Fluctuation corrections to the hydrodynamical equations of superfluid helium

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We consider nonlocal corrections to the hydrodynamical equations of He II caused by long-wavelength fluctuations. Using the hydrodynamical equations of He II we obtain an equation for the matrix of the binary correlators of fluctuating quantities. We consider two frequency regions: $\omega \ll c_2^2/\Gamma$ and $c_2^2/\Gamma \ll \omega \ll c_1^2/\Gamma$ (Γ is a combination of kinetic coefficients). In the first region the matrix equation reduces to the set of the usual kinetic equations. In the second frequency region one must solve the matrix equation. We find the fluctuation corrections to the kinetic coefficients in both frequency regions. Using the expressions for those corrections we find the contribution of the long-wavelength fluctuations, which has a nontrivial frequency dependence, to the velocity and damping of first and second sound. In the first frequency region this contribution is small, but in the second one there are terms which diverge near the λ- point and give grounds for expecting an experimental observation of the effect. We show in an Appendix that in the low-frequency region the kinetic matrix equation reduces to the set of the usual kinetic equations for any system for which the dissipationless hydrodynamical equations are in Hamiltonian form.

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INTRODUCTION

We consider in the present paper non-local corrections to the hydrodynamical equations of He II which are caused by long-wavelength fluctuations. We undertook a study of superfluid helium in connection with the important information which follows from the interpretation of hydrodynamic processes in He II. This problem has been studied earlier by Andreev[1] for a normal liquid.

We assume in what follows that there is in He II a macroscopic motion with wavevector k and frequency ω. We understand by long-wavelength fluctuations hydrodynamic fluctuations with characteristic wavevector $q \gg k$, and by short-wavelength fluctuations the phonons and rotons. We are dealing with temperatures $\gtrsim 1$ K so that the short-wavelength fluctuations have wavelengths $\sim a$ (a is the atomic size).

The short-wavelength fluctuations make a basic contribution to the thermodynamic quantities and correspondingly determine basically the local hydrodynamical equations. However, the short-wavelength fluctuations contribute non-local terms $\sim ka$ to the equations, while the long-wavelength terms are $\sim k/q$ so that the contribution of the latter to the non-local corrections dominate, as $qa \ll 1$. The present paper is devoted to a calculation of these corrections and an elucidation of their role.

HYDRODYNAMICS OF He II

It is well known that the hydrodynamic state of He II is characterized by the locally given densities of energy E, momentum $\mathbf{g}$, mass ρ and the superfluid velocity $\mathbf{v}_s$. Correspondingly, the hydrodynamical equations are a set of conservation laws and an equation for the superfluid velocity:

$$\partial E/\partial t+\nabla \mathbf{Q}=0, \quad \partial g_i/\partial t+\nabla_k \Pi_{ik}=0,$$
$$\partial \rho/\partial t+\nabla \mathbf{g}=0, \quad \partial \mathbf{v}_s/\partial t+\nabla \Phi=0. \tag{1}$$

By virtue of the Galilean invariance the mass flux density is the same as the momentum density so that the problem of the search for the hydrodynamical equations reduces to the construction of the energy flux density $\mathbf{Q}$, the stress tensor Π_{ik} and the potential Φ for the superfluid velocity. To do this it is necessary to know the way the entropy density S depends on the variables enumerated above. The differential of the entropy density S has the form

$$TdS=dE-\mu d\rho-\mathbf{v}_n d\mathbf{g}-\mathbf{j}d\mathbf{v}_s. \tag{2}$$

Here T is the temperature, μ the chemical potential, and $\mathbf{v}_n$ the normal velocity. By virtue of the Galilean invariance

$$\mathbf{j}=\mathbf{g}-\rho\mathbf{v}_n. \tag{3}$$

For the pressure P we have the following equation:

$$P=\mu\rho+TS+\mathbf{v}_n\mathbf{g}-E. \tag{4}$$

The dissipationless terms in Eqs. (1) have the following form:

$$\mathbf{Q}_r=(P+E)\mathbf{v}_n+\mu\mathbf{j}, \quad \Phi_r=\mu+v_s v_n,$$
$$\Pi_{rik}=P\delta_{ik}+v_{nk}g_i+j_k v_{si}. \tag{5}$$

Moreover,[2] there are dissipative terms in Eqs. (1) which can be written as follows:

$$\mathbf{Q}_d=-\varkappa\nabla T-\beta_1(\nabla\mathbf{v}_n)-\beta_2(\nabla\mathbf{j}),$$
$$\Phi_d=-\beta_2\nabla\ln T-\zeta_3\nabla\mathbf{j}-\zeta_1\nabla\mathbf{v}_n,$$
$$\Pi_{dik}=-\eta(\nabla_k v_{ni}+\nabla_i v_{nk}-{}^2/_3\delta_{ik}\nabla\mathbf{v}_n)$$
$$-\delta_{ik}(\beta_1\nabla\ln T+\zeta_1\nabla\mathbf{j}+\zeta_2\nabla\mathbf{v}_n). \tag{6}$$

Here η is the first viscosity coefficient, ζ the second viscosity coefficient, and $\varkappa$ the thermal conductivity coefficient. Moreover, we took into account the β coefficients which in the usual case are small, as they are proportional to the relative velocity. However, we retained these coefficients, bearing in mind the fluctuation corrections considered below.

Up to terms of third order in the relative velocity $\mathbf{w}=\mathbf{v}_n-\mathbf{v}_s$ the expansion of the energy density has the form

$$E=\frac{\rho_s}{\rho_n}\left(\frac{1}{2}\rho v_s^2-\mathbf{v}_s\mathbf{g}\right)+\frac{1}{2\rho_n}g^2+E_0. \tag{7}$$

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Here $\rho_s = \rho - \rho_n$ is the superfluid mass density, ρ_n and E_0 are functions of ρ and $\sigma = S/\rho$. In this approximation

$$\mathbf{j} = -\rho_s \mathbf{w}, \quad \mathbf{g} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s. \tag{8}$$

Up to first order in $\mathbf{w}$ the temperature and pressure can be expressed in terms of E_0 as follows:

$$T_0 = \frac{1}{\rho}\left(\frac{\partial E_0}{\partial \sigma}\right)_\rho, \quad P_0 = \rho\left(\frac{\partial E_0}{\partial \rho}\right)_\sigma - E_0. \tag{9}$$

In the same approximation the specific heat at constant pressure is

$$C_P = \frac{1}{\rho}\left(\frac{\partial S}{\partial \ln T_0}\right)_{P_0}. \tag{10}$$

It will be convenient for us to introduce the following dimensionless parameters

$$\gamma = -\left(\frac{\partial \ln \rho}{\partial \ln \sigma}\right)_{P_0}, \quad z = -\left(\frac{\partial \ln \sigma}{\partial \ln \rho}\right)_{T_0}. \tag{11}$$

The first of them, $\gamma \sim 10^{-2}$, is small by virtue of the fact that the thermodynamic quantities of He II depend weakly on the temperature. With the same parameter is also connected the anomalous smallness of the thermal expansion coefficient of He II:

$$\partial \ln \rho / \partial T \approx -C_P \gamma / \sigma T. \tag{12}$$

In what follows we shall take γ to be the small expansion parameter. Taking the smallness of γ into account the expressions for the velocities of first, c_1, and second, c_2, sound in He II have the form[2]

$$c_1^2 = \left(\frac{\partial P_0}{\partial \rho}\right)_\sigma, \quad c_2^2 = \frac{\rho_s}{\rho_n}\frac{T\sigma^2}{C_P}. \tag{13}$$

We shall consider temperatures $\gtrsim 1$ K for which the ratio $c_2^2/c_1^2 \sim 10^{-2}$, which is also connected with the smallness of γ. We note that Eq. (13) for c_1 is true up to terms in γ^2.

We shall express the answer in terms of the following dimensionless parameters:

$$\begin{gathered} y_1 = 1 - \partial \ln \rho_n / \partial \ln \rho, \quad y_2 = \partial \ln \rho_n / \partial \ln \sigma, \\ \varphi_1 = \partial \ln c_1 / \partial \ln \rho, \quad \varphi_2 = \partial \ln c_1 / \partial \ln \sigma, \\ \varphi_3 = \partial \ln C_P / \partial \ln \rho, \quad \varphi_4 = \partial \ln C_P / \partial \ln \sigma. \end{gathered} \tag{14}$$

