "integral," method starts from the expression (3) for the change of the self-interaction of the electron under acceleration, an expression that determines the real and imaginary parts of the shift in the final state. This expression allows one to indicate those accelerated motions of the charge which lead to the real part of the mass shift and the accompanying logarithmic singularity in the imaginary part of the shift, and consequently to an infrared singularity $d\omega/\omega$ of the emission probability spectrum. These motions are characterized by a definite relation between the natural parameters of the 4-trajectory (its curvature and the two torsions), and by means of external forces or fields acting on the charge they are characterized by a sufficiently lengthy action on the charge of the constant electric field which does the work.

In order that a mass shift should appear it is essential that the spin of the self-field should equal to one. For a scalar charge, i. e., the source of a scalar field, no accelerated motion leads to a mass shift or to a singularity in the emission spectrum.

We have determined the mass shift of the electron in a motion in an electric field with arbitrary transverse momentum. Its real part does not depend on $p_\perp$, and

 0038-5646/81/040659-10$02.40

twice the imaginary part for $p_\perp \gg m$ tends to the speed of emission of a charge in a crossed field.

In the more sensitive "differential" method the mass shift is determined by the change of the Lagrange function of the self-field of the charge under acceleration, i. e., by the state of the self-field at the instant of time under consideration. The mass shift turns out to be an odd function of the velocity of the charge, changing sign at the turning point and tending to the limit determined in the "integral" method as $v \to 1$. A variable mass shift is also exhibited by the scalar charge, but its limit as $v \to 1$ vanishes.

The expression of the variable mass shift in terms of the Lagrange function of the field makes it possible to interpret it as a reactive energy of aperiodic oscillations of the self-field as the charge is accelerated.

The spectral functions of the variable mass shifts of electromagnetic and scalar charges coincide respectively with the one-dimensional Bose and Fermi distributions corresponding to an effective temperature $T = 2w_0/\pi$ determined by the acceleration w_0. The enhancement of low-frequency mass excitations of the electromagnetic field is due to the current-vector structure of the electromagnetic interaction. The effective temperature coincides with the one obtained by interpreting pair production by the electric field as a thermal excitation. This coincidence is produced by a common cause: both the mass shift and the pair production occur only if the constant electric field acts on the charge for a sufficiently long time, doing work on it which exceeds $2m$.

2. THE CHANGE OF THE SELF-INTERACTION OF THE ELECTRON UNDER ACCELERATION

The classical part of the mass shift follows directly from the expression for the change of the self-interaction of the electron in the external field[1]:

$$\Delta W_{\rm cl} = -\Delta m_{\rm cl}\tau = \frac{i}{2}\int d^4x d^4x' j_\alpha(x) j_\alpha(x') \Delta^c(x-x',\mu)\Big|_0^T$$
$$= \frac{1}{2} ie^2 \int d\tau\, d\tau' \dot{x}_\alpha(\tau)\dot{x}_\alpha(\tau') \Delta^c(x(\tau)-x(\tau'),\mu)\Big|_0^T. \quad (3)$$

It is thus determined by the square of the interval $(x-x')^2$ between the emission and absorption of the photon, and the relative Lorentz factor $\dot{x}_\alpha \dot{x}_\alpha'$. For any 4-trajectory $x_\alpha(\tau)$ these quantities are subject to the relation

$$\dot{x}_\alpha(\tau)\dot{x}_\alpha(\tau') = -\frac{1}{2}\frac{\partial^2 (x-x')^2}{\partial\tau\partial\tau'}. \quad (4)$$

We shall consider 4-trajectories for which the distance between any two points is a function only of the length of the path between them, i. e., such that $(x(\tau) - x(\tau'))^2 = 2f(\tau-\tau')$, and consequently $\dot{x}_\alpha \dot{x}_\alpha' = f''(\tau-\tau')$. Such 4-trajectories have constant curvature (acceleration) $a = (\ddot{x}_\alpha^2)^{1/2}$, since $\ddot{x}_\alpha^2 = -f^{(4)}(0)$. As a result of this the mass shift will be a functional of f. All other local invariants of the trajectory,[3,4] such as the first and second torsion,[2)] are also constant, and are determined by the derivatives of even order of the function $f(u)$ at the origin.

For a charge in hyperbolic motion the squared interval $(x-x')^2$ and the relative Lorentz factor $\dot{x}_\alpha \dot{x}_\alpha'$ have the expressions

$$(x-x')^2 = -2w_0^{-2}[\operatorname{ch} w_0(\tau-\tau')-1], \quad \dot{x}_\alpha \dot{x}_\alpha' = -\operatorname{ch} w_0(\tau-\tau'). \quad (5)$$

Therefore, if one uses in the expression (3) for the self-interaction the propagator

$$D^c = (2\pi)^{-2}[(x-x')^2 + i\delta]^{-1},$$

corresponding to $\mu = 0$, then the mass shift

$$\Delta m_{\rm cl} = -i\frac{\alpha w_0}{\pi}\int_0^\infty du \left[\frac{\operatorname{ch} u}{2(\operatorname{ch} u - 1)} - \frac{1}{u^2}\right] \quad (6)$$

will have an infrared divergence, since the relative Lorentz factor increases as $u \equiv w_0(\tau-\tau') \to \infty$ just as rapidly as the squared interval.

In order to remove the infrared divergence we use a photon propagator with $\mu \neq 0$. We then obtain for the mass shift[3)] (cf. Ref. 1)

$$\Delta m_{\rm cl} = \frac{\alpha w_0}{2\pi}\int_0^\infty dz e^{-i\lambda/2z}\left[e^{iz}K_1(iz) - \left(\frac{\pi}{2iz}\right)^{1/2}\right]$$
$$= \frac{\alpha w_0}{2\pi}\left[-\pi - i\left(\ln\frac{4}{\gamma^2\lambda} - 1\right) + \ldots\right]; \quad (7)$$

the ... denote terms which vanish for $\lambda = (\mu/w_0)^2 \to 0$. In addition to a finite $\operatorname{Im}\Delta m_{\rm cl}$ which increases logarithmically for $\lambda \to 0$ there appeared a term $\operatorname{Re}\Delta m_{\rm cl} = -\alpha w_0/2$ which does not depend on $\lambda \ll 1$. The integration interval $\lambda \lesssim z \lesssim 1$ is important for the formation of both parts of $\Delta m_{\rm cl}$, hence

$$w_0^{-2} \lesssim -(x-x')^2 \lesssim \mu^{-2} \quad \text{or} \quad 1 \lesssim w_0(\tau-\tau') \lesssim 2\ln(w_0/\mu),$$

i. e., the proper time of formation of these terms is of the order of, or much larger than, the reciprocal of the acceleration—exhibiting the characteristic infrared extension of the formation region.

The spectrum of the total probability of classical emission

$$dw_{\mathbf{k}} = \frac{d\mathscr{E}_{\mathbf{k}}}{\omega} = |j_\alpha(k)|^2 \frac{d^3k}{16\pi^3\omega}, \quad j_\alpha(k) = e\int_{-\infty}^{\infty} d\tau \dot{x}_\alpha(\tau) e^{-ikx(\tau)}, \quad (8)$$

obtained for a uniformly accelerated (UA) electron by Nikishov and the author,[5]

$$dw_{\mathbf{k}} = \frac{\alpha}{\pi^2 a^2} K_1^2\left(\frac{k_\perp}{a}\right)\frac{d^3k}{\omega}, \quad (9)$$

exhibits the characteristic infrared singularity $d^3k/k_\perp^2\omega$. Integrating the spectrum with respect to $k_\parallel$, account being taken of the relation $k_\parallel = k_\perp \sinh w_0$ between the longitudinal component $k_\parallel$ of the wave vector and the proper time τ of emission on the electron trajectory,[4)] yields, upon dividing by the total emission time $\Delta\tau$, the spectrum of the emission probability per unit proper time:

$$d\dot{w}_{k_\perp\varphi} = \frac{\alpha}{\pi^2 a} K_1^2\left(\frac{k_\perp}{a}\right) k_\perp dk_\perp d\varphi. \quad (10)$$

This spectrum maintains its infrared singularity $dk_\perp/k_\perp$. Further integration with respect to $k_\perp$, φ leads to a probability of emission per unit proper time

$$\dot{w} = \frac{\alpha a}{\pi}\left(2\ln\frac{2a}{\gamma k_{\perp min}} - 1\right) \quad (11)$$

in agreement with the general formula $\dot{w} = -2\operatorname{Im}\Delta m$, see Eq. (1).

