Semiclassical approximation for stimulated bremsstrahlung

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In the semiclassical impact parameter approximation calculations are made of the cross sections of stimulated multiphoton bremsstrahlung when a particle is scattered by a force center. The case of electron scattering in a Coulomb field is considered in detail. At low electron energies, a strong difference from the Born approximation is found, this being due to the allowance made for the adiabaticity of the radiation process. In the case of a weak radiation field in the adiabatic limit simple expressions are obtained for the cross sections of stimulated bremsstrahlung, these being analogous to the Kramers expression for spontaneous emission.

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Calculation of the cross sections of stimulated bremsstrahlung (and its inverse) in a strong monochromatic field are of interest in connection with the study of plasma heating by laser radiation, and also the possibility of obtaining negative absorption in transitions between continuum states.[1,2] Recently,[3] the first measurements were made of the cross sections of stimulated bremsstrahlung in the case of electron scattering on Ar and H_2. The emission and absorption of up to five photons in one scattering was observed. The cross sections of stimulated bremsstrahlung in strong fields were calculated for the first time by Bunkin and Fedorov[4] in the Born approximation in the field of the scatterer and with Volkov wave functions for the electron in the field of the wave. In following studies, this approximation was widely used by other authors. Unfortunately, no calculations have yet been made of the cross sections of stimulated bremsstrahlung in the approximation that takes into account the interaction of the electron with the field of the scatterer exactly and with the radiation field perturbatively. Corresponding calculations for multiphoton ionization of atoms have been made to very high orders of perturbation theory.[5,6] So far, it has been possible to take into account the effect of both the fields on the electron only in the case of scattering on a model δ-function potential.[7] But the case of greatest interest is the Coulomb field, when the condition of applicability of the Born approximation[8] reduces to the requirement that the parameter

$$\eta=Ze^2/\hbar v, \tag{1}$$

where Ze is the charge of the scatterer and v is the relative velocity, be small.

In the present paper, on the basis of the semiclassical impact parameter theory, we attempt to go beyond the Born approximation. In the case of the Coulomb potential, this theory is valid for large n, precisely where the Born approximation breaks down. In Sec. 3, we consider the case of a weak radiation field, and in the adiabatic limit we obtain simple expressions for the cross sections of stimulated bremsstrahlung, these being analogous to the Kramers formula in the case of spontaneous bremsstrahlung.

1. IMPACT PARAMETER APPROXIMATION FOR STIMULATED BREMSSTRAHLUNG (EMISSION AND ABSORPTION)

Our point of departure is the Hamiltonian of Ref. 9, which describes scattering of a particle on a potential $V(r)$ and simultaneously its interaction with the field of an electromagnetic wave in the dipole approximation:

$$H=\frac{p^2}{2m}+V(r)-\frac{e}{mc}\mathbf{Ap}+\hbar\omega b^+b+\frac{e^2}{2mc^2}\mathbf{A}^2, \tag{2}$$

$$\mathbf{A}=c(2\pi\hbar/V\omega)^{1/2}\mathbf{e}(b+b^+), \tag{3}$$

where ω is the frequency of the wave, $\mathbf{e}$ is the unit vector in the direction of its polarization, b^+ and b are the creation and annihilation operators, and V is the quantization volume.

We now assume that the scattering of the particle on the potential can be described classically. In the general case, this assumption is justified for fast particles and smooth potentials; in the case of the Coulomb field, it is justified for large values of the parameter η, i.e., for slow particles. It is shown in the Appendix that if, first, the motion of the particle in the potential $V(r)$ can be described classically and, second, in the region important for the process of stimulated bremsstrahlung the kinetic energy of the particle is greater than both the energy of its interaction with the radiation field and the photon energy, then instead of the Schrödinger equation with the Hamiltonian (2) one can use the simpler equation

$$i\hbar\frac{\partial\chi}{\partial t}=-e\left(\frac{2\pi\hbar}{V\omega}\right)^{1/2}\mathbf{ev}(t)\,(be^{-i\omega t}+b^+e^{i\omega t})\chi, \tag{4}$$

where $\mathbf{v}(t)$ is the time-dependent classical velocity of the particle as it moves in the field $V(r)$. If the second condition is satisfied, we can in the region important for the bremsstrahlung ignore the perturbation of the particle's trajectory caused by its interaction with the radiation field. The region that makes an important contribution to the bremsstrahlung is a region of atomic order. Since the kinetic energy of the particle in an attractive field is equal to the sum of its total energy and the potential energy $|V(r)|$, in this region it is not less than the atomic energy. Therefore, the second