We also introduce their linear combinations:

$$\begin{gathered} \theta_1 = \varphi_1 + \frac{\sigma}{C_P}(1-z), \quad \theta_2 = 1 + \frac{C_P}{\sigma}\left(1 - \frac{\rho}{\rho_s} y_2\right), \\ \theta_3 = \frac{1}{2}\left(\frac{\rho}{\rho_s} y_1 - \varphi_3 - \frac{\sigma}{C_P} z\right), \\ \theta_4 = \frac{1}{2}\left(\frac{\rho}{\rho_s}(y_1 + y_2) - 2 + \frac{\sigma}{C_P}(1-z) + \varphi_4 - \varphi_3\right), \\ \theta_5 = y_1 + y_2 - 1 + \frac{\sigma}{C_P}(1-z), \quad \theta_6 = \frac{1}{2}\left(\frac{\rho}{\rho_s} y_1 + \varphi_3 - \frac{\sigma}{C_P} z\right), \\ \theta_7 = \frac{1}{2}\left(\frac{\sigma}{C_P}(1-z) - \varphi_4 + \varphi_3 + \frac{\rho}{\rho_s} y_1 + \frac{\rho_s - \rho_n}{\rho_s} y_2\right), \\ \theta_8 = \frac{\rho}{\rho_s} y_1 + \frac{\rho}{\rho_s} y_2 - 1 + \frac{\sigma}{C_P}(1-z), \quad \theta_9 = \frac{\sigma}{C_P}(1-z) - \varphi_3 - 1 + \varphi_4, \\ \theta_{10} = \frac{1}{2}\left(\frac{\rho_s}{\rho} + 2\frac{C_P}{\sigma} - \frac{C_P}{\sigma}\left(3 + \frac{\rho_n}{\rho}\right) y_2\right), \\ \theta_{11} = \frac{1}{2}\left(1 + \frac{\rho_n}{\rho}\right)\left(y_1 + z y_2 + \frac{\rho_s}{\rho_n}\right), \\ \theta_{12} = \frac{\rho_s}{\rho z} + \frac{C_P}{\sigma z}(1 - y_2), \quad \theta_{13} = \varphi_3 - \varphi_4 - \frac{2}{3}\frac{\rho_n}{\rho_s} y_2, \\ \theta_{14} = \frac{\rho}{\rho_s} y_1 - \frac{\sigma}{C_P} z + \frac{2}{3}, \quad \theta_{15} = \frac{\sigma}{C_P}(1-z) + \frac{\rho}{\rho_s} y_1 + \left(1 + \frac{\rho_n}{3\rho_s}\right) y_2. \end{gathered} \tag{15}$$

We note that none of the parameters introduced are small of order γ, except $\varphi_2 \sim \gamma$.

FLUCTUATION CORRELATORS

We shall look for equations for the mass, energy, and momentum densities and for the superfluid velocity taking long-wavelength fluctuations into account. We shall denote averages over them through the brackets $\langle \cdots \rangle$. We shall denote by ρ, E, $\mathbf{g}$, and $\mathbf{v}_s$ average quantities and by the letter δ their fluctuations. Thus, by definition

$$\langle \delta\rho \rangle = \langle \delta E \rangle = \langle \delta \mathbf{g} \rangle = \langle \delta \mathbf{v}_s \rangle = 0.$$

This equation guarantees us the natural conditions under which the integrals of ρ, E, and $\mathbf{g}$ have the meaning of the total mass, energy, and momentum and also curl $\mathbf{v}_s = 0$.

To find the fluctuation corrections to Φ, $\mathbf{Q}$, and Π_{ik} we must expand them in the fluctuations $\delta\rho$, δE, $\delta\mathbf{g}$, and $\delta\mathbf{v}_s$, and average. When averaged the linear terms vanish and the quadratic one gives the required fluctuations. By virtue of the assumed smallness of the wavevector k we shall neglect the dissipative terms in the expansions of Φ, $\mathbf{Q}$, and Π_{ik}. We find from (5)

$$\begin{aligned} \mathbf{Q}_{\text{fl}} ={}& \frac{c_1^2}{\rho}\langle \delta\rho\, \delta\mathbf{g} \rangle + \frac{T}{\rho_n}\left[\frac{\rho_n}{\rho}\frac{\sigma}{C_P}\left(z + \frac{\rho_s}{\rho_n}\right) + 1 - y_2\right]\langle \delta S\, \delta\mathbf{g} \rangle \\ &+ \frac{\rho_s}{\rho_n}\sigma T\left[\frac{\rho}{\rho_s}(y_1 + y_2) - \frac{\sigma}{C_P}(1-z)\right]\langle \delta\rho\, \delta\mathbf{v}_s \rangle \\ &- T\frac{\rho_s}{\rho_n}\left(1 + \frac{\sigma}{C_P} - \frac{\rho}{\rho_s} y_2\right)\langle \delta S\, \delta\mathbf{v}_s \rangle, \\ \Phi_{\text{fl}} ={}& \frac{c_1^2}{2\rho^2}\left(2\varphi_1 - 1 + \frac{\sigma}{C_P}(1-z)\right)\langle \delta\rho^2 \rangle + \frac{\gamma c_1^2}{2S^2 z}(\varphi_4 - \varphi_3 - 1)\langle \delta S^2 \rangle \\ &+ \frac{\gamma c_1^2}{\rho S}\left(\frac{2\varphi_2}{\gamma} - 2 - \frac{2\varphi_3}{z} + \frac{2}{z} - \frac{\varphi_4}{z} - \frac{\sigma}{C_P} z\right)\langle \delta\rho\, \delta S \rangle \\ &+ \frac{1}{2\rho\rho_n}\theta_5 \langle \delta\mathbf{g}^2 \rangle + \frac{1}{\rho_n}\left(-\theta_5 + \frac{\rho_n}{\rho}\frac{\sigma}{C_P}(1-z)\right)\langle \delta\mathbf{g}\, \delta\mathbf{v}_s \rangle \\ &+ \frac{\rho_s}{2\rho_n}\left[\frac{\rho}{\rho_s}(y_1 + y_2) - 1 + \frac{\sigma}{C_P}(1-z)\right]\langle \delta\mathbf{v}_s^2 \rangle, \\ P_{\text{fl}} ={}& \frac{y_1}{2\rho_n}\langle (\delta\mathbf{g} - \rho\delta\mathbf{v}_s)^2 \rangle + \left(\varphi_1 - \frac{1}{2}\frac{\sigma}{C_P} z\right)\frac{c_1^2}{\rho}\langle \delta\rho^2 \rangle \\ &- \frac{T\varphi_3}{2C_P\rho}\langle \delta S^2 \rangle + \frac{\gamma c_1^2}{S}\left(-1 + \frac{2\varphi_2}{\gamma} - \frac{\sigma}{C_P} z + \frac{\varphi_3}{z}\right)\langle \delta\rho\, \delta S \rangle \\ &- \frac{z\sigma}{2\rho_n C_P}\langle \delta\mathbf{g}^2 \rangle + \frac{z\sigma}{C_P}\frac{\rho_s}{\rho_n}\langle \delta\mathbf{g}\, \delta\mathbf{v}_s \rangle - \frac{z\sigma}{C_P}\frac{\rho}{2}\frac{\rho_s}{\rho_n}\langle \delta\mathbf{v}_s^2 \rangle, \\ \Pi_{ik\,\text{fl}} ={}& P_{\text{fl}}\delta_{ik} + \frac{1}{\rho_n}\langle \delta g_i\, \delta g_k \rangle - \frac{\rho_s}{\rho_n}\langle \delta g_i\, \delta v_{sk} + \delta v_{sk}\delta g_i \rangle + \rho\frac{\rho_s}{\rho_n}\langle \delta v_{si}\, \delta v_{sk} \rangle. \end{aligned} \tag{16}$$

We have introduced in these formulae instead of δE a more convenient quantity—the fluctuation in the entropy density[1)] δS.

The fluctuation corrections to the transfer terms are thus determined by the binary correlators $(\delta\rho, \delta S, \delta\mathbf{v}_s, \delta\mathbf{g})$, and we shall denote this set by a single symbol a_α. We introduce the following notation for the matrix of binary correlators:

$$A_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) = \langle a_\alpha(\mathbf{r}_1), a_\beta(\mathbf{r}_2) \rangle. \tag{17}$$

As usual, we change to the variables $\mathbf{r} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ and $\mathbf{r}_1 - \mathbf{r}_2$ and Fourier transform with respect to the latter variable, denoting the corresponding wavevector by $\mathbf{q}$. We understand in what follows by A the function $A(\mathbf{r}, \mathbf{q})$. As regards the corrections to the hydrodynamic equations, it follows from (16) that they are determined by the matrix (17) with coincident spatial variables:

$$A(\mathbf{r},\mathbf{r})=\int\frac{d^3q}{(2\pi)^3}A(\mathbf{r},\mathbf{q})=\int d\tau\,A(\mathbf{r},\mathbf{q}). \tag{18}$$

We consider the equilibrium expression for the matrix A_0, the components of which we shall list as follows:

$$A_0(\mathbf{r},\mathbf{q})=\begin{pmatrix} B_1 & B_2 \\ B_2^{\tau} & B_3 \end{pmatrix}. \tag{19}$$

The matrix B_1 refers here to the correlators of the scalar quantities $(\delta\rho, \delta S)$, its components are scalars; the matrix B_3 refers to the correlators of vector quantities $(\delta\mathbf{v}_s, \delta\mathbf{g})$, its components are second rank tensors; correspondingly the matrix B_2 refers to correlators of scalar and vector quantities, its components are vectors.