3. THE CONDITION FOR THE APPEARANCE OF A MASS SHIFT AND OF AN INFRARED SINGULARITY IN THE EMISSION SPECTRUM

The infrared behavior of the emission probability and of the spectrum of a UA charge is unique to this kind of motion. Usually an infrared singularity appears in a spectrum of total probability of emission when the acceleration occurs on a finite portion of the 4-trajectory of the charge, and outside this portion the motion is considered to be free (see Ref. 6, §9, and Ref. 7, §9). If the accelerated motion of the charge takes place on an unbounded interval of the 4-trajectory, the spectrum of the total probability may or may not have an infrared singularity, although the probability itself increases without bound on account of the proportionality between the proper time and the acceleration, and in this case it makes sense to talk about the probability of emission per unit proper time. We consider two examples.

For a charge moving uniformly on a circumference in a magnetic field with a 4-acceleration of constant magnitude

$$a=e\eta p_\perp m^{-2}=((eFp)^2)^{1/2}m^{-2}$$

we have

$$(x-x')^2=-\frac{m^2+p_\perp^2}{m^2}(\tau-\tau')^2+4\left(\frac{p_\perp}{e\eta}\right)^2\sin^2\frac{e\eta}{2m}(\tau-\tau'),$$
$$\dot{x}_\alpha\dot{x}_\alpha'=-1-2\frac{p_\perp^2}{m^2}\sin^2\frac{e\eta}{2m}(\tau-\tau'). \tag{12}$$

The interval $(x-x')^2$ increases sufficiently rapidly compared to $\dot{x}_\alpha\dot{x}_\alpha'$, there is no infrared singularity, and Eq. (3) yields

$$\Delta m_{\rm cl}=-i\frac{\alpha a}{2\pi}\frac{m}{p_\perp}\int_0^\infty du\left[\frac{1+2(p_\perp/m)^2\sin^2u}{u^2+(p_\perp/m)^2(u^2-\sin^2u)}-\frac{1}{u^2}\right], \tag{13}$$

i.e., $\mathrm{Re}\,\Delta m_{\rm cl}=0$, and the rate of emission $\dot{w}=-2\,\mathrm{Im}\,\Delta m_{\rm cl}$ is a function of $p_\perp/m$ and equals $2\alpha a p_\perp/3m$ for $p_\perp\ll m$ and $5\alpha a/2\sqrt{3}$ for $p_\perp\gg m$.

The spectrum of total probability has the form $dw_{\mathbf{k}}=\Delta\tau d\dot{w}_{\mathbf{k}}$; the spectrum of the rate of emission is discrete:

$$d\dot{w}_{\mathbf{k}_s}=\frac{\alpha}{2\pi}\gamma\omega_s[\mathrm{ctg}^2\theta J_s^2(z)+v^2J_s'^2(z)]d\Omega,$$
$$\omega=\omega_s=se\eta/m\gamma,\quad z=k_\perp p_\perp/e\eta=sv\sin\theta,\quad s=1,2,\ldots, \tag{14}$$

does not contain infrared singularities and is related to intensity spectrum of radiation[2] of Schott by means of

$$dI_{\mathbf{k}_s}=(1-v^2)^{1/2}\omega_s d\dot{w}_{\mathbf{k}_s}.$$

For a charge executing infinite motion in a crossed field with constant magnitude of the 4-acceleration

$$a=eFp_-m^{-2}=((eFp)^2)^{1/2}m^{-2},$$

we have

$$(x-x')^2=-(\tau-\tau')^2-a^2(\tau-\tau')^4/12,\quad \dot{x}_\alpha\dot{x}_\alpha'=-1-a^2(\tau-\tau')^2/2. \tag{15}$$

In spite of the strong growth of the Lorentz factor, there is no infrared divergence, since the interval grows even faster. In this case

$$\Delta m_{\rm cl}=-i\cdot 5\alpha a/4\sqrt{3}. \tag{16}$$

We have again $\mathrm{Re}\,\Delta m_{\rm cl}=0$ and the rate of radiation $\dot{w}=-2\,\mathrm{Im}\,\Delta m_{\rm cl}$ coincides, as it should,[8] with the limiting rate of emission in a magnetic field for $p_\perp\gg m$, if one considers the magnitudes of the 4-accelerations in both cases to be equal.

The spectrum of the total emission probability has the form[5,9]

$$dw_{\mathbf{k}}=\frac{\alpha}{\pi^2a^2}\left(\frac{2}{u}\right)^{2/3}\left[t^2\Phi^2(y)+\left(\frac{2}{u}\right)^{2/3}\Phi'^2(y)\right]\frac{d^3k}{\omega},$$
$$y=\left(\frac{u}{2}\right)^{2/3}(1+t^2), \tag{17}$$

where $\Phi(y)$ is the Airy function, and

$$u=eFk_-/ma^2,\quad t=(p_2k_--p_-k_2)/mk_-.$$

If one uses the representation $d^3k/\omega=dk_1dk_2dk_-/k_-$ and integrates the spectrum (17) over the "dummy" variable k_1, taking into account the relation $\tau=p_-k_1/mak_-$ between k_1 and the proper time τ of emission,[5,9] then, upon dividing by the emission time $\Delta\tau=p_-\Delta k_1/mak_-$ we obtain the spectrum of the radiation rate

$$d\dot{w}_{k_2k_-}=\frac{2\alpha a}{\pi^2}\left(\frac{u}{2}\right)^{1/3}\left[t^2\Phi^2(y)+\left(\frac{2}{u}\right)^{2/3}\Phi'^2(y)\right]du\,dt. \tag{18}$$

Integration with respect to k_2 yields the radiation rate spectrum in the variable k:

$$d\dot{w}_{k_-}=-\frac{\alpha a}{\pi}\left[\Phi_1(u^{2/3})+\frac{2}{u^{2/3}}\Phi'(u^{2/3})\right]du,\quad \Phi_1(x)=\int_x^\infty dz\Phi(z). \tag{19}$$

The integration with respect to u leads to the rate $\dot{w}=-2\,\mathrm{Im}\,\Delta m$, in agreement with Eq. (16).

The spectra (17)–(19) exhibit a low-frequency singularity of the integrable type. For example, the spectrum (19) has a singularity of the type $k_-^{-2/3}dk_-$.

In distinction from the extended formation region for $\Delta m_{\rm cl}$ of a UA charge, the proper time of formation in the last two examples is smaller or of the order of the reciprocal acceleration: $\tau-\tau'\sim a^{-1}v$ for (13) and $\tau-\tau'\sim a^{-1}$ for (16). Thus, the infrared singularity is determined by the behavior (integrability) of the quantity $\dot{x}_\alpha\dot{x}_\alpha'(x-x')^{-2}$ for large relative proper times $\tau-\tau'$. In this connection it is essential that the source of electromagnetic field of the particle is a 4-current, i.e., that the spin of the self-field is unity.

4. THE ROLE OF THE SPIN OF THE SELF-FIELD

For a scalar field, for instance, the source is not a 4-vector current density, but rather a scalar charge density, and in this case the quantity $\dot{x}_\alpha\dot{x}_\alpha'$ should be replaced by one.[5)] This decreases the rate of emission of radiation, and softens the singularity of the spectrum of the probability of emission in the low-frequency region. We list the results for the mass shift and spectrum of emission probability of a scalar charge (i.e., the source of a scalar field) for different motions with constant acceleration $a=(\ddot{x}_\alpha^2)^{1/2}$.