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condition in the case of attractive fields will be satisfied even for slow particles provided the frequency and strength of the radiation field are less than the atomic frequencies and strengths. In the case of the attractive Coulomb field, besides this restriction on the frequency and field strength of the radiation, it is necessary that the absolute magnitude of the particle's energy be less than its energy in the first Bohr orbit in order to ensure that the motion is classical.[8] The solution of Eq. (4) is the problem of calculating the probabilities of excitation of the states of the wave field, i.e., the states of an oscillator when the particle moves along a classical trajectory in the field of the scatterer. The solution of Eq. (4) is well known[10] and is

$$\chi=\exp\{ib\nu(t)+ib^+\nu^+(t)+ig(t)\}, \tag{5}$$

$$\nu(t)=\frac{e}{\hbar}\left(\frac{2\pi\hbar}{V\omega}\right)^{1/2}\int_{t_0}^{t} dt\mathbf{e}\mathbf{v}(t)e^{-i\omega t}, \tag{6}$$

where the function $g(t)$ does not contain the variables b and b^+ and is unimportant for what follows.

Knowing χ, we can find the probability amplitude A_{ls} for excitation of the field state $|l\rangle$ at the time t if the field at t_0 is in the state $|s\rangle$. This amplitude can be expressed in terms of Laguerre polynomials[10]:

$$A_{ls}=\langle l|\chi|s\rangle=(i\nu)^{s-l}\left(\frac{l!}{s!}\right)^{1/2}L_l^{s-l}\ (\nu\nu^+)\exp\left(-\frac{\nu\nu^+}{2}+ig\right),\quad s\geq l. \tag{7}$$

For $s<l$, the positions of s and l in the expression (7) should be interchanged. We shall be interested in a radiation field for which many photons are present in the initial and final states. We introduce the number $n=s-l$, which is equal to the number of absorbed ($n>0$) and emitted ($n<0$) photons, and we use the asymptotic behavior[11] for the Laguerre polynomials when $s,l\gg1$. Then[9]

$$A_n=J_n(\sqrt{4s\nu\nu^+})\exp(ig+in\beta+{}^1/{}_2i\pi n),\quad \nu/\nu^+=e^{2i\beta}. \tag{8}$$

Here, J_n is a Bessel function of integral index.

The differential cross section of scattering with the absorption of n photons is equal to the product of the classical differential cross section for scattering of the particle in the field of the scatterer and the probability of excitation $|A_n|^2$ of the state n:

$$d\sigma_n=J_n^2(|B|)d\sigma_c; \tag{9}$$

$$B=\frac{e}{\hbar\omega}\int_{-\infty}^{\infty}dt\mathbf{E}_0\mathbf{v}(t)e^{-i\omega t}, \tag{10}$$

and

$$E_0=(8\pi\hbar\omega s/V)^{1/2} \tag{11}$$

in the limit s, $V\to\infty$ but at constant ratio of these quantities goes over into the classical intensity amplitude of an electromagnetic wave. It can be seen that the integral (10) is none other than the classical integral which determines the Fourier component with frequency ω for dipole radiation in collisions.[12]

It should be noted that our treatment is not completely classical, although the motion of the particle in the potential $V(r)$ is described classically and the electromagnetic field in the final expression (10) is also treated in the classical limit. Equation (4) describes excitation of the states of the field of the electromagnetic wave and is quantum mechanical. Considering large population numbers, we arrive at the classical limit for the radiation field, though the change in the states of this field, i.e., the emission and absorption of photons, is still described quantum mechanically.

If the particle before and after the scattering, which occurs at the time $t=0$, is free and moves with the velocities $\mathbf{v}_i$ and $\mathbf{v}_f$, respectively, it is readily seen that

$$|B|=\frac{eE_0}{\hbar\omega^2}(\mathbf{v}_i\mathbf{e}-\mathbf{v}_f\mathbf{e}). \tag{12}$$

Expressing v_f by means of the quantum-mechanical law of energy conservation,

$$mv_f^2/2=mv_i^2/2+n\hbar\omega, \tag{13}$$

which does not follow from our classical treatment, we arrive at the same argument of the Bessel function as in the case of the Born approximation.[1,4]

2. THE CASE OF A COULOMB FIELD

We now consider in more detail the case of electron scattering in an attractive Coulomb field. The motion of the electron in the focal coordinate system is given by the expressions (Ref. 13)

$$x=a(\varepsilon-\operatorname{ch}u),\quad y=a(\varepsilon^2-1)^{1/2}\operatorname{sh}u,\quad z=0,$$
$$t=\frac{a}{v}(\varepsilon\operatorname{sh}u-u),\quad a=\frac{Ze^2}{mv^2},\quad \varepsilon=\frac{1}{\sin(\vartheta/2)}. \tag{14}$$