It is well known[3] that the classical equilibrium distribution of fluctuations is given by the function

$$\exp\left[-\int d^3r(\Delta E-\mu\Delta\rho-T\Delta S-\mathbf{v}_n\Delta\mathbf{g})T^{-1}\right].$$

In the quadratic approximation this distribution function gives the following equilibrium expression for the matrix B (written down up to terms linear in the velocities):

$$B_1=\begin{pmatrix} \dfrac{\rho T}{c_1^2} & \dfrac{\rho T\sigma}{c_1^2}(1-z) \\ \dfrac{\rho T\sigma}{c_1^2}(1-z) & \rho C_P \end{pmatrix},\quad B_3=\begin{pmatrix} \dfrac{T}{\rho_s}\delta_{ik}^{\parallel} & T\delta_{ik}^{\parallel} \\ T\delta_{ik}^{\parallel} & T\rho_s\delta_{ik}^{\parallel}+T\rho_n\delta_{ik} \end{pmatrix},$$

$$B_2=\begin{pmatrix} \dfrac{T}{c_1^2}\left(1+\dfrac{\rho_n}{\rho_s}(y_1+zy_2)\right)\mathbf{w}_{\parallel} & \dfrac{T}{c_1^2}\left(\rho\mathbf{v}_n-\rho_s\mathbf{w}_{\perp}\left(1+\dfrac{\rho_n}{\rho_s}(y_1+zy_2)\right)\right) \\ -\dfrac{C_P}{\sigma}\dfrac{\rho_n}{\rho_s}y_2\mathbf{w}_{\parallel} & \dfrac{\rho_n C_P}{\sigma}y_2\mathbf{w}_{\perp}+\dfrac{\gamma\rho}{z}\dfrac{C_P}{\sigma}(1-z)\mathbf{v}_s \end{pmatrix}. \tag{20}$$

We have introduced here the following notation:

$$\delta_{ik}^{\parallel}=\frac{q_iq_k}{q^2},\quad \delta_{ik}^{\perp}=\delta_{ik}-\frac{q_iq_k}{q^2},\quad w_{i\parallel}=\delta_{ik}^{\parallel}w_k,\quad w_{i\perp}=\delta_{ik}^{\perp}w_k. \tag{21}$$

EQUATION FOR THE CORRELATORS

We now consider the equation of motion for the matrix A. Expanding (5), (6) up to first order terms and changing to the variation in the entropy density we find the equation

$$\frac{\partial a_\alpha}{\partial t}+\nabla_j(M_{j\alpha\beta}a_\beta)=\nabla_j(\Gamma_{jk\alpha\beta}\nabla_k a_\beta)+f_\alpha. \tag{22}$$

Here the f_α are the random forces and in the dissipative term we have dropped terms containing the gradient of Γ which are unimportant for what follows. The matrices $M_{j\alpha\beta}$ and $\Gamma_{jk\alpha\beta}$ have the following form:

$$M_{j\alpha\beta}=\begin{pmatrix} 0 & 0 & 0 & \delta_{jn} \\ m_{21j} & (1-y_2)w_j+v_{sj} & -S\rho_s\delta_{jn}/\rho_n & S\delta_{jn}/\rho_n \\ \dfrac{c_1^2}{\rho}\delta_{ji} & c_1^2\gamma\left(1-\dfrac{1}{z}\right)\delta_{ji}/\rho\sigma & m_{33jin} & m_{34jin} \\ c_1^2\delta_{ji} & \gamma c_1^2\delta_{ji}/\sigma & m_{43jin} & m_{44jin} \end{pmatrix}, \tag{23}$$

where

$$m_{21j}=(y_1+y_2-1)\sigma w_j-\sigma\frac{\rho}{\rho_n}v_{sj},\quad m_{33jin}=(v_{sn}+w_n(1-y_1-y_2))\delta_{ij},$$

$$m_{34jin}=\frac{1}{\rho}w_n(y_1+y_2-1)\delta_{ij},\quad m_{43jin}=-\delta_{ij}\rho w_ny_1-\rho_s(\delta_{nj}w_i+\delta_{ni}w_j),$$

$$m_{44jin}=\delta_{kj}v_{si}+\delta_{ni}v_{sj}+\delta_{ij}w_ny_1+\delta_{nj}w_i+\delta_{ni}w_j; \tag{24}$$

$$\Gamma_{jk\alpha\beta}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ \varkappa\sigma z\delta_{jk}/C_P\rho & \varkappa\delta_{jk}/\rho C_P & 0 & 0 \\ 0 & 0 & \Gamma_{33jkni} & \Gamma_{34jkni} \\ 0 & 0 & \Gamma_{43jkni} & \Gamma_{44jkni} \end{pmatrix},$$

with

$$\Gamma_{33jkni}=\frac{\rho_s}{\rho_n}(\zeta_3\rho-\zeta_4)\delta_{ji}\delta_{kn},\quad \Gamma_{34jkni}=\frac{1}{\rho_n}(\zeta_4-\rho_s\zeta_3)\delta_{ji}\delta_{kn},$$

$$\Gamma_{43jkni}=\frac{\rho_s}{\rho_n}\left(\rho\zeta_1-\zeta_2-\frac{4}{3}\eta\right)\delta_{ji}\delta_{kn},$$

$$\Gamma_{44jkni}=\frac{\eta}{\rho_n}\delta_{jk}\delta_{in}+\frac{1}{\rho_n}\left(\zeta_2-\rho_s\zeta_1+\frac{1}{3}\eta\right)\delta_{ji}\delta_{kn}.$$

The components of a are here numbered as follows:

$$a_\alpha=(\delta\rho,\ \delta S,\ \delta v_{si},\ \delta g_i),\quad a_\beta=(\delta\rho,\ \delta S,\ \delta v_{sn},\ \delta g_n).$$

To find the equation for A we differentiate the matrix (17) with respect to time and use (22) to replace the time-derivatives under the averaging sign. Changing then to Fourier components with respect to q and expanding the equation obtained up to first order in k/q we find

$$\frac{\partial A}{\partial t}+iq_jM_jA-iq_iAM_i^{\tau}+\frac{1}{2}\nabla_iM_iA+\frac{1}{2}\nabla_iAM_i^{\tau}-\frac{1}{2}q_j\nabla_iM_j\frac{\partial}{\partial q_i}A$$
$$-\frac{1}{2}q_j\frac{\partial}{\partial a}A\nabla_iM_j^{\tau}=-q_jq_k\Gamma_{jk}\delta A-q_jq_k\delta A\Gamma_{jk}^{\tau}. \tag{25}$$

Here $\delta A=A-A_0$ is the deviation from the local equilibrium value. For the correlator $\langle f_\alpha, a_\beta\rangle$ we took the local function which guarantees the relaxation of the matrix A to its equilibrium value.

We linearize Eq. (25) with respect to δA and transfer all terms with δA to the left and the others to the right-hand side. The deviation δA is small inasfar as deviations from equilibrium in macroscopic motions are small. In the linear case in which we are interested the coefficient of δA must thus be taken to be equal to their equilibrium (homogeneous) values. Assuming that the macroscopic motion occurs with wavevector k and frequency ω we find

$$i\omega\delta A-q_jq_k\Gamma_{jk}\delta A-q_jq_k\delta A\Gamma_{jk}^{\tau}-iq_j(M_j\delta A-\delta AM_j^{\tau})-\frac{1}{2}M_jk_j\delta A$$
$$-\frac{1}{2}\delta Ak_jM_j^{\tau}=\frac{\partial}{\partial t}A_0+\frac{1}{2}\nabla_j(M_jA_0)+\frac{1}{2}\nabla_j(A_0M_j^{\tau})$$
$$-\frac{1}{2}q_j\nabla_iM_j\frac{\partial}{\partial q_i}A_0-\frac{1}{2}q_j\frac{\partial}{\partial q_i}A_0\nabla_iM_j^{\tau}. \tag{26}$$

We now perform a similarity transformation of Eq. (26), multiplying it from the left by the matrix Ξ and from the right by the matrix Ξ^{T}. The left-hand side then retains its form except for the substitution

$$\delta A\to\Xi\delta A\Xi^{\tau},\quad M\to\Xi M\Xi^{-1},\quad \Gamma\to\Xi\Gamma\Xi^{-1}.$$

We choose the matrix Ξ corresponding to the change from a_α to normal coordinates:[4]

$$\Xi=\begin{pmatrix} \left(\dfrac{c_1}{2\rho q}\right)^{1/2} & \dfrac{\gamma}{\sigma}\left(\dfrac{c_1}{2\rho q}\right)^{1/2} & 0 & (2\rho c_1q)^{-1/2}\dfrac{\mathbf{q}}{q} \\ \left(\dfrac{c_1}{2\rho q}\right)^{1/2} & \dfrac{\gamma}{\sigma}\left(\dfrac{c_1}{2\rho q}\right)^{1/2} & 0 & -(2\rho c_1q)^{-1/2}\dfrac{\mathbf{q}}{q} \\ -\left(\dfrac{\rho_n}{\rho_s}\dfrac{c_2}{2\rho q}\right)^{1/2} & \dfrac{1}{\sigma}\left(\dfrac{\rho_n}{\rho_s}\dfrac{c_2}{2\rho q}\right)^{1/2} & -\left(\dfrac{\rho_s}{\rho_n}\dfrac{\rho}{2qc_2}\right)^{1/2}\dfrac{\mathbf{q}}{q} & \left(\dfrac{\rho_s}{2\rho_n\rho qc_2}\right)^{1/2}\dfrac{\mathbf{q}}{q} \\ -\left(\dfrac{\rho_n}{\rho_s}\dfrac{c_2}{2\rho q}\right)^{1/2} & \dfrac{1}{\sigma}\left(\dfrac{\rho_n}{\rho_s}\dfrac{c_2}{2\rho q}\right)^{1/2} & \left(\dfrac{\rho_s}{\rho_n}\dfrac{\rho}{2qc_2}\right)^{1/2}\dfrac{\mathbf{q}}{q} & -\left(\dfrac{\rho_s}{2\rho_n\rho qc_2}\right)^{1/2}\dfrac{\mathbf{q}}{q} \\ 0 & 0 & 0 & \delta_{\perp} \end{pmatrix}. \tag{27}$$