For hyperbolic motion

$$\Delta m_{\rm cl}=-i\frac{\alpha a}{2\pi},\quad dw_{\mathbf{k}}=\frac{\alpha}{\pi^2a^2}K_0^2\left(\frac{k_\perp}{a}\right)\frac{d^3k}{\omega}. \tag{21}$$

In distinction from Eqs. (7), (9), and (10), there are no infrared singularities in the probability and spectrum of the scalar radiation, $\mathrm{Re}\,\Delta m_{\rm cl}=0$, the proper time of formation of Δm_{cl} is of the order of the reciprocal acceleration: $\tau - \tau' \sim a^{-1}$. The expression (21) for Δm_{cl} has been obtained by Zel'nikov and Frolov.[11]

For uniform motion on a circle

$$\Delta m_{cl} = i\frac{\alpha a}{2\pi}\frac{m}{p_\perp}\int_0^\infty du\left[\frac{1}{u^2+(p_\perp/m)^2(u^2-\sin^2 u)} - \frac{1}{u^2}\right], \qquad (22)$$

$$d\dot{w}_{\mathbf{k}_s} = \frac{\alpha}{2\pi}(1-v^2)^{1/2}\omega_s J_s^2(z)\,d\Omega.$$

For the nonrelativistic and ultrarelativistic limits the rate of emission $\dot{w}$ equals respectively $\alpha a p_\perp/3m$ and $\alpha a/2\sqrt{3}$.

In the motion along a 4-trajectory of an electromagnetic charge in a crossed field:

$$\Delta m_{cl} = -i\alpha a/4\sqrt{3},$$

$$d\dot{w}_{\mathbf{k}_2\mathbf{k}_-} = \frac{2\alpha a}{\pi^2}\left(\frac{u}{2}\right)^{2/3}\Phi^2(y)\,dt\,du, \quad d\dot{w}_{k_-} = \frac{\alpha a}{\pi}\Phi_1(u^{2/3})\,du. \qquad (23)$$

In the last two examples the proper time of formation of Δm_{cl} is smaller or of the order of the reciprocal acceleration: $\tau - \tau' \sim a^{-1}v$ for Eq. (22) and $\tau - \tau' \sim a^{-1}$ for Eq. (23), i.e., the same as for the electromagnetic charge. Therefore Δm_{cl}^{sc} is of the same order as Δm_{cl}^{em}, but the spectra of the scalar and electromagnetic radiations are substantially different in the low-frequency region $\omega \lesssim a$, where the role of the relative Lorentz factor $\dot{x}_\alpha \dot{x}_\alpha'$, i.e., of the structure of the interaction due to the spin of the self-field, has the most influence. For another manifestation of the spin of the self-field, see the book by Lightman *et al.*, Ref. 10 (Problem 12.5).

Zel'nikov and Frolov[11] have utilized an equation of the type (3) for the self-interaction of a UA source of a field of spin s (see the book by Schwinger[12]) and have obtained an expression for the mass shift differing from Eq. (7) by replacing the index 1 in the Macdonald function by the index s and a common factor $(-1)^{s-1}$. It follows from this expression that $\mathrm{Re}\,\Delta m_{cl} = (-1)^s s\alpha a/2$ and does not depend on $\lambda \to 0$, and $\mathrm{Im}\,\Delta m_{cl}$ for $s \geq 2$ grows according to a power law as λ^{-s+1}. This is caused by the appearance in the self-interaction of the Lorentz factor $\dot{x}_\alpha \dot{x}_\alpha'$ raised to the power s. The author has shown that for uniform motion around a circle of a source of a field of spin 2 $\mathrm{Re}\,\Delta m_{cl} = 0$, and $\mathrm{Im}\,\Delta m_{cl}$ is finite for $\lambda \to 0$, but is positive, which violates unitarity. Apparently the indicated shortcomings in the behavior of $\mathrm{Im}\,\Delta m_{cl}$ for $s \geq 2$ are caused by the fact that the tensor sources used for a field with $s \geq 2$ do not satisfy a conservation law for $\ddot{x}_\alpha(\tau) \neq 0$, in distinction from the always conserved 4-current source of a field of spin 1.

5. THE MASS SHIFT FOR AN ARBITRARY MOTION OF A CHARGE IN AN ELECTROMAGNETIC FIELD

For a charge moving in an electromagnetic field of a general form (when both field invariants are not zero) we have

$$(x-x')^2 = 2\frac{p_\perp^2}{e^2\eta^2}\left[1-\cos\frac{e\eta}{m}(\tau-\tau')\right] - 2\frac{m^2+p_\perp^2}{e^2\varepsilon^2}\left[\mathrm{ch}\frac{e\varepsilon}{m}(\tau-\tau')-1\right],$$

$$\dot{x}_\alpha\dot{x}_\alpha' = \frac{p_\perp^2}{m^2}\cos\frac{e\eta}{m}(\tau-\tau') - \frac{m^2+p_\perp^2}{m^2}\mathrm{ch}\frac{e\varepsilon}{m}(\tau-\tau'). \qquad (24)$$

Here ε and η are the magnitudes of the electric and magnetic fields in the reference frame in which they are parallel; $p_\perp$ is the conserved magnitude of the momentum perpendicular to the field in this frame. The interval and the Lorentz factor increase at the same rate as $\tau - \tau' \to \infty$ and therefore Δm_{cl} has an infrared singularity. As can be seen from Eq. (24), it is important for the infrared singularity that the charge be subjected to the action of a constant electric field ε during a proper time span not smaller than $m/e\varepsilon$.

We note that the curvature (acceleration) a, and the first and second torsions τ_1 and τ_2 of the 4-trajectory are given in this general case by the formulas

$$a = \gamma\frac{e}{m}(\varepsilon^2+v^2\eta^2)^{1/2}, \quad \tau_1 = v\gamma^2\frac{e^2(\eta^2+\varepsilon^2)}{m^2 a}, \quad \tau_2 = \frac{e^2\eta\varepsilon}{m^2 a}, \qquad (25)$$

where v and $\gamma = (1-v^2)^{-1/2}$ are the velocity and the Lorentz factor corresponding to the transverse momentum $p_\perp = mv\gamma$. The inverse relations which express the fields ε and η and the transverse velocity v in terms of the natural parameters of the world line are also important:

$$\frac{e\varepsilon}{m}, \frac{e\eta}{m} = ((l^2+a^2\tau_2^2)^{1/2} \pm l)^{1/2}, \quad v = 2a\tau_1(2(l^2+a^2\tau_2^2)^{1/2}+a^2+\tau_1^2+\tau_2^2)^{-1}, \qquad (26)$$

where $2l = a^2 - \tau_1^2 - \tau_2^2$. As we have seen, a necessary and sufficient condition for infrared divergence of Δm_{cl} is that $\varepsilon \neq 0$. In the language of the natural parameters of the world line this means that either the second torsion is nonzero $\tau_2 \neq 0$, or the curvature exceeds the first torsion, $a^2 > \tau_1^2$ if $\tau_2 = 0$.

Since the expression for the mass shift in this general case is rather complicated, we list it for the general motion of a charge in a purely electric field ($\eta = 0$, $\varepsilon \neq 0$, $p_\perp \neq 0$):

$$\Delta m_{cl} = \frac{\alpha w_0}{2\pi}\int_0^\infty dz\exp\left(-\frac{i\lambda\gamma^2}{2z}\right)\left\{e^{iz}\int_0^\infty du(\mathrm{ch}\,u - v^2) \times\exp\left[-iz\left(\mathrm{ch}\,u-\frac{v^2u^2}{2}\right)\right] - \left(\frac{\pi}{2iz\gamma^2}\right)^{1/2}\right\}. \qquad (27)$$

Here, again, v and $\gamma = (1-v^2)^{-1/2}$ are respectively the velocity and Lorentz factor corresponding to the transverse momentum $p_\perp = mv\gamma$. This formula generalizes Eq. (7) to the case of a nonvanishing transverse momentum if $\eta = 0$ [as can be seen from Eq. (24), at $p_\perp = 0$ the shift is not sensitive to the magnetic field, and Eq. (7) remains in force if in addition to the electric field ε there is a magnetic field $\eta \neq 0$].