Here, ϑ is the scattering angle. The calculation of the integral (10) is completely analogous to the calculation of the integrals in the case of spontaneous bremsstrahlung in the case of scattering in a Coulomb field[12] or in the case of dipole Coulomb excitation of nuclei.[14] As a result, we obtain

$$|B|=\gamma S; \tag{15}$$

$$S=2\xi e^{\pi\xi/2}[e_x^2K_{i\xi}'^2(\xi\varepsilon)+e_y^2(1-1/\varepsilon^2)K_{i\xi}^2(\xi\varepsilon)]^{1/2}, \tag{16}$$

$$\gamma=eE_0v/\hbar\omega^2,\quad \xi=\omega a/v, \tag{17}$$

where $K_{i\xi}$ is a MacDonald function of imaginary index, and the prime denotes the derivative with respect to the argument; e_x and e_y are the projection of the vector $\mathbf{e}$ onto the focal coordinate axes. For what follows, it is convenient to express e_x and e_y in terms of the projections of $\mathbf{e}$ onto the directions of the incident and the scattered electron. For this, we introduce a coordinate system in which $\mathbf{e}$ and the unit vectors $\mathbf{n}_i$ and $\mathbf{n}_f$ in the direction of the incident and scattered electron have the coordinates

$$\mathbf{e}=(\sin\alpha,0,\cos\alpha),\quad \mathbf{n}_i=(0,0,1),$$
$$\mathbf{n}_f=(\sin\vartheta\cos\varphi,\ \sin\vartheta\sin\varphi,\ \cos\vartheta). \tag{18}$$

Then $\mathbf{n}_i\mathbf{e}=\cos\alpha,\quad \mathbf{n}_f\mathbf{e}=\cos\alpha\cos\vartheta+\sin\alpha\sin\vartheta\cos\varphi,$

$$e_x=\frac{\mathbf{n}_i\mathbf{e}-\mathbf{n}_f\mathbf{e}}{2\sin(\vartheta/2)},\quad e_y=\frac{\mathbf{n}_i\mathbf{e}+\mathbf{n}_f\mathbf{e}}{2\cos(\vartheta/2)}. \tag{19}$$

Using (9) and (15)–(19) and the fact that in the Coulomb case $d\sigma_c$ is the Rutherford cross section, we obtain finally the expression for the differential cross section of n-photon emission (absorption):

$$\frac{d\sigma_n}{d\Omega}=\frac{a^2\varepsilon^4}{4}J_n^2(\gamma S), \tag{20}$$

$$S=\xi e^{\pi\xi/2}[\varepsilon^2(\mathbf{n}_i\mathbf{e}-\mathbf{n}_f\mathbf{e})^2K_{i\xi}'^2(\xi\varepsilon)+(\mathbf{n}_i\mathbf{e}+\mathbf{n}_f\mathbf{e})^2K_{i\xi}^2(\xi\varepsilon)]^{1/2}. \tag{21}$$

In the case of scattering in a repulsive field, we have the same expressions (20) and (21), but the sign of the argument of the exponential in (21) is negative.

The cross sections (20) contain two dimensionless parameters, γ and ξ. The parameter γ is related to the intensity of the alternating field, and ξ determines the degree of adiabaticity of the process, since $\xi\varepsilon$ is the ratio of the collision time to the oscillation period of the field. The parameter ξ is specific for the Coulomb field, and in the Born approximation vanishes. Indeed, at small ξ and $\varepsilon \sim 1$, so that $\xi\varepsilon \ll 1$, the function $K'_{i\xi}$ in the expression (21) can be replaced approximately by $(\xi\varepsilon)^{-1}$, and the function $K_{i\xi}$ equated to zero.[11] As a result, for S we obtain the Born value, equal to $\mathbf{n}_i \cdot \mathbf{e} - \mathbf{n}_f \cdot \mathbf{e}$. At small scattering angles, when $\varepsilon \gg 1$, such a transition is impossible because of the long-range nature of the Coulomb field. Note that for large ξ the cross sections for attractive and repulsive fields will be very different (in the Born approximation, they are equal). If ξ is large, then for a repulsive field S will be exponentially small, and virtually no emission or absorption will occur.