The first two rows correspond to first sound, the second two to second sound, and the last one to vortex oscillations. The transformation by means of the matrix (27) diagonalizes the matrix M:

$$\operatorname{diag}(\Xi M\Xi^{-1})=q^{-1}(c_1\mathbf{q}, -c_1\mathbf{q},\ c_2\mathbf{q}, -c_2\mathbf{q},\ 0). \tag{28}$$

The structure of Eq. (28) shows that on the left-hand side of the transformed Eq. (26) we must have the coefficients $c_1 q, c_2 q$ in front of the off-diagonal terms of $\Xi\delta A\Xi^T$. We shall assume that in the essential integration region we have the inequality

$$c_2 q \gg \Gamma q^2. \tag{29}$$

We shall give below the condition which this imposes on the frequency ω. It follows from the inequality $c_1 > c_2$ that a relation similar to (29) occurs also for the first sound velocity. Moreover, we assume that $q \gg k \sim \omega/c$ so that on the left-hand side of the equation the coefficients in front of the off-diagonal elements of $\Xi\delta A\Xi^T$ are much larger than those in front of the diagonal ones. As all component on the right-hand side of Eq. (26) are of the same order we reach the conclusion that we can neglect the off-diagonal terms in $\Xi\delta A\Xi^T$ when evaluating the fluctuation corrections.

Therefore the only important terms are

$$\operatorname{diag}\Xi\delta A\Xi^T = (\delta n_1(\mathbf{q}),\ \delta n_1(-\mathbf{q}),\ \delta n_2(\mathbf{q}),\ \delta n_2(-\mathbf{q}),\ \delta f_{ik}(\mathbf{q})). \tag{30}$$

The diagonal terms are written down here, taking into account the structure of Ξ, δn_1 has the meaning of the distribution function of the first sound fluctuations, δn_2 that of the second sound fluctuations, and δf that of the distribution function of the vortex fluctuations.

We now proceed to solving Eq. (26). We are interested in the linear regime so that we must split off on the right-hand side of (26) the terms linear in the deviations (in the macroscopic motion) which is done starting from Eqs. (20), (23). After this $\nabla \to i\mathbf{k}$; as regards the time-derivative, we replace it taking into account the linearized Eqs. (1) in which we take into account only the dissipationless contributions (5) as we assume that not only the contribution from the fluctuation terms (16), but also those from the dissipative terms (6) is small, which is guaranteed by inequality (29). After this we must perform with the right-hand side of (26) the same procedure as with the left-hand side, i.e., multiplying from the left by Ξ and from the right by Ξ^T. By virtue of the earlier indicated structure of $\Xi\delta A\Xi^T$ we shall be interested only in the diagonal terms in the right-hand side thus obtained.

Using the structure of the matrix $\Xi M\Xi^{-1}$ and also performing the transformation $\Gamma \to \Xi\Gamma\Xi^{-1}$ we find that the diagonal terms of the transformed Eq. (26) have the following form:

$$(\omega - c_1 k_\parallel + 2i\Gamma_1 q^2)\delta n_1 = \frac{T}{c_1 q}\Big[c_1 k_\parallel \delta \ln T - \Big(\varphi_1 - \frac{z\sigma}{C_P}\Big)\mathbf{k}\mathbf{v}_n + v_{n\parallel}k_\parallel + \frac{1}{\rho}\theta_1 \mathbf{k}\mathbf{j}\Big], \tag{31}$$

$$(\omega - c_2 k_\parallel + 2i\Gamma_2 q^2)\delta n_2 = \frac{T}{c_2 q}\Big(c_2 k_\parallel \delta \ln T + \theta_3 \mathbf{k}\mathbf{v}_n + v_{n\parallel}k_\parallel + \frac{1}{\rho}\theta_4 \mathbf{k}\mathbf{j}\Big), \tag{32}$$

$$\Big(\omega + 2i\frac{\eta}{\rho_n}q^2\Big)\delta f_{ik} = T\delta_{\perp ik}\Big[\Big(y_1 - \frac{z\sigma}{C_P}\Big)\mathbf{k}\mathbf{v}_n + \frac{1}{\rho}\theta_5 \mathbf{k}\mathbf{j}\Big] + T(v_{n\perp i}k_{\perp k} + k_{\perp i}v_{n\perp k}). \tag{33}$$

In these formuale occur the quantities Γ_1 and Γ_2 which determined the first and second sound damping and which are linear combinations of kinetic coefficients (we give below the explicit expression).

It is well known[4] that the dissipationless hydrodynamical equations of He II can be written in canonical form. Hence it follows in correspondence with Appendix I that the equations for the distribution functions of the fluctuations have the form of the usual kinetic Eqs. (A23).[2)] The dispersion laws for first and second sound must be taken in the approximation which is linear in the relative velocity:[5]

$$\lambda_1 = c_1 q + \rho_n \mathbf{v}_n \mathbf{q}/\rho + \rho_s \mathbf{v}_s \mathbf{q}/\rho,$$
$$\lambda_2 = c_2 q + \mathbf{v}_n \mathbf{q}(1 + \rho_s/\rho - y_2) + \mathbf{v}_s \mathbf{q}(y_2 - \rho_s/\rho). \tag{34}$$

Linearizing Eqs. (A23) and using (34) and the equilibrium expressions

$$n_{10} = \frac{T}{\lambda_1 - \mathbf{v}_n \mathbf{q}},\quad n_{20} = \frac{T}{\lambda_2 - \mathbf{v}_n \mathbf{q}} \tag{35}$$

leads to the same Eqs. (31), (32).

CORRECTIONS TO THE HYDRODYNAMICAL EQUATIONS

Equation (30) for $\Xi\delta A\Xi^T$ enables us to express the fluctuation corrections to the hydrodynamical equations in terms of the distribution functions for the first and second sound and the vortex motion fluctuations. Multiplying (30) from the left by Ξ^{-1} and from the right by $(\Xi^T)^{-1}$ we can find δA after which we find, using (16), (18),

$$\Phi_{\mathrm{fl}} = \frac{1}{\rho}\Big(\theta_1 N_1 + \theta_4 N_2 + \frac{1}{2}\theta_5 F\Big),\quad \mathbf{Q}_{\mathrm{fl}} = c_1 \mathbf{N}_1 + c_2\theta_2 \mathbf{N}_2,$$
$$\Pi_{\mathrm{fl}ik} = \Big[\Big(\varphi_1 - \frac{\sigma}{C_P}z\Big)N_1 + \theta_3 N_2 + \frac{1}{2}\Big(y_1 - \frac{\sigma}{C_P}z\Big)F\Big]\delta_{ik} + N_{1ik} + N_{2ik} + F_{ik}. \tag{36}$$

We have introduced here the following notation for the integrals:

$$N_1 = \int d\tau\, q c_1 \delta n_1,\quad N_2 = \int d\tau\, q c_2 \delta n_2,$$
$$\mathbf{N}_1 = \int d\tau\, \mathbf{q} c_1 \delta n_1,\quad \mathbf{N}_2 = \int d\tau\, \mathbf{q} c_2 \delta n_2,$$
$$N_{1ik} = \int d\tau \frac{q_i q_k}{q} c_1 \delta n_1,\quad N_{2ik} = \int d\tau\, q_i \frac{q_k}{q} c_2 \delta n_2, \tag{37}$$
$$F = \int d\tau\, \delta f_{ii},\quad F_{ik} = \int d\tau\, \delta f_{ik}.$$

To evaluate the integrals (37) we must substitute the distribution functions (31) to (33) into the integrands. We then get formally divergent integrals[3)] but their divergent part gives an unimportant renormalization of the transfer terms which is independent of $\omega, \mathbf{k}$. We shall therefore only be interested in the pole part of the integrals (37) which gives a nontrivial $\omega, \mathbf{k}$ dependence of the transfer terms (36). We note that if the usual kinetic coefficients are small by virtue of being proportional to the relative velocity, the fluctuation corrections to β have the same order as the other corrections, as by virtue of the non-local nature β can be directed along the unique vector $\mathbf{k}/k$: $\vec{\beta} = \bar{\beta}\mathbf{k}/k$.