In the case $\lambda\gamma^2 \ll 1$ the equation (27) takes the form of the right-hand side of Eq. (7):

$$\Delta m_{cl} = \frac{\alpha w_0}{2\pi}\left\{-\pi - i\left[2\ln\frac{2w_0}{\gamma_E\mu\gamma} + J(v^2)\right]\right\}. \qquad (28)$$

Here $\gamma_E = 1.781\ldots$ is Euler's constant, $\gamma = (1-v^2)^{-1/2}$, and the function

$$J(v^2) = \int_0^\infty du\left(\frac{1-v^2+v^2u^2/2}{\mathrm{ch}\,u-1-v^2u^2/2} - \frac{2}{u^2}\right) \qquad (29)$$

and has the following asymptotic properties:

$$J(v^2) = \begin{cases} -1+(2/3+2\pi^2/9)v^2+\ldots, & v^2 \ll 1 \\ 5\pi\gamma/2\sqrt{3}-3.895+\ldots, & 1-v^2 \ll 1 \end{cases}. \qquad (30)$$

Thus, $\mathrm{Re}\,\Delta m_{cl} = -\alpha e\varepsilon/2m$ and does not depend on v.

As regards $\operatorname{Im}\Delta m_{\rm cl}$, as $v\to 1$ the term involving the J function becomes leading, and the rate of radiation tends to that of a charge in a crossed field, provided one identifies the corresponding accelerations, see Eq. (16):

$$\dot{w}=-2\operatorname{Im}\Delta m_{\kappa\pi}|_{v\to 1}=5\alpha a/2\sqrt{3},\quad a=e\varepsilon\gamma/m. \tag{31}$$

We also list the spectrum of emission probabilities, determined from the general formula (8) by Nikishov and the author[5]:

$$dw_{\mathbf{k}}=\frac{m^2e^{\pi\nu}}{4\pi^3e^2}\left\{\left[\left(1-\frac{v^2}{z^2}\right)\gamma^2-1\right]K_{i\nu}^2(z)+\gamma^2K_{i\nu}'^2(z)\right\}\frac{d^3k}{\omega}, \tag{32}$$

where $z=k_\perp m\gamma/e\varepsilon$, $\nu=k_\perp p_\perp\cos\varphi/e$, and φ is the angle between the vectors $k_\perp$ and $p_\perp$.

We make two remarks. 1) A nontrivial integration of this spectrum with respect to $k_\perp$ must yield $-2\operatorname{Im}\Delta m_{\rm cl}$ as given by Eq. (28). The infrared singularity appears on account of the behavior $K_i'(z)\sim -z^{-1}$ as $k_\perp\to 0$, and the integration with respect to $k_\perp$ from $k_{\perp\min}$ to infinity leads to the logarithmic term in (28), with the photon mass μ replaced by the lower limit $k_{\perp\min}$. This is essential for a physical interpretation of the emission probability. 2) The spectrum (8), and in particular (32), does not involve an ultraviolet singularity, in distinction from the expression (3) for $\Delta m_{\rm cl}$, where such a singularity leads to the need of a subtraction, essential for both the real and imaginary parts of $\Delta m_{\rm cl}$.

6. THE MASS SHIFT DETERMINED BY THE STATE OF THE SELF-FIELD

Another method of determining the mass shift consists in finding the correction ΔL to the Lagrange function L of a charge in the external field $A_\alpha^{\rm ext}$

$$L=-m(1-v^2)^{1/2}+e\mathbf{A}^{\rm ext}\mathbf{v}-eA_0^{\rm ext}, \tag{33}$$

correction which takes into account the interaction of the charge with its proper field

$$\Delta L=\left[\int dVA_\alpha j_\alpha+\int dV\frac{E^2-H^2}{2}\right]_0^F. \tag{34}$$

Here A_α, $\mathbf{E}$, and $\mathbf{H}$ are the four-potential and the field strengths of the self-field of the charge. Since for uniform motion of the charge the interaction with the self-field is already taken into account and included in the observable mass of the charge, the correction to the Lagrange function takes into account only the change produced by the external field of the interaction with the self-field, and therefore represents the difference between the quantity in brackets in the external field and in the vacuum, for identical positions and velocities of the charge at the instant of time under consideration (the indices F and 0 on the bracket denote this). It can be seen from the expression for L that if one considers ΔL as a correction for a given coordinate and velocity, i.e., $\Delta L=(\Delta L)_{x,v}$, then it can be interpreted as a change of L on account of a change of the mass of the charged particle:

$$(\Delta L)_{x,v}=-\Delta m(1-v^2)^{1/2}. \tag{35}$$

Thus,

$$\Delta m=-\gamma(\Delta L)_{x,v}=-\gamma\left[\int dVA_\alpha j_\alpha+\frac{1}{2}\int dV(E^2-H^2)\right]^F. \tag{36}$$

The expression

$$m_{\rm em}=-\gamma\left[\int dVA_\alpha j_\alpha+\frac{1}{2}\int dV(E^2-H^2)\right] \tag{37}$$

could be called the electromagnetic mass of the charge. Making use of the relation

$$A_\alpha j_\alpha=-(E^2-H^2)-\frac{\partial}{\partial x_\alpha}(F_{\alpha\beta}A_\beta), \tag{38}$$

which follows from the Maxwell equations, one can represent m_{em} in the form

$$m_{\rm em}=\gamma\left[\frac{1}{2}\int dV(E^2-H^2)+\int dV\partial_\alpha(F_{\alpha\beta}A_\beta)\right] \tag{39}$$

or, alternatively, in the form

$$m_{\rm em}=\frac{1}{2}\gamma\left[-\int dVA_\alpha j_\alpha+\int dV\partial_\alpha(F_{\alpha\beta}A_\beta)\right]. \tag{40}$$

For a charge at rest or in uniform motion the divergence term vanishes, and in this case the electromagnetic mass is given by the expressions

$$m_{\rm em}=\gamma\frac{1}{2}\int dV(E^2-H^2)=-\gamma\frac{1}{2}\int dVA_\alpha j_\alpha. \tag{41}$$

The first of these expressions coincides with the result of Butler,[13] who proposed for the 4-momentum of the self-field of a uniformly moving charge the expression

$$G_\alpha=\left(\gamma\int dV\frac{E^2-H^2}{2}\right)u_\alpha,$$

where u_α is the 4-velocity of the charge. Earlier, Eq. (41) had been used by Tsytovich[14] for macroscopic renormalization of the mass of a uniformly moving electron in a medium. For a UA charge the divergence term does no longer vanish, and one must use one of the equations (37), (39), (40) for the electromagnetic mass.

In cylindrical coordinates the retarded field of a UA charge moving along the z axis according to the law $z=z_e=(w_0^{-2}+t^2)^{1/2}$ is determined by

$$\begin{gathered}E_z=-\pi^{-1}ew_0^{-2}(w_0^{-2}+t^2-z^2+\rho^2)\xi^{-3}\theta(z+t),\\ E_\rho=2\pi^{-1}ew_0^{-2}z\rho\xi^{-3}\theta(z+t)+(2\pi)^{-1}e\rho(w_0^{-2}+\rho^2)^{-1}\delta(z+t),\\ H_\varphi=2\pi^{-1}ew_0^{-2}t\rho\xi^{-3}\theta(z+t)-(2\pi)^{-1}e\rho(w_0^{-2}+\rho^2)^{-1}\delta(z+t),\\ E_\varphi=H_z=H_\rho=0,\quad \xi=[(w_0^{-2}+t^2-z^2-\rho^2)^2+4w_0^{-2}\rho^2]^{1/2},\end{gathered} \tag{42}$$

and the retarded potential is

$$\begin{gathered}A_0=e\frac{z(w_0^{-2}+\rho^2+z^2-t^2)-\xi t}{4\pi\xi(z^2-t^2)}\theta(z+t)-\frac{e}{4\pi}\ln(1+w_0^2\rho^2)\delta(z+t),\\ A_z=e\frac{t(w_0^{-2}+\rho^2+z^2-t^2)-\xi z}{4\pi\xi(z^2-t^2)}\theta(z+t)+\frac{e}{4\pi}\ln(1+w_0^2\rho^2)\delta(z+t),\\ A_\rho=A_\varphi=0.\end{gathered} \tag{43}$$

The solution obtained by Born[15] differs from the one above by the absence of the delta-function terms and the replacement of the theta-function by one. It coincides with the retarded solution of the Maxwell equations only in the region $z>-t$. The retarded solution of Schott[16] differs from the one given here only by the absence of the delta-function terms, which where proposed by Bondi and Gold,[17] in order that the solutions should satisfy Maxwell's equations also on the plane $z=-t$; cf. also the paper by Fulton and Rohrlich.[18]