It follows from the expression (20) that in the impact parameter approximation the cross sections are symmetric with respect to the sign of n, i.e., the cross sections for the absorption and emission of n photons are equal. This is a reflection of the fact that in the classical treatment we ignore the influence of the energy loss on the motion of the incident particle. In a first approximation, these losses can be taken into account in the following phenomenological manner. In the expression (21) it is natural to introduce in front of the vector $\mathbf{n}_f$ the factor v_f/v_i, where v_f is determined from the energy conservation law (13), and then symmetrize the parameters a and ξ by the method described in Ref. 14. Calculations showed that in the case of Coulomb excitation of nuclei the symmetrized cross sections are significantly better than the unsymmetrized ones. However, in our case there are many inelastic channels, and the symmetrization procedure is not so unambiguous. To verify its effectiveness, quantum-mechanical calculations are needed. Since such calculations are not currently available, we shall in what follows simply use the expressions (20) and (21) without phenomenological corrections.

3. THE CASE OF A WEAK RADIATION FIELD

In a weak radiation field $\gamma \ll 1$, and we can restrict ourselves to the first few terms of the series expansions of the Bessel functions. Since the emission and absorption cross sections are equal, we shall in what follows for brevity speak of emission alone. Restricting ourselves to just the first term in the series expansion of the Bessel function and integrating over the angular variables of the scattered electron, we can represent the cross section of n-photon emission in the form

$$\sigma_n(\alpha)=\frac{a^2\gamma^{2n}\xi^{2n}e^{n\pi\xi}}{(n!)^2}\int_1^\infty d\varepsilon\,\varepsilon^{-2n+1}\int_0^{2\pi} d\varphi\{[\cos\alpha-(\varepsilon^2-1)^{1/2}\sin\alpha\cos\varphi]^2K_{i\xi}'^2(\xi\varepsilon)+(1-\varepsilon^{-2})[(\varepsilon^2-1)^{1/2}\cos\alpha+\sin\alpha\cos\varphi]^2K_{i\xi}^2(\xi\varepsilon)\}^n. \quad (22)$$

We consider first in more detail single-photon emission. In this case, the integral over ε can be calculated explicitly on the basis of the expressions for the indefinite integrals of a product of two modified Bessel functions.[11] As a result,

$$\sigma_1(\alpha)=\pi a^2\gamma^2\xi^2e^{\pi\xi}[R_1\sin^2\alpha+R_2(3\cos^2\alpha-1)]; \quad (23)$$

$$R_1=\int_1^\infty d\varepsilon\,\varepsilon\left[K_{i\xi}'^2(\xi\varepsilon)+\left(1-\frac{1}{\varepsilon^2}\right)K_{i\xi}^2(\xi\varepsilon)\right]=-\frac{1}{\xi}K_{i\xi}(\xi)K_{i\xi}'(\xi), \quad (24)$$

$$R_2=\int_1^\infty d\varepsilon\,\varepsilon\left[\frac{1}{\varepsilon^2}K_{i\xi}'^2(\xi\varepsilon)+\left(1-\frac{1}{\varepsilon^2}\right)^2K_{i\xi}^2(\xi\varepsilon)\right]$$

$$=K_{i\xi}'^2(\xi)+\frac{1}{2i}\left[K_\mu(\xi)\frac{\partial^2K_\mu(\xi)}{\partial\mu\,\partial\xi}-\frac{\partial K_\mu(\xi)}{\partial\xi}\frac{\partial K_\mu(\xi)}{\partial\mu}\right]_{\mu=i\xi}. \quad (25)$$

Note that there is a direct correspondence between the cross section (23) and the classical bremsstrahlung cross section for scattering of a particle in a Coulomb field. If the classical quantity E_0 in the expression (23) is replaced by (11) with $s=1$, which corresponds to spontaneous emission, and it is multiplied by the number of states $Vd\mathbf{k}(2\pi)^{-3}$ of the electromagnetic field and integrated over the angular variables of the emitted photon, we arrive at the classical bremsstrahlung cross section.[12-14]

In the case of low frequencies ω, it follows from the behavior of the MacDonald function[11] for $\xi \ll 1$ that the cross section σ_1 contains the classical logarithm:

$$\sigma_1(\alpha)=\pi a^2\gamma^2[\sin^2\alpha\ln(2/\gamma'\xi)+3\cos^2\alpha-1], \quad (26)$$

where $\gamma'=1.781\ldots$.