As a result of the calculations described above we get the following expressions:

$$\bar{\beta}_2 = \frac{i-1}{16\pi\rho}T\omega^{1/2}\Big(\frac{1}{\Gamma_1^{3/2}}c_1\theta_1 J_2' + \frac{1}{\Gamma_2^{3/2}}c_2\theta_2 J_2''\Big), \tag{38}$$

$$\bar{\beta}_1 = \frac{i-1}{16\pi}T\omega^{1/2}\Big[\frac{c_1}{\Gamma_1^{3/2}}\Big(\Big(\varphi_1 - z\frac{\sigma}{C_P}\Big)J_2' + J_4'\Big) + \frac{c_2\theta_2}{\Gamma_2^{3/2}}(\theta_3 J_2'' + J_4'')\Big], \tag{39}$$

$$\tilde{\varkappa} = \frac{i-1}{16\pi}\omega^{1/2}\Big(\frac{c_1^2}{\Gamma_1^{3/2}}J_3' + \frac{c_2^2\theta_2^2}{\Gamma_2^{1/2}}J_3''\Big), \tag{40}$$

$$\tilde{\eta}=\frac{i-1}{16\pi}\omega^{1/2}T\left(\frac{7}{15}\frac{\rho_n^{3/2}}{\eta^{3/2}}+\frac{J_3'-J_5'}{2\Gamma_1^{3/2}}+\frac{J_3''-J_5''}{2\Gamma_2^{3/2}}\right), \tag{41}$$

$$\rho\tilde{\zeta}_1=\frac{i-1}{16\pi}\omega^{1/2}T\left(\frac{\theta_1}{\Gamma_1^{3/2}}\left(\left(\varphi_1-z\frac{\sigma}{C_P}\right)J_1'+J_3'\right)+\frac{\theta_4}{\Gamma_2^{3/2}}(\theta_3J_1''+J_3'')+\frac{\rho_n^{3/2}}{\eta^{3/2}}\left(y_1-z\frac{\sigma}{C_P}+\frac{2}{3}\right)\theta_5\right), \tag{42}$$

$$\tilde{\zeta}_2=\frac{i-1}{16\pi}\omega^{1/2}T\left[\frac{1}{\Gamma_1^{3/2}}\left(\left(\varphi_1-z\frac{\sigma}{C_P}\right)^2J_1'+2\left(\varphi_1-z\frac{\sigma}{C_P}-\frac{1}{3}\right)J_3'+\frac{5}{3}J_5'\right)+\frac{1}{\Gamma_2^{3/2}}\left(\theta_3^2J_1''+2\left(\theta_3-\frac{1}{3}\right)J_3''+\frac{5}{3}J_5''\right)+\left(y_1-z\frac{\sigma}{C_P}+\frac{2}{3}\right)^2\left(\frac{\rho_n}{\eta}\right)^{3/2}\right], \tag{43}$$

$$\rho^2\tilde{\zeta}_3=\frac{i-1}{16\pi}\omega^{1/2}T\left[\frac{1}{\Gamma_1^{3/2}}\theta_1^2J_1'+\frac{1}{\Gamma_2^{3/2}}\theta_4^2J_1''+\frac{\rho_n^{3/2}}{\eta^{3/2}}\theta_5^2\right]. \tag{44}$$

The functions $J(x)$ are introduced in Appendix II, and we have denoted by a single prime functions with argument c_1k/ω and by two primes functions of argument c_2k/ω. We note that all corrections to the kinetic coefficients consist of three terms referring, respectively, to first sound (with Γ_1 in the denominator), to second sound (with Γ_2 in the denominator) and to vortex fluctuations (with η/ρ_n in the denominator).

If we consider a sound wave as the macroscopic motion we have $\omega\sim ck$. It follows in that case from (31) to (33) that the integration in the integrals (37) is over a region with characteristic

$$q\sim\left(\frac{c_1k}{\Gamma_1}\right)^{1/2},\left(\frac{\omega}{\Gamma_2}\right)^{1/2},\left(\frac{\omega\rho_n}{\eta}\right)^{1/2}.$$

In the temperature range considered, $\gtrsim 1$ K, $\Gamma_1\sim\Gamma_2\sim\eta/\rho_n$ so that the inequality (29) can be rewritten in the form

$$c_2^2\gg\Gamma_2\omega, \tag{45}$$

which is the same as the condition that second sound is weakly damped at the frequency ω. The same inequality (45) guarantees the satisfying of the necessary condition[4] $q\gg k$.

FLUCTUATION CORRECTIONS NEAR THE λ-POINT

Above we considered that the condition $\omega\ll c_1q, c_2q$ was satisfied. However, when we approach the λ-point the second sound velocity decreases which is connected with the decrease in ρ_s and the intermediate region

$$c_2q\ll\omega\ll c_1q$$

therefore becomes larger and more interesting for us to study. We shall study this region in the present section. The characteristic q which occur in the integrals are $q\sim(\Gamma/\omega)^{1/2}$ so that the inequality indicated above can be rewritten in the form

$$c_2^2\ll\Gamma\omega\ll c_1^2. \tag{46}$$

This condition means that it is no longer possible to assume that c_2q is a quantity large compared to ω so that in the matrix $\Xi\delta A\Xi^T$ it is necessary to consider many more terms which we write in the form

$$\Xi\delta A\Xi^T=\begin{pmatrix}\delta n_1(\mathbf{q}) & & \\ & \delta n_1(-\mathbf{q}) & \\ & & \begin{matrix}\delta n_{11}\delta n_{12}\delta n_{13k}\\ \delta n_{21}\delta n_{22}\delta n_{23k}\\ \delta n_{31i}\delta n_{32i}\delta f_{ik}\end{matrix}\end{pmatrix}. \tag{47}$$

Performing now the transformation of δA and substituting (16), (18) we find a generalization of (36):

$$\begin{aligned}Q_{fl}&=c_1N_1+c_2\theta_2N_2+\frac{\gamma c_1^2\rho}{\rho_n}\theta_{12}N_3,\\ \Phi_{fl}&=\frac{\theta_1}{\rho}N_1+\frac{\theta_5}{2\rho}F+\frac{\theta_8}{2\rho}N_{22}+\frac{\rho_n}{2\rho\rho_s}\theta_9N_{21},\\ \Pi_{fl\,ik}&=\left[(\varphi_1-\psi)N_1-\frac{1}{3}\varphi_3N_{21}+\frac{1}{2}\left(\frac{\rho}{\rho_s}y_1-\psi\right)N_{22}+\frac{1}{2}(y_1-\psi)F\right]\delta_{ik}+N_{1ik}+N_{2ik}+F_{ik}+N_{4ik}.\end{aligned} \tag{48}$$

We have here introduced the notation

$$\begin{aligned}N_{21}&=\frac{1}{2}\int d\tau\, c_2q(\delta n_{11}+\delta n_{22}+\delta n_{12}+\delta n_{21}),\\ N_{22}&=\frac{1}{2}\int d\tau\, c_2q(\delta n_{11}+\delta n_{22}-\delta n_{12}-\delta n_{21}),\\ \mathbf{N}_2&=\frac{1}{2}\int d\tau\, c_2\mathbf{q}(\delta n_{11}+\delta n_{21}-\delta n_{12}-\delta n_{22}),\\ \mathbf{N}_3&=\int d\tau\left(\frac{\rho_s q}{2\rho_n\rho c_2}\right)^{1/2}(\delta\mathbf{n}_{13}+\delta\mathbf{n}_{22}),\\ N_{2ik}&=\frac{1}{2}\int d\tau\, c_2q\delta_{ik}^{\parallel}(\delta n_{11}+\delta n_{22}-\delta n_{12}-\delta n_{21}),\\ N_{4ik}&=\int d\tau\left(\frac{\rho_s qc_2}{2\rho\rho_n}\right)^{1/2}\left[\frac{q_i}{q}(\delta n_{13k}-\delta n_{23k})+(\delta n_{13i}-\delta n_{23i})\frac{q_k}{q}\right].\end{aligned} \tag{49}$$

Multiplying Eq. (26) from the left by Ξ and from the right by Ξ^T, substituting the expression (47) and using inequality (46) we can find the equations

$$\delta n_{12}=\delta n_{21}, \tag{50}$$

$$(\omega+2iq^2\Gamma_2)(\delta n_{11}-\delta n_{22})=2\frac{k_\parallel}{q}\theta_2\delta T, \tag{51}$$

$$(\omega+2iq^2\Gamma_3)(\delta n_{11}+\delta n_{22}+\delta n_{12}+\delta n_{21})=\frac{2T}{c_2q}\left(-\varphi_3\mathbf{kv}+\frac{\mathbf{kj}}{\rho}(\theta_4-\theta_7)-\frac{\rho_n}{\rho_s}y_2k_\parallel j_\parallel\right), \tag{52}$$

$$(\omega+2iq^2\Gamma_4)(\delta n_{11}+\delta n_{22}-\delta n_{12}-\delta n_{21})=\frac{2T}{c_2q}\left[(\theta_3+\theta_6)\mathbf{kv}_n+2k_\parallel v_{n\parallel}+\frac{1}{\rho}(\theta_4+\theta_7)\mathbf{kj}+\frac{\rho_ny_2}{\rho\rho_s}k_\parallel j_\parallel\right], \tag{53}$$

$$(\omega+2iq^2\Gamma_5)(\delta\mathbf{n}_{12}+\delta\mathbf{n}_{13})=\left(\frac{2\rho_n\rho c_2}{\rho_sq}\right)^{1/2}\left(\theta_{10}\mathbf{k}_\perp\delta T+\theta_{11}\mathbf{k}_\perp\frac{T}{\rho c_1^2}\delta P\right), \tag{54}$$

$$(\omega+2iq^2\Gamma_6)(\delta\mathbf{n}_{12}-\delta\mathbf{n}_{13})=T\left(\frac{2\rho\rho_n}{\rho_sqc_2}\right)^{1/2}\frac{\rho_s}{\rho}\left(2v_{n\parallel}\mathbf{k}_\perp+\frac{1}{\rho}(1+0.5y_2)j_\parallel\mathbf{k}_\perp-\frac{1}{\rho}k_\parallel\mathbf{j}_\perp\right). \tag{55}$$

Here

$$\Gamma_3=\frac{1}{\rho}\frac{\varkappa}{C_P},\quad \Gamma_4=\frac{\rho_s}{\rho\rho_n}\left(\frac{4}{3}\eta+\zeta_2-2\rho\zeta_1+\rho^2\zeta_3\right),\quad \Gamma_5=\frac{1}{2}(\Gamma_3+\eta/\rho_n),\quad \Gamma_6=\frac{1}{2}(\Gamma_4+\eta/\rho_n). \tag{56}$$

We note that the characteristic quantity in Eq. (53) is $q\sim(\omega/\Gamma_4)^{1/2}$ which as is clear from (56), increases in proportion to $(\rho_n/\rho_s)^{3/2}$ as the λ-point is approached, so that the justification of neglecting terms with c_2q requires a more detailed verification which shows that condition (46) is sufficient for neglecting the terms with c_2q which might appear in Eqs. (51) to (53).