Thus, the field of a UA charge differs from zero only in the half-space $z\geqslant -t$, and is singular on the boundary plane $z=-t$. The field lines of the electric and magnetic fields and the flow lines of the energy flux are mutually orthogonal and in the region $z > -t$ they are along the circles which form the coordinate lines of a bispherical coordinate system. If one denotes r_1 and r_2 the distances from an arbitrary field point to the charge $z = z_e$ and its mirror image $z = -z_e$ in the $z = 0$ plane, and defines $\psi = \ln(r_2/r_1)$ and χ as the angle between the two segments r_1, r_2, there ψ and χ together with the azimuthal angle φ of the cylindrical coordinates form a bispherical coordinate system, being related with the cylindrical coordinates of a point by the relations

$$z=z_e \operatorname{sh}\psi/(\operatorname{ch}\psi-\cos\chi), \quad \rho=z_e\sin\chi/(\operatorname{ch}\psi-\cos\chi).$$

In terms of these coordinates the electric and magnetic field strengths at any point with $z > -t$ are given by the expression

$$E=\frac{e(1-v^2)(\operatorname{ch}\psi-\cos\chi)^2}{4\pi z_e^2(1-v^2\sin^2\chi)^{3/2}}, \quad H=v\sin\chi E. \tag{44}$$

The lines of the electric field $\mathbf{E}$ are along arcs of circle passing through the charge and its mirror image in the plane $z = 0$, i.e., along circles with constant χ, φ, and variable ψ. When they reach the plane $z = -t$ the $\mathbf{E}$-lines suffer a break and go out radially in the $z = -t$ plane to infinity. The lines of the magnetic field $\mathbf{H}$ are latitude circles with variable φ (and $\psi, \chi =$ const). Finally, the energy flux lines of the Poynting vector $\mathbf{E}\times\mathbf{H}$ form the χ-circles ($\psi, \varphi =$ const).

At the boundary $z = -t$ the field energy becomes infinite and flows with constant power $2\alpha w_0^2/3$ into the volume $z > -t$. The energy excess in the volume $z > -t$ over the energy of a uniformly moving charge increases linearly in time:

$$\frac{1}{2}\int dV(E^2+H^2)\Big|_0^F=\frac{2}{3}\alpha w_0^2 t. \tag{45}$$

A similar behavior is exhibited by the z-component of the field momentum. In particular,

$$\int dV[\mathbf{E}\times\mathbf{H}]_z\Big|_0^F=-\frac{2}{3}\alpha w_0^2 t, \tag{46}$$

so that the excess of energy-momentum of the field in the volume $z > -t$ over the energy momentum of the field of a uniformly moving charge forms an isotropic 4-vector (a 4-vector of zero length, or null-vector).

It follows from Eq. (44) that the field becomes lightlike only for $v\sin\chi$ close to one, i.e., only near the spherical surface $\chi = \pi/2$ passing through the charge and centered at the coordinate origin, and only in this case is the electron ultrarelativistic.

Making use of the field (42) and carrying out an integration over all space, we obtain

$$\frac{1}{2}\int dV(E^2-H^2)\Big|_0^F=\frac{\alpha w_0}{2\gamma}f_1(v), \tag{47}$$

$$f_1(v)=\frac{1}{2}\left(v+\frac{1}{v}-\frac{1}{2v^2\gamma^4}\ln\frac{1+v}{1-v}\right)=\frac{\partial}{\partial x}(x\operatorname{cth}x), \quad x=2w_0\tau=2\operatorname{Arth}v_1$$

with the Bondi-Gold terms not contributing to the integral.

The computation of the divergence term is conveniently carried out separately for the spatial and temporal parts of the divergence of the 4-vector $F_{\alpha\beta}A_\beta = (\mathbf{A}\times\mathbf{H} + A_0\mathbf{E}, i\mathbf{A}\cdot\mathbf{E})$:

$$\int dV\operatorname{div}([\mathbf{A}\times\mathbf{H}]+A_0E)=\frac{\alpha}{\delta}-\frac{2}{3}\alpha w_0^2 t+\ldots, \tag{48}$$

$$\int dV\frac{\partial}{\partial t}(\mathbf{AE})=-\frac{\alpha}{\delta}+\frac{2}{3}\alpha w_0^2 t-\alpha w_0\gamma^{-1}f_1(v)+\ldots.$$

The integration is over the half-space occupied by the field $-t+\delta \leq z \leq \infty$, $\delta \to 0$, excluding a thin layer of thickness δ near the plane $z = -t$; the dots ... denote terms which vanish as $\delta \to 0$. If one extends the integration to the plane $z = -t$, the Bondi-Gold terms lead to the appearance in the integrands on the left-hand sides of the above equations of singular derivatives

$$\frac{\partial}{\partial z}[-E_z^B e\ln(1+w_0^2\rho^2)\theta(z+t)\delta(z+t)],$$

$$\frac{\partial}{\partial t}[E_z^B e\ln(1+w_0^2\rho^2)\theta(z+t)\delta(z+t)],$$

respectively, which in sum yield zero, on account of the fact that

$$E_z^B=-\pi^{-1}ew_0^{-2}(w_0^{-2}+\rho^2+t^2-z^2)\xi^{-3}$$

depends only on $t^2 - z^2$ and is not singular on the line $z^2 - t^2 = 0$.

Thus the sum, i.e., the divergence term as a whole, does not depend on the Bondi-Gold terms and of the limit $\delta \to 0$:

$$\int dV\partial_\alpha(F_{\alpha\beta}A_\beta)=-\alpha w_0 f_1(v)/\gamma. \tag{49}$$

We also stress the fact that the divergence term does not require subtraction, since it already vanishes for a uniformly moving charge.

It is striking that the divergence term is twice as large as the change in the Lagrange function of the field and has the opposite sign. On account of the relation (38) this means that

$$\int dVA_\alpha j_\alpha\Big|^F=0. \tag{50}$$

This fact is no accident. If one substitutes into the left-hand side of Eq. (50) the solution

$$A_\alpha^{\text{ret}}(x)=\int d^4x' D^{\text{ret}}(x-x')j_\alpha(x'),$$

then the resulting integrand $D^{\text{ret}}(x-x')j_\alpha(x)j_\alpha(x')$ differs from zero only for $x = x'$, since D^{ret} has support on the light cone $(x-x')^2 = 0$, and the product of the currents has support on the timelike trajectory of the charge $(x-x')^2 < 0$. But at the point $x = x'$ the quantity $D^{\text{ret}}j_\alpha j_\alpha$ coincides with its expression in the absence of the field, and therefore the difference $(F, 0)$ vanishes. This reasoning is not applicable to the uniform motion of a charge in a medium, since the propagators in the medium and in vacuum are different.

Thus, the mass shift of a UA charge is given by the formula[1]

$$\Delta m_{cl}=-\gamma\frac{1}{2}\int dV(E^2-H^2)|_0^F=\gamma\frac{1}{2}\int dV\partial_\alpha(F_{\alpha\beta}A_\beta)=-\frac{1}{2}\alpha w_0 f_1(v). \tag{51}$$

It is an odd function of the velocity of the electron and varies rapidly near the turning point[6)] from the value $\alpha w_0/2$ to the value $-\alpha w_0/2$. Since $w_0 = e\varepsilon/m$, one may say that the accelerated electron has an electric dipole moment equal to $\alpha(e\hbar/2mc)f_1$ directed along the velocity, such that the interaction energy with the electric field as the charge goes through the turning point decreases from the maximal value to the minimal value.