Substituting the solutions (50) to (55) into (48), (49) we are able to obtain expressions for the fluctuation fluxes. The integrals (49) are determined by functions given in Appendix II. As a result we obtain for the kinetic coefficients fluctuation contributions which formally do not satisfy Onsager's symmetry principle. However, one can easily restore this symmetry by using the zeroth order equations of motion for the deviations of various quantities from their equilibrium values.[5)] Finally, the fluctuation corrections to the kinetic coefficients have the form

$$
\begin{gathered}
\tilde{\tilde{\beta}}_1=\tilde{\tilde{\beta}}_2=0,\\
\tilde{\varkappa}=\frac{i-1}{24\pi}\frac{\omega^{1/2}}{\Gamma_5^{3/2}}\theta_{12}\gamma\frac{\rho}{\rho_n}\left(c_1^2\theta_{10}+\theta_{11}\frac{\omega^2}{k^2}\frac{C_P}{\sigma z}\right)+c_2^2\theta_2^2\frac{i-1}{48\pi}\frac{\omega^{1/2}}{\Gamma_2^{3/2}},\\
\tilde{\tilde{\zeta}}_1=\frac{T\omega^{1/2}}{32\pi\rho}(i-1)\left(-\frac{\varphi_3\theta_{13}}{\Gamma_3^{3/2}}+\frac{\theta_{14}\theta_{15}}{\Gamma_4^{3/2}}\right),\\
\tilde{\tilde{\zeta}}_2=\frac{T\omega^{1/2}}{32\pi}(i-1)\left(\frac{\varphi_3^2}{\Gamma_3^{3/2}}+\frac{\theta_{14}^2}{\Gamma_4^{3/2}}\right)+\frac{T\omega^{1/2}}{32\pi}(i-1)\\
\times\frac{c_2^2k^2}{\omega^2}\left[\frac{\varphi_3(1+\varphi_3-\varphi_4+\theta_{13})}{\Gamma_3^{3/2}}+\frac{\theta_{14}(\theta_8-\theta_{15})}{\Gamma_4^{3/2}}\right],\\
\tilde{\tilde{\zeta}}_3=\frac{T\omega^{1/2}}{32\pi\rho^2}(i-1)\left[\frac{\theta_{13}(\varphi_4-1-\varphi_3)}{\Gamma_3^{3/2}}+\frac{\theta_8'\theta_{15}}{\Gamma_4^{3/2}}\right],\\
\eta=\frac{T\omega^{1/2}}{40\pi}(i-1)\left(\frac{1}{3\Gamma_4^{3/2}}+\frac{1}{\Gamma_6^{3/2}}\right)+\frac{T\omega^{1/2}}{80\pi}(i-1)\frac{\rho_n}{\rho_s}y_2\left(\frac{1}{3\Gamma_4^{3/2}}+\frac{\rho_s}{2\rho_n\Gamma_6^{3/2}}\right)\frac{c_2^2k^2}{\omega^2}.
\end{gathered}
\tag{57}
$$

We have dropped from these expressions contributions caused by first sound and by vortex fluctuations which have the same form as in (38) to (44).

SOUND WAVES

We now find the corrections to the speed and damping of first and second sound which are caused by fluctuations. We assume here that both the damping of sound waves and the fluctuation corrections to the dispersion law are small so that we may assume that in the zeroth approximation $\omega=c_1k$ and $\omega=c_2k$.

We can find the first and second sound dispersion law expressions most simply from the linearized Eq. (22) by a method known from quantum mechanics. The term with the matrix M which gives the first and second sound velocities plays the role of the zeroth order Hamiltonian in the wave Eq. (22) and the term with the matrix Γ plays the role of the perturbing Hamiltonian. We can find the perturbing terms in the first and second sound dispersion laws as the diagonal components of the kinetic term in the representation in which Mq is diagonal, i.e., in the representation $\Xi\Gamma\Xi^{-1}$. As a result we obtain for the first and second sound dispersion laws corrections characterized by the following coefficients of k^2 in the damping rates:

$$
\begin{gathered}
\Gamma_1=\frac{1}{2\rho}\left[\frac{\sigma^2}{C_P^2}\frac{z}{c_1^2}\varkappa T+\frac{4}{3}\eta+\zeta_2+\gamma\rho\left(\frac{4}{3}\eta_1-\frac{\rho_s}{\rho_n}\zeta_1\right)\right.\\
\left.+\frac{\sigma z}{C_Pc_1}\beta_1-\frac{\gamma^2c_1}{\sigma T}\rho\frac{\rho_s}{\rho_n}\beta_2\right],
\end{gathered}
\tag{58}
$$

$$
\Gamma_2=\frac{1}{2\rho}\left[\frac{\varkappa}{C_P}+\frac{\rho_s}{\rho_n}\left(\frac{4}{3}\eta+\zeta_2+\rho^2\zeta_3-2\rho\zeta_1\right)+2\frac{c_2}{\sigma T}(\beta_1-\rho\beta_2)\right].
\tag{59}
$$

We must distinguish in Eqs. (58), (59) between the contributions from the usual kinetic coefficients and the fluctuation corrections. When we considered above the equations for the matrix of the binary correlators of fluctuating quantities we had in mind Eqs. (58), (59) with the usual kinetic coefficients. We obtain the fluctuation corrections by substituting into (58), (59) the expressions for the fluctuation corrections to the kinetic coefficients.

Putting $\omega=c_1k$ for first and $\omega=c_2k$ for second sound and using the expressions for the integrals J given in Appendix II we can obtain explicit formulae for the coefficients (38) to (44). Substitution of the expressions obtained into (58), (59) yields for the sound dispersion laws corrections which we express in terms of the corrections to the sound speeds and to the imaginary parts of the wavevectors:

$$
\tilde{c}_1=\frac{1}{48\pi}\frac{T}{c_1}\frac{\omega^{3/2}}{\rho\Gamma_1^{3/2}}\varphi,\qquad
\tilde{\alpha}_1=-\frac{1}{48\pi}\frac{T}{c_1^3}\frac{\omega^{5/2}}{\rho\Gamma_1^{3/2}}\varphi;
\tag{60}
$$

$$
\tilde{c}_2=\frac{\omega^{3/2}}{32\pi}\frac{c_1^{3/2}}{c_2^{3/2}C_P\rho\Gamma_1^{3/2}}\left(1-\frac{4}{9}\frac{C_P}{\sigma}\right),\qquad
\tilde{\alpha}_2=-\frac{\omega^{3/2}}{112\pi}\frac{c_1^{3/2}}{c_2^{1/2}C_P\rho\Gamma_1^{3/2}}
\tag{61}
$$

Here

$$
\begin{gathered}
\varphi=\sqrt{2}\left(\frac{211}{1155}+\frac{22}{35}\varphi_1-\frac{92}{105}\frac{\sigma}{C_P}z+\varphi_1^2-\frac{12}{5}\varphi_1\frac{\sigma}{C_P}z+\frac{12}{7}\frac{\sigma^2}{C_P^2}z^2\right)\\
+\left(\frac{\Gamma_1}{\Gamma_2}\right)^{3/2}\left(\frac{3}{2}\theta_3^2+\theta_3+\frac{3}{10}\right)\\
+\frac{\rho_n^{3/2}\Gamma_1^{3/2}}{\eta^{3/2}}\left(\frac{3}{2}\left(y_1-\frac{\sigma}{C_P}z\right)^2+2\left(y_1-\frac{\sigma}{C_P}z\right)+\frac{8}{5}\right).
\end{gathered}
\tag{62}
$$

We note that the corrections to the first sound dispersion law are written down up to terms of orders $(c_2/c_1)^2$ and to the second sound dispersion law up to terms of orders $(c_2/c_1)^{3/2}$.

We can similarly evaluate the corrections to the first sound dispersion law near the λ-point under condition (46). For this it is necessary to substitute Eq. (57). As a result we get the same formalue (60), in which now

$$
\begin{gathered}
\varphi=\sqrt{2}\left(\frac{211}{1155}+\frac{22}{35}\varphi_1-\frac{92}{105}\frac{\sigma}{C_P}z+\varphi_1^2-\frac{12}{5}\varphi_1\frac{\sigma}{C_P}z+\frac{12}{7}\frac{\sigma^2}{C_P^2}z^2\right)\\
+\frac{\rho_n^{3/2}\Gamma_1^{3/2}}{\eta^{3/2}}\left(\frac{3}{2}\left(y_1-\frac{\sigma}{C_P}z\right)^2+2\left(y_1-\frac{\sigma}{C_P}z\right)+\frac{8}{5}\right)\\
+\Gamma_1^{3/2}\left[\frac{3\varphi_3^2}{4\Gamma_3^{3/2}}+\left(\frac{4}{15}+\frac{3}{4}\theta_{14}^2\right)\frac{1}{\Gamma_4^{3/2}}+\frac{4}{5\Gamma_6^{3/2}}\right].
\end{gathered}
\tag{63}
$$

We note that there is in this expression a term, caused by Γ_4, which diverges near the λ-point like $(\rho_n/\rho_s)^{3/2}$ and which is connected with the fluctuations in the superfluid velocity.

CONCLUSION

As seen, the expression for the first-sound distribution function changes in the limit as $\rho_s\to 0$ to the expression for the sound fluctuations distribution function in a normal liquid;[1] moreover, if we correct the expression for the vortex fluctuations in Ref. 1, we answer for the vortex fluctuations distribution function also changes into the expression for a classical liquid in the limit as $\rho_s\to 0$. Correspondingly, the contributions to the fluctuation corrections, caused by first sound and vortex fluctuations, change in the limit as $\rho_s\to 0$ to the expression for a classical liquid.

The situation is more complicated with the contribution from the second sound fluctuations, since it corresponds above the transition point to a purely damped mode that is responsible for specific entropy oscillations. Correspondingly, it is impossible to take the limit as $\rho_s\to 0$ in the region described by inequality (45), since (45) gives $\omega\to 0$ as $\rho_s\to 0$. It is also impossible to take the limit as $\rho_s\to 0$ in the region described by inequality (46), since the corrections obtained formally diverge as $\rho_s \to 0$. The reason is that we have neglected fluctuations in the absolute magnitude of the order parameter which become important in the immediate vicinity of the λ-point.

We now give estimates of the corrections to the sound dispersion laws at $T = 2$ K for which inequality (45) gives

$$\omega \ll 10^{10}\ \text{sec}^{-1}.$$

We have for the relative changes in the velocities and in the imaginary parts of the wavevectors

$$\frac{\tilde{c}_1}{c_1} \sim \frac{\omega^{3/2}}{10^{20}} \varphi c^{\prime/_2}, \quad \frac{\tilde{\alpha}_1}{\alpha_1} \sim \frac{\omega^{1/_2}}{10^7} \varphi c^{\prime 1/2};$$
$$\frac{\tilde{c}_2}{c_2} \sim -\frac{\omega^{3/2}}{10^{17}} c^{\prime/_2}, \quad \frac{\tilde{\alpha}_2}{\alpha_2} \sim \frac{\omega^{1/2}}{10^7} c^{\prime 1/2}.$$

The change in the first-sound velocity is thus completely unimportant. As for the other quantities, the corrections to them for limiting frequencies of some tens of MHz are of order 10^{-4}. Such corrections are, apparently, beyond the limits of possibilities of experimental techniques, but the non-trivial frequency dependence of these corrections enables us to hope for their experimental detection.

The fluctuation corrections to the kinetic coefficients diverge in the region determined by inequality (46) because Γ_4 tends to zero as $\rho_s \to 0$; this enables us to hope for an experimental observation of the fluctuation corrections to the first sound velocity and damping near the λ-point.

APPENDIX 1

THE KINETIC EQUATION

We saw that in order to find the fluctuation corrections to the hydrodynamical equations we must solve the matrix equation for the matrix of binary correlators of fluctuating quantities. We show now that if the dissipationless hydrodynamical equations are in canonical form the matrix kinetic equation reduces to the set of the usual kinetic equations for each type of long-wavelength fluctuations.

Let (p, β) be the complete set of canonically conjugate variables. It was shown in Ref. 4 that the Hamiltonian density for the hydrodynamical equations for He II can be written in the form $H(p, \nabla\beta)$. To avoid unwieldiness we restrict ourselves to this form of function for the Hamiltonian density although the final answer—the form of the kinetic equation—is retained for any way that H depends on its variables. It is well known that the canonical equations for p, β have the form

$$\frac{\partial}{\partial t} p = \nabla \frac{\partial H}{\partial \nabla \beta}, \quad \frac{\partial}{\partial t} \beta = -\frac{\partial H}{\partial p}.$$

Linearizing these equations and adding kinetic terms and the random forces f, we find

$$\frac{\partial}{\partial t}\delta p = \nabla_i \left(\frac{\partial^2 H}{\partial \nabla_i \beta^\tau \partial \nabla_k \beta} \nabla_k \delta\beta \right)$$
$$+ \nabla \left(\frac{\partial^2 H}{\partial \nabla \beta^\tau \partial p} \delta p \right) + \nabla \left(\gamma_{pp} \nabla \frac{\partial H}{\partial p^\tau} \right) + \nabla \left(\gamma_{p\beta} \nabla \frac{\partial H}{\partial \nabla \beta^\tau} \right) + f_p, \tag{A1}$$

$$\frac{\partial}{\partial t}\delta\beta = \frac{\partial^2 H}{\partial p^\tau \partial \nabla \beta} \nabla \delta\beta + \frac{\partial^2 H}{\partial p^\tau \partial p} \delta p + \nabla \left(\gamma_{p\beta}^\tau \nabla \frac{\partial H}{\partial p^\tau} \right) + \nabla \left(\gamma_{\beta\beta} \nabla \frac{\partial H}{\partial \nabla \beta^\tau} \right) + f_\beta. \tag{A2}$$

Here γ is the kinetic-coefficients matrix.

We now introduce the binary correlators matrix:

$$\langle \delta p(\mathbf{r}_1) \delta p^\tau(\mathbf{r}_2) \rangle, \quad \langle \delta p(\mathbf{r}_1) \delta \beta^\tau(\mathbf{r}_2) \rangle,$$
$$\langle \delta \beta(\mathbf{r}_1) \delta p^\tau(\mathbf{r}_2) \rangle, \quad \langle \delta \beta(\mathbf{r}_1) \delta \beta^\tau(\mathbf{r}_2) \rangle. \tag{A3}$$

As usual, we change to $\mathbf{r} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ and Fourier transform with respect to $\mathbf{r}_1 - \mathbf{r}_2$. Denoting the Fourier component of the matrix (A3) by $A(\mathbf{q})$ and using the fact that $\delta p, \delta\beta$ are real, the matrix $A(\mathbf{q})$ is Hermitean. We define another two Hermitean matrices:

$$J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Delta = \begin{pmatrix} \dfrac{\partial^2 H}{\partial p^\tau \partial p} & iq \dfrac{\partial^2 H}{\partial p^\tau \partial \nabla \beta} \\ -iq \dfrac{\partial^2 H}{\partial \nabla \beta^\tau \partial p} & q_i \dfrac{\partial^2 H}{\partial \nabla_i \beta^\tau \partial \nabla_k \beta} q_k \end{pmatrix}. \tag{A4}$$

We note that $J^2 = 1$.

We define now the matrix Ω:

$$\Omega = J\Delta. \tag{A5}$$

We consider the problem of the eigenvalues of the matrix Ω, which can be seen from (A1), (A2) to have the meaning of eigenfrequencies of local oscillations of the system. It follows from (A5) that the eigenvalues of Ω are determined from the equation

$$\det(\Delta - \lambda J) = 0. \tag{A6}$$

Because Δ and J are Hermitian, this is an equation for λ with real coefficients, i.e., its solutions are either real or pairs of complex conjugate numbers. The latter would indicate local instability of the system, so that we reject that possibility. Thus, all λ are real.

Let now x_a be the eigenvector of Ω which corresponds to the eigenvalue λ_a. By virtue of (A5) we have

$$\Delta x_a = \lambda_a J x_a. \tag{A7}$$

Using the analogous equation for x_b, the fact that Δ and J are Hermitian and the λ are real, we find that

$$(\lambda_a - \lambda_b) x_b^+ J x_a = 0.$$

Thus, when $\lambda_a \neq \lambda_b$ we have the orthogonality condition

$$x_b^+ J x_a = 0. \tag{A8}$$

We shall assume that the vectors x_a are normalized to unity:

$$x_a^+ J x_a = \pm 1. \tag{A9}$$

However, because of the indefiniteness of the metric given by J it is also possible to have for an eigenvector x' of the matrix Ω

$$x'^+ J x' = 0. \tag{A10}$$

We introduce a vector z given by the conditions

$$z^+ J x' = i, \quad z^+ J x_a = 0. \tag{A11}$$

Here the x_a are all eigenvectors of Ω, except x'. We consider the vector z':

$$z'=\Omega z.$$

We take the conjugate of this definition and multiply it from the right by Jx. As a result we get by virtue of (A8), (A11)

$$z'^{+}Jx_{a}=0,\quad z'^{+}Jx'=i\lambda.$$

Comparing this with (A11) we are led to $z'=\lambda z$.

Therefore z, and also any linear combination of z and x', is an eigenvector of Ω corresponding to the eigenvalue λ, i.e., the condition (A10) can hold only when there is degeneracy. We introduce a linear combination x'' of the vectors x' and z, for which we have

$$x''^{+}Jx''=0,\quad x''^{+}Jx'=i\,. \tag{A12}$$

We note that x' and x'' are not uniquely defined. We can still perform a rotation in the x', x''-space: this is a linear transformation of x', x'' which does not change the form of (A10), (A12).