7. THE MASS DENSITY AND THE MAXWELL STRESSES OF THE SELF-FIELD

We note that $(E^2-H^2)/2$ is the mass density of the electromagnetic field of the electron. Indeed, at each point of space the electromagnetic field moves like a compressible fluid in the direction of the Poynting vector with a velocity $\mathbf{u}$ determined by the relation

$$\mathbf{u}(1+u^2)^{-1}=[\mathbf{E}\times\mathbf{H}](E^2+H^2)^{-1}.$$

For the field of an electron we always have $\mathbf{E}\cdot\mathbf{H}=0$, $E^2>H^2$, and therefore $u=H/E$. In particular, for the field of a UA electron $u=v\sin\chi$ [cf. Eq. (44)]. In a Lorentz frame moving with velocity u relative to the original frame the fields $\mathbf{E}'$ and $\mathbf{H}'$ are parallel, their Poynting vector vanishes, and the field energy density coincides with the mass density. It can be written in terms of the fields $\mathbf{E}$ and $\mathbf{H}$ in the original frame:

$$\frac{1}{2}(E'^2+H'^2)=\left[\left(\frac{E^2+H^2}{2}\right)^2-[\mathbf{E}\times\mathbf{H}]^2\right]^{1/2}=\left[\left(\frac{E^2-H^2}{2}\right)^2+(\mathbf{EH})^2\right]^{1/2}. \qquad (52)$$

For the field of an electron the right-hand side equals $(E^2-H^2)/2$.

The mass density (52) of the field is an eigenvalue of the energy-momentum tensor of the field, which, in the frame where the Poynting vector vanishes, has the diagonal form

$$T_{\alpha\beta}'={}^1/_2(E'^2+H'^2)\ \mathrm{diag}\,(1,\ 1,\ -1,\ -1), \qquad (53)$$

if the third space axis is chosen along the common direction of the fields $\mathbf{E}'$ and $\mathbf{H}'$. The pressure along this distinguished direction is negative (i.e., a tensile stress). In the original coordinate frame the energy-momentum tensor is not diagonal, on account of $\mathbf{E}\times\mathbf{H}\neq 0$, but an appropriate choice of the spatial axes allows one to diagonalize its spatial part, the maxwellian stress tensor T_{ik}. Then two of the principal stresses will have opposite signs and be equal in magnitude to Lorentz-invariant mass density (52). The third principal stress equals the energy density of the field and is not Lorentz invariant.

For the field of a UA charge the Maxwell stress tensor is diagonal in the bispherical coordinate system

$$T_{ik}=\mathrm{diag}\,(T_{\psi\psi},\ T_{\varphi\varphi},\ T_{\chi\chi}); \qquad (54)$$

the principal stresses are along the electric and magnetic field vectors and along the Poynting vector:

$$T_{\psi\psi}=-T_{\varphi\varphi}=-(E^2-H^2)/2, \quad T_{\chi\chi}=(E^2+H^2)/2. \qquad (55)$$

The stress along the electric field is negative, i.e., is a tension. The volume integral of this tension for an accelerated electron and $v>0$ is larger than for a uniformly moving electron, which indicates a stronger coupling of the charge to the field and explains why the mass shift is negative, cf. Eq. (51).

The mass density of a UA electron

$$\frac{1}{2}(E^2-H^2)=\frac{2\alpha}{\pi}w_0^{-4}[(w_0^{-2}+t^2-z^2-\rho^2)^2+4w_0^{-2}\rho^2]^{-2}\theta(z+t), \qquad (56)$$

in addition to the singularity at the location of the charge, will exhibit for $v\to 1$ a maximum at the point $z=-t$, $\rho=0$ with a transverse width $\Delta\rho\sim w_0^{-1}$ and a longitudinal width $\Delta z\sim w_0^{-1}\gamma^{-1}$. This maximum gives the main contribution to the integral (47), since the contribution from the region near the charge coincides with the field mass of a uniformly moving charge and is cancelled in the subtraction. In general, on account of the symmetry of the field at any time $t>0$, the following relation holds between the contributions of the regions $-t\leq z\leq 0$, $0\leq z\leq t$, and $t\leq z<\infty$:

$$\frac{1}{2}\int\limits_{-t<z<0}dV(E^2-H^2)=\frac{1}{2}\int\limits_{0<z<t}dV(E^2-H^2)$$
$$=-\frac{1}{2}\int\limits_{t<z<\infty}dV(E^2-H^2)|_0^F=\frac{\alpha w_0}{2\gamma}f_1(v); \qquad (57)$$

the whole subtraction term is referred to the integral over the third region. The energy of the field, (45), has has a similar volume distribution.

The relation (57) does not change under a Lorentz transformation with velocity k along the z axis which transforms the hyperplane $t=C$ into the inclined hyperplane

$$t=-kz+C(1-k^2)^{1/2},$$

which, as before, is tangent to the hyperbolic cylinder $t^2-z^2=C^2$ with the boundaries of the region again given by its intersection with the invariant planes $t=\mp z$ and the tangency with hyperbolic cylinder $t^2-z^2=C^2$. For $k=v$ the charge is at rest in the transformed frame, and the integral (51) accumulates its value $-\alpha w_0 f_1(v)/2$ on the hyperplane $t=v(w_0^{-1}-z)$, so that the dependence of the mass shift on the velocity is equivalent to its dependence on the slope of the integration 3-hyperplane.

8. THE OSCILLATION OF THE PROPER MASS OF A SCALAR CHARGE DURING ACCELERATION

Although the real part of the mass shift of a UA scalar charge vanishes according to Eq. (21), in distinction from the case of an electromagnetic charge, we derive more physical information by considering the mass shift of a scalar charge in terms of the field, i.e., in terms of the Lagrange function of the field, as we did in Sec. 6 for the electromagnetic charge, since this approach yields the time dependence of the mass shift.

It is obvious that for the source of a scalar field the analog of Eq. (34) will have the form

$$\Delta L=\left[-\int dV\varphi\rho+\frac{1}{2}\int dV\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2\right]_0^F. \qquad (58)$$

On the basis of the considerations expounded in Sec. 6, the contribution of the first term vanishes, so that one needs to take into account only the volume integral of the Lagrange density of the scalar field. For an arbitrary motion of the source the potential φ of the scalar field is connected with the zero component A_0 of the electromagnetic potential by means of the relation $\varphi=A_0(1-v^2(t'))^{1/2}$, where $v(t')$ is the 3-velocity of the source at the retarded instant of time. Determining for the UA motion of the charge the factor $(1-v^2)^{1/2}$ as a function of the coordinates of the field point at time t, and using the expression (43) for A_0, we obtain

$$\varphi=(2\pi)^{-1}ew_0^{-1}\xi^{-1}\theta(z+t). \quad (59)$$

On account of the jump in φ at the boundary $z=-t$ the energy is infinite there and flows in with constant power $\alpha w_0^2/3$ into the volume $z>-t$. Therefore the excess of the field energy in the volume $z>-t$ over the field energy of a charge in uniform motion grows linearly with time:

$$\frac{1}{2}\int dV\left[\left(\frac{\partial\varphi}{\partial\mathbf{x}}\right)^2+\left(\frac{\partial\varphi}{\partial t}\right)^2\right]_0^T=\frac{1}{3}\alpha w_0^2 t. \quad (60)$$

For the density of the Lagrange function for the field we obtain

$$\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial\mathbf{x}}\right)^2-\left(\frac{\partial\varphi}{\partial t}\right)^2\right]=\frac{2\alpha w_0^{-1}}{\pi\xi^4}(\rho^2+z^2-t^2)\theta(z+t). \quad (61)$$

This expression is finite at the boundary $z=-t$, in distinction from the energy density. The integration of this function over the 3-volume is facilitated considerably if one takes into account the fact that for a charge in uniform motion $(\partial\varphi/\partial x_\alpha)^2=E^2-H^2$, and for a UA charge

$$(\partial\varphi/\partial x_\alpha)^2=E^2-H^2-2\pi^{-1}\alpha w_0^{-2}\partial\xi^{-2}/\partial z^2.$$

Therefore

$$-\frac{1}{2}\int dV\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2\Big|_0^T=-\frac{1}{2}\int dV(E^2-H^2)\Big|_0^T-\frac{\alpha w_0^{-2}}{\pi}\int dV\frac{\partial\xi^{-2}}{\partial z^2}, \quad (62)$$

i.e., the change of the Lagrange function of the scalar field is equal to the already determined variation of the Lagrange function of the electromagnetic field, (47), minus a simple term which does not require subtractions, which is equal to

$$\frac{\alpha w_0^{-2}}{\pi}\int dV\frac{\partial\xi^{-2}}{\partial z^2}=\frac{\alpha w_0}{2\gamma}\left(\frac{1}{v}-\frac{1}{2v^2\gamma^2}\ln\frac{1+v}{1-v}\right)=\frac{\alpha w_0}{2\gamma}\frac{\partial}{\partial y}(y\,\mathrm{cth}\,y); \quad (63)$$

where v and $\gamma=(1-v^2)^{-1/2}$ are the velocity and the Lorentz factor of the charge at the time t, and $y=w_0\tau=\mathrm{Artanh}\,v$.

As a result of this the change in the Lagrange function of the scalar field has the form

$$-\frac{1}{2}\int dV\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2\Big|_0^T=\frac{\alpha w_0}{2\gamma}f_0(v),$$

$$f_0(v)=-\frac{1}{2v\gamma^2}+\frac{1+v^2}{4v^2\gamma^2}\ln\frac{1+v}{1-v}=-\frac{\partial}{\partial x}\left(\frac{x}{\mathrm{sh}\,x}\right), \quad (64)$$

where $x=2w_0\tau=2\,\mathrm{Artanh}\,v$, as in Eq. (47). We call attention to the fact that the functions of proper time f_0 and f_1 are related by

$$f_0(x)=f_1(x)-f_1(x/2). \quad (65)$$

We note that the expression (58) for ΔL can be transformed by means of the relation

$$-\varphi\rho=-\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2+\frac{\partial}{\partial x_\alpha}\left(\varphi\frac{\partial\varphi}{\partial x_\alpha}\right), \quad (66)$$

which follows from the wave equation for φ, cf. Eq. (38). The resulting equations differ from the electromagnetic ones by the substitution

$$A_\alpha j_\alpha\to-\varphi\rho,\quad E^2-H^2\to\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2,\quad F_{\alpha\beta}A_\beta\to-\varphi\frac{\partial\varphi}{\partial x_\alpha},\quad f_1\to f_0.$$

The expression for the shift can again be reduced to a divergence term, which does not require subtractions. The spatial and temporal parts of the divergence term do not contain singular contributions, owing to the finiteness of φ at the boundary $t=-z$, and their terms linear in t are half as large as in electrodynamics, since the energy flux through the boundary $z=-t$ into the volume $z>-t$, is half as large [cf. Eq. (60)].

Thus, the mass shifts $\Delta m=-\gamma\Delta L$ of the scalar and electromagnetic field are described by the simple odd functions of the velocity or proper time

$$\Delta m^{\mathrm{em}}=-\frac{1}{2}\alpha w_0\frac{\partial}{\partial x}\left(\frac{x}{\mathrm{th}\,x}\right),\quad \Delta m^{\mathrm{sc}}=\frac{1}{2}\alpha w_0\frac{\partial}{\partial x}\left(\frac{x}{\mathrm{sh}\,x}\right), \quad (67)$$

where $x=2w_0\tau=2\,\mathrm{Artanh}\,v$. Near the turning point they are both linear in v or in τ:

$$\Delta m^{\mathrm{em}}=-2\alpha w_0^2\tau/3+\dots,\quad \Delta m^{\mathrm{sc}}=-\alpha w_0^2\tau/3+\dots, \quad (68)$$

and the rates of mass change at $\tau=0$ coincide with the intensities of the emission of the charges

$$I^{\mathrm{em}}=2\alpha a^2/3,\quad I^{\mathrm{sc}}=\alpha a^2/3. \quad (69)$$

An important distinction arises for $v\to\pm1$: whereas the mass shift of the electromagnetic charge tends monotonically to the nonzero value $\alpha w_0/2$, that of the scalar field attains at $w_0\tau\simeq\pm0.8$ the extremum $\mp0.3098w_0/2$ and then tends to zero. The limiting values of the shift for $v\to+1$ are given by the formulas of Secs. 2 and 4.

Thus, the scalar field, just as the electromagnetic field, exhibits inertia: as the charge passes near the turning point the mass of the self-field suffers an oscillation around its value which corresponds to no acceleration. The sign of this oscillation, cf. Eq. (68), is closely related to causality and can be changed only if one replaces retarded solutions by advanced ones.

9. MASS SHIFT, REACTIVE ENERGY, AND THE SPECTRUM

This oscillation of the mass is analogous to the oscillations of the reactive energy in an AC circuit. Indeed, the instantaneous power of an alternating current consists of the active and reactive parts:

$$JU={}^1/_2J_0^2R(1+\cos2\omega t)-{}^1/_2J_0^2X\sin2\omega t, \quad (70)$$

where J_0 is the amplitude of the current J, R, and X are the active and reactive parts of the impedance, respectively. One usually characterizes the active power through its mean value $\frac{1}{2}J_0^2R$, and the reactive power through its peak value $\frac{1}{2}J_0^2X$. If one replaces the real current and voltage by their complex representations

$$J=J_0e^{-i\omega t},\quad \hat U=U_0e^{-i(\omega t+\varphi)},$$

then the so-called complex power

$${}^1/_2J\hat U={}^1/_2J_0^2(R+iX) \quad (71)$$

characterizes both the active and reactive parts.

We represent the active power as the mean density of the rate of radiation multiplied by $\hbar\omega$:

$${}^1/_2J_0^2R=\hbar\omega\langle dw/dt\rangle, \quad (72)$$

and the peak value of the reactive power as ω times the difference between the amplitudes of the electric and magnetic energies of the field, or as -2ω times the mean Lagrange function of the field:

$$\frac{1}{2}J_0^2X=\omega\int dV\frac{H_0^2-E_0^2}{2}=-2\omega\left\langle\int dV\frac{E^2-H^2}{2}\right\rangle. \qquad (73)$$

The complex power of the current will have the form

$$\frac{1}{2}\hat{J}\hat{U}^{*}=2i\omega\left\{-\left\langle\int dV\frac{E^2-H^2}{2}\right\rangle-i\frac{1}{2}\hbar\left\langle\frac{dw}{dt}\right\rangle\right\}, \qquad (74)$$

i.e., up to the factor $2i\omega\gamma^{-1}$ (γ is the Lorentz factor, introduced artificially via the relation $dt=\gamma d\tau$) it coincides with the shift of the self-mass. It is clear that $\mathrm{Re}\,\Delta m$ or the mean Lagrange function is the amplitude of the oscillations of the reactive energy [cf. Eq. (73)]. The relation (74) is a special case of the theorem about the complex Poynting vector (cf. Ref. 19, §2.20).

In the case considered here, however, the mass oscillation is essentially aperiodic. In this connection we recall the famous Heaviside theorem,[20] which refers to the other extreme case—the work done by suddenly switched on electromotive forces during the transient period.[7)]

The total work performed by suddenly switched on electromotive forces till the end of the transient regime differs from the work it would do during the same time for a steady-state power corresponding to the final value of the current density by twice the excess of the steady-state electric energy over the magnetic energy:

$$A(t)=I_{st}t+2L_{st},\quad L=\int dV\frac{E^2-H^2}{2}. \qquad (75)$$

It is essential that, independent of the complicated character of the transient period, the work done to the end of it, i.e., over a sufficiently long time interval t, is determined by two quantities: the steady-state power loss and the steady-state Lagrange function of the field, cf. Eq. (72) and (73). The transient period is caused by the inertia of the system: if there is no transient period the system exhibits no inertia, and then $A(t)=I_{st}t$ for all $t>0$. Therefore, if $L_{st}\neq 0$ the system is manifestly inertial and during the transient regime the source must give up the additional energy $2L_{st}$ over and above the noninertial energy production $I_{st}t$. If $L_{st}=0$, then integrally, from the energy point of view, the system behaves like an inertia-free system, although it may exhibit a transient period, only after which the linear energy-production $A(t)=I_{st}t$ becomes valid. There is a close analogy between these two cases and the manifestation of the inertia of the fields of electromagnetic and scalar charges. It is striking that the inertial properties of the field, characterized in macroscopic electrodynamics by the inductance and capacitance of the system, manifest themselves for such an elementary system as a point charge.

The reactive oscillations of the self-field which exist during the acceleration of the charges are an important physical concept in the understanding of transient processes in electrodynamics and in chromodynamics.