By virtue of the definition of A and Δ we have the conditions

$$A^{*}=A(-\mathbf{q}),\quad \Delta^{*}=\Delta(-\mathbf{q}). \tag{A13}$$

From the second of Eqs. (A13), (A7), and $J^{*}=-J$ it follows that if x is an eigenvector of $\Omega(\mathbf{q})$ corresponding to the eigenvalue λ then x^{*} is an eigenvector of $\Omega(-\mathbf{q})$ corresponding to the eigenvalue $-\lambda$. The eigenvectors of Ω can thus also be renumbered as $x_a^{*}(-\mathbf{q})$, i.e.,

$$x_{b}^{*}(-\mathbf{q})=\sum_{b}x_{b}(\mathbf{q})S_{ba}. \tag{A14}$$

When there is no degeneracy one can have only a single number with absolute magnitude unity in each row and in each column of the matrix S. Since $x^{*}(-\mathbf{q})$ corresponds to the eigenvalue $-\lambda(-\mathbf{q})$, the diagonal elements of the matrix S can be non-vanishing only, if for any eigenvalues $\lambda(\mathbf{q})=-\lambda(-\mathbf{q})$. In particular, it follows from this that if the matrix A is diagonal, pairs of functions of $\mathbf{q}$ and $-\mathbf{q}$ stand on its diagonal.

We now find the form of the matrix A in equilibrium. It is well known that the local equilibrium fluctuation distribution is given by the function[3]

$$\exp\left(-\int d^{3}rR/T\right). \tag{A15}$$

Here R is a linear combination of ΔH and the deviations from equilibrium of the densities of the conserved quantities. In the quadratic approximation R is a positive definite quadratic form of the fluctuating quantities $(\delta p, \nabla\delta\beta)$:

$$R=(\delta p^{\tau}\ \delta\beta^{\tau})\,\Phi(\mathbf{r},-i\nabla)\begin{pmatrix}\delta p\\ \delta\beta\end{pmatrix}+\mathrm{div}.$$

In the approximation in zeroth order in k/q, $A_0(q)$ determines the matrix $\Phi(\mathbf{r},\mathbf{q})$. As the matrix Φ is determined by a linear combination of the conserved quantities we have the relation

$$\Omega^{+}\Phi-\Phi\Omega=0, \tag{A16}$$

which for the particular case $\Phi=\Delta$ can be varied directly. Thus, Ω and Φ are reduced simultaneously to diagonal form. Using this we find from (A15) the equilibrium distribution

$$A_{0}=\sum_{a}\frac{T}{\Phi_{a}}x_{a}x_{a}^{+}. \tag{A17}$$

Here the Φ_a are the corresponding eigenvalues of Φ. We note that

$$\Omega A_{0}-A_{0}\Omega^{+}=0. \tag{A18}$$

We differentiate now A with respect to time. Using the equations of motion (A1) and (A2) we can obtain a kinetic equation for A:

$$\partial A/\partial t=-i\Omega A+iA\Omega^{+}-{}^{1}/{}_{2}\{\Omega,A\}+{}^{1}/{}_{2}\{A,\Omega^{+}\}\\ -\Gamma_{ik}q_{i}q_{k}\delta A-q_{j}q_{k}\delta A\Gamma_{jk}^{\tau}. \tag{A19}$$

Here $\delta A=A-A_0$ is the deviation from the equilibrium value, Γ the matrix of the kinetic coefficients and the $\{\ldots\}$ are Poisson brackets:

$$\{\Omega,A\}=\frac{\partial\Omega}{\partial\mathbf{q}}\frac{\partial A}{\partial\mathbf{r}}-\frac{\partial\Omega}{\partial\mathbf{r}}\frac{\partial A}{\partial\mathbf{q}}. \tag{A20}$$

We have written Eq. (A19) up to terms in first order in k/q. The correlator containing the random forces is chosen such that it guarantees the relaxation of A to its equilibrium value. By virtue of (A18) we can replace $A\to\delta A$ in the first two terms on the right-hand side of (A19).

We now introduce the Hermitian matrix B:

$$A=\sum_{ab}x_{a}B_{ab}x_{b}^{+}. \tag{A21}$$

We substitute this expression into (A19) and multiply it from the left by $x_a^{+}J$ and from the right by Jx_b. Using the fact that the x_a are eigenvectors of Ω and the orthogonality condition (A8) we find that the first two terms on the right-hand side of (A19) give

$$\lambda_{a}\delta B_{ab}-\lambda_{b}\delta B_{ab}.$$

If $\lambda_a=\lambda_b$ (in particular, for the diagonal components), this difference banishes. If $\lambda_a\neq\lambda_b$, the presence of this term leads to the fact that the component B_{ab} is small of order ω/λ (let us say, or order ω/cq in He II) as compared to the diagonal ones; in what follows we shall neglect these components. The diagonal terms of the matrix B we shall denote by n—they are real because of the Hermiticity. In the degenerate case the off-diagonal terms of B are of the same orders as the diagonal ones. In that case we perform a linear transformation of the x_a which gets rid of these off-diagonal terms. This is, however, impossible under condition (A10). Retaining (A10), (A12) we can reduce the corresponding terms in A to the form

$$n'(x'x'^{+}+x''x''^{+})+in''(x'x''^{+}-x''x'^{+}). \tag{A22}$$

By virtue of the Hermiticity of B the quantities n' and n'' are real.

We now find the equation for the n_a which have the meaning of distribution functions for the fluctuations. If we perform the operation with Eq. (19) which we described above, the derivatives of all x, except x_a, cancel in the equation we obtain for n_a by virtue of (A8). The time-derivative in (A19) gives the following term with derivatives of x_a:

$$\pm n_{a}\left(x_{a}^{+}J\frac{\partial x_{a}}{\partial t}+\frac{\partial x_{a}^{+}}{\partial t}Jx_{a}\right).$$

It vanishes as the terms in the brackets combine into the derivative of (A9). Similarly all other terms with derivatives of x_a cancel, and we get for the n_a the usual kinetic equation

$$\partial n_a/\partial t=\{n_a,\ \lambda_a\}-2\Gamma_{aij}q_iq_j\delta n_a. \tag{A23}$$

Here Γ_a is a diagonal component of the matrix Γ.

The situation is somewhat more complicated if there are eigenvectors satisfying condition (H10). In that case, after substituting (A22) into (A19) we multiply the ensuing equation to the left by $x'^{+}J$, to the right by Jx', and add to it the same multiplied to the left by $x''^{+}J$ and to the right by Jx''. The terms with derivatives of other x_a cancel by virtue of (A8). One can easily check that terms with derivatives of x', x'' also cancel by virtue of the normalization conditions (A10), (A12) and we are led to the same kinetic Eq. (A23) for n'. Similarly, multiplying (A19) from the right by Jx'', to the left by $x'^{+}J$ and subtracting (A19), multiplied from the left by $x''^{+}J$ and to the right by Jx' we get for n'' a kinetic equation which is the same as (A23).

APPENDIX 2

We give the expressions for the integrals:

$$J_n=\frac{1}{2}\int_{-1}^{1}d\xi\,\xi^{n-1}(1-x\xi)^{1/2},$$

$$J_1=\frac{1}{3x}f_1,\quad J_2=\frac{1}{x^2}\left(\frac{1}{3}f_1-\frac{1}{5}f_2\right),$$

$$J_3=\frac{1}{x^3}\left(\frac{1}{3}f_1-\frac{2}{5}f_2+\frac{1}{7}f_3\right),\quad J_4=\frac{1}{x^4}\left(\frac{1}{3}f_1-\frac{3}{5}f_2+\frac{3}{7}f_3-\frac{1}{9}f_4\right),$$

$$J_5=\frac{1}{x^5}\left(\frac{1}{3}f_1-\frac{4}{5}f_2+\frac{6}{7}f_3-\frac{4}{9}f_4+\frac{1}{11}f_5\right),$$

$$f_m(x)=(1+x)^{m+1/2}-(1-x)^{m+1/2}.$$

In the case $x>1$ we choose for the functions containing the root of $1-x$ the sheet of the complex plane on which $\sqrt{(-1)}=i$.

[1)] We must bear in mind that $\langle\delta S\rangle\neq 0$.

[2)] The inequality (29) guarantees that this equation to satisfied in this case.

[3)] If we take the quantum nature of the fluctuation distribution functions into account, the divergence of these integrals vanishes.

[4)] We note that it is impossible to evaluate by the method developed here the corrections to the dispersion relation of the vortex oscillations as it follows from (33) that for the integration region $q\sim(\omega p_n/\eta)^{1/2}\sim k$ is characteristic, i.e., the inequality $q\gg k$ is not satisfied.

[5)] Although such a procedure is not unique, it does not affect the physical results in the region where the theory is applicable.

[6)] In tensor notation

$$\frac{\partial}{\partial t}\delta p_\alpha=\nabla_i\left(\frac{\partial^2 H}{\partial\nabla_i\beta_\alpha\partial\nabla_k\beta_\beta}\nabla_k\delta\beta_\beta\right)+\dots\ .$$

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