In conclusion we list the interesting spectral representations for the variable mass shifts (67) of the electromagnetic and scalar charges (cf. the integrals 3.911.1–2 in Ref. 22):

$$\Delta m^{\mathrm{em,sc}}(\tau)=-\frac{\alpha}{2}T\frac{\partial}{\partial T}\int_0^\infty\frac{d\omega\sin\omega\tau}{e^{\omega/T}\mp 1}. \qquad (76)$$

These spectral functions coincide with the derivatives of the one-dimensional Bose- and Fermi-distributions corresponding to a temperature $T=2w_0/\pi$ determined by the acceleration, and being four times as large as the Davies-Unruh temperature (cf. Refs. 23, 24). The same temperature $T=2e\varepsilon/m\pi$ arises if one interprets the production of pairs by an electric field in the vacuum with probability $\exp(-\pi m^2/e\varepsilon)$,[25] as a thermal excitation of the lowest state of the pair with energy $2m$ and probability $e^{-2m/T}$. This coincidence of temperature is not accidental, since both phenomena (the pair production and the reactive mass oscillations) are produced by the same cause: a sufficiently lengthy action of an electric field on the charge, doing work $\gtrsim 2m$.

The author is grateful to V. L. Ginzburg, D. A. Kirzhnits, A. I. Nikishov, and V. P. Frolov for discussions and valuable remarks.

Note added in proof (*3 February 1981*): The integral

$$S_1(\lambda)=\int_0^\infty dz\,e^{-i\lambda/2z}\left[e^{iz}K_1(iz)-\left(\frac{\pi}{2iz}\right)^{1/2}\right], \qquad (A.1)$$

which according to Eq. (7) determines the mass shift of a UA source of a massive vector field, can be expressed in terms of products of modified cylinder functions $K_\nu(x)$, $L_\nu(x)$:

$$\mathrm{Re}\,S_1(\lambda)=-\pi x^2[I_1'(x)K_1(x)+{}^1/_2I_0(x)K_2(x)+{}^1/_2I_2(x)K_0(x)]+\pi x,$$
$$\mathrm{Im}\,S_1(\lambda)=x^2[K_1^2(x)-K_0(x)K_2(x)],\quad x=\sqrt{\lambda}=\mu/w_0. \qquad (A.2)$$

The expression for the integral

$$S_0(\lambda)=-\int_0^\infty dz e^{-i\lambda/2z}\left[e^{iz}K_0(iz)-\left(\frac{\pi}{2iz}\right)^{1/2}\right], \qquad (A.3)$$

which determines the classical shift of the mass of a UA source of the massive vector field differs from (A.2) by decreasing all the indices by one unit ($\nu\to\nu-1$) and changing the sign of the real part. According to these formulas for $\sqrt{\lambda}>0$, not only the imaginary parts but also the real parts of the mass shifts are nonvanishing and negative.

1) We use a natural system of units with $\hbar=c=1$, and Heaviside units for the electromagnetic quantities, so that $\alpha=e^2/4\pi\hbar c=1/137$, $\beta=\hbar e\varepsilon/m^2c^3$, etc. The magnitude of the 4-acceleration for uniformity accelerated (UA) motion will be denoted by w_0.

2) For the squares of the first and second torsions we have the expressions: $\tau_1^2=\dddot{x}^2a^{-2}+a^2$, $\tau_2^2=(\dddot{x}^2-\dddot{x}^4a^{-2})\tau_1^{-2}a^{-2}$. We note that $a^2\tau_1^2$ coincides, up to a factor, with the square of the 4-force of radiation damping.

3) The integral containing the Macdonald function K_1 contains traces of the proper-time representation of the Δ^c-function: the variable z is related to the proper time t of the photon by means of the relation $z^{-1}=2w_0^2t$.

4) This relation leads to $dk_{\parallel}/\omega=\omega_0 d\tau$.

5) For the scalar self-interaction and spectrum of emission scalar quanta one must replace in Eqs. (3), (8) the 4-current

density $j_\alpha(x)$ by the scalar charge density $\rho(x)$:

$$j_\alpha(x)=e\int d\dot{\tau}x_\alpha(\tau)\delta(x-x(\tau))\to\rho(x)=e\int d\tau\delta(x-x(\tau)) \quad (20)$$

and $j_\alpha(k)\to\rho(k)$, denoting the total charge at rest by the letter e, in order that the force of interaction of slow charges should coincide with the Coulomb force $e^2/4\pi r^2$ apart from the sign.

6)At the instants t equal to 0, w_0^{-1}, $2w_0^{-1}$, when v equals respectively 0, $2^{-1/2}$, $2/5^{1/2}$, the mass shift in units of its limiting value $-w_0/2$ equals respectively 0, 0.841, 0.970.

7)The Heaviside theorem is mentioned on the last page of Stratton's book,[19] but has without good reasons been forgotten in other textbooks; its proof was rederived by Lorentz in Ref. 21.

[1]V. I. Ritus, Zh. Eksp. Teor. Fiz. **75**, 1560 (1978); Erratum, *ibid.* **76**, 383 (1976) [Sov. Phys. JETP **48**, 788 (1978)].

[2]L. D. Landau and E. M. Lifshitz, Teoriya polya, Nauka, Moscow, 1973 [Engl. Transl. Classical Theory of Fields, Pergamon, London, 1975].

[3]J. L. Synge, Proc. Roy. Irish Acad. **A65**, 27 (1967).

[4]E. Honig, E. L. Schücking, and C. V. Vishveshvara, J. Math. Phys. **15**, 774 (1974).

[5]A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **56**, 2035 (1969) [Sov. Phys. JETP **29**, 1093 (1969)].

[6]I. Bialynicky–Birula and Z. Bialynicka–Birula, Quantum Electrodynamics, Pergamon Press, Oxford, 1975.

[7]W. E. Thirring, Principles of Quantum Electrodynamics (F. Deuticke Vienna, Germ. Ed. 1956), Acad. Press, N. Y., 1958 [Russian Transl. Vysshaya Shkola, Moscow, 1964].

[8]A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **46**, 776 (1964) [Sov. Phys. JETP **19**, 529 (1964)].

[9]A. I. Nikishov, Kvantovaya elektrodinamika yavlenii v intensivnom pole (Qauntum electrodynamics of phenomena in intensive fields), Trudy FIAN, **111**, p. 152 (1979).

[10]A. Lightman, W. Press, R. Price, and S. Teukolsky, Problem Book in General Relativity and Gravitation, Princeton University Press, 1975 [Russ. Transl. Mir, 1979].

[11]V. P. Frolov, and A. I. Zel'nikov, in Intern. Conf. on General Relativity and Gravitation, Jena, 1980, p. 555.

[12]J. Schwinger, Sources, Particles, Fields, vol. 1, Addison–Wesley, 1969 [Russ. Transl. Mir, 1973].

[13]J. W. Butler, Amer. J. Phys. **37**, 1258 (1969).

[14]V. N. Tsytovich, Zh. Eksp. Teor. Phys. **42**, 457 (1962) [Sov. Phys. JETP **15**, 320 (1962)].

[15]M. Born, Ann. Phys. (Leipzig) **30**, 1 (1909).

[16]G. A. Schott, Electromagnetic Radiation, Cambridge U. Press, 1912.

[17]H. Bondi and T. Gold, Proc. Roy. Soc. **A229**, 416 (1955).

[18]T. Fulton and F. Rohrlich, Ann. Phys. (NY) **9**, 499 (1960).

[19]J. A. Stratton, Electromagnetic Theory, McGraw Hill, NY, 1941 [Russian Translation: Gostekhizdat, 1948].

[20]O. Heaviside, The Electrician, April 25, p. 490 (1885); Electrical Papers, v. 1, p. 462, London, 1892.

[21]H. A. Lorentz, Proc. Nat. Acad. Sci. (USA) **8**, 333 (1922).

[22]I. S. Gradshtein and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedenii (Tables of Integrals, Sums, Series, and Products), Gostekhizdat, Moscow, 1962 [Various English Translations, Academic Press (and others), various years].

[23]P. C. W. Davies, J. Phys. **A8**, 609 (1975).

[24]W. G. Unruh, Phys. Rev. D14, 870 (1976).

[25]W. Heisenberg and H. Euler, Z. Physik **98**, 714 (1936).

Translated by M. E. Mayer