At high frequencies ($\xi \gg 1$), it is necessary to use the asymptotic expansions[11,15] for the MacDonald functions. As a result,

$$R_1=\frac{\pi}{3^{1/2}\xi^2}e^{-\pi\xi}, \quad R_2=\frac{2\pi}{3^{1/2}\xi^2}e^{-\pi\xi} \quad (27)$$

and the cross section $\sigma_1(\alpha)$ also takes on the simple form

$$\sigma_1(\alpha)=3^{-1/2}\pi^2a^2\gamma^2(3\cos^2\alpha+1). \quad (28)$$

We now consider the cross section σ_n for emission of n photons ($n>1$) in the limiting cases of low and high frequencies. At low frequencies, the MacDonald functions $K'_{i\xi}(\xi\varepsilon)$ and $K_{i\xi}(\xi\varepsilon)$ can be replaced by $K'_0(\xi\varepsilon)$ and $K_0(\xi\varepsilon)$. Further, in the integral over ε we make the change of variables $z=\xi\varepsilon$ and split the integral over z from ξ to ∞ into two integrals: from ξ to 1 and from 1 to ∞. The second integral is a polynomial in ξ^2. In the first integral, the functions $K'_0(z)$ and $K_0(z)$ can be represented in the form of a series.[11] It is readily seen that the main contribution to this integral is given by the z^{-1} term of the series for the function $K'_0(z)$. Retaining only this term, we find that (22) becomes

$$\sigma_n(\alpha)=\frac{a^2\gamma^{2n}\xi^{2n-2}}{(n!)^2}\int_\xi^1 dz\,z^{-4n+1}\int_0^{2\pi} d\varphi[\xi\cos\alpha-(z^2-\xi^2)^{1/2}\sin\alpha\cos\varphi]^{2n}. \quad (29)$$

The integrals over z and φ can be readily calculated if the binomial in the integrand is represented in the form of a sum. Retaining only the principal term in ξ, we finally find that at low frequencies

$$\sigma_n(\alpha)=\frac{\pi a^2\gamma^{2n}}{(n!)^2}P_n(\alpha), \quad n>1. \quad (30)$$

where $P_n(\alpha)$ is a polynomial and can be expressed in terms of the hypergeometric function as follows:

$$P_n(\alpha)=\frac{\cos^{2n}\alpha}{2n-1}F(-n,-n+{}^1/_2,-2n+2,-\mathrm{tg}^2\alpha),$$

$$P_n(0)=\frac{1}{2n-1},\quad P_n\left(\frac{\pi}{2}\right)=\frac{1}{2^{2n-1}(n-1)}. \tag{31}$$

It can be seen that the adiabaticity parameter ξ is itself completely absent in (30), and the cross section is proportional to ω^{-4n}.

To find the cross sections σ_n in the limit of high frequencies, we note that in the integral representation of the MacDonald function[11]

$$K_{i\xi}(\xi\varepsilon)=\frac{e^{-\pi\xi/2}}{2}\int_{-\infty}^{\infty}du\exp(i\xi\varepsilon\,\mathrm{sh}\,u-i\xi u) \tag{32}$$

the region of small u is important at large ξ. Therefore, replacing $\sinh u$ by $u+u^3/6$ and using the integral representation[15] for the Airy function $\mathrm{Ai}(x)$, we find that

$$K_{i\xi}(\xi\varepsilon)\approx\pi e^{-\pi\xi/2}\left(\frac{2}{\xi\varepsilon}\right)^{1/3}\mathrm{Ai}\left[\left(\frac{2\xi^2}{\varepsilon}\right)^{1/3}(\varepsilon-1)\right], \tag{33}$$

$$K_{i\xi}'(\xi\varepsilon)\approx\pi e^{-\pi\xi/2}\left(\frac{2}{\xi\varepsilon}\right)^{2/3}\frac{2\varepsilon+1}{3\varepsilon}\mathrm{Ai}'\left[\left(\frac{2\xi^2}{\varepsilon}\right)^{1/3}(\varepsilon-1)\right]. \tag{34}$$

If we substitute (33) and (34) in (22), replace the variable ε by the variable $x=2^{1/3}\xi^{2/3}(\varepsilon-1)$, and everywhere in (22) retain only the principal term of the asymptotic expansion in inverse powers of ξ, we obtain the following expression for $\sigma_n(\alpha)$ in the limit of high frequencies:

$$\sigma_n(\alpha)=\frac{a_0^2\eta^2}{(n!)^2}\frac{E_0^{2n}}{\omega^{(10n+2)/3}}G_n(\alpha), \tag{35}$$

$$G_n(\alpha)=\pi^{2n}2^{(4n-1)/3}\int_0^\infty dz\int_0^{2\pi}d\varphi[\mathrm{Ai}'^2(z)\cos^2\alpha+z\,\mathrm{Ai}^2(z)\sin^2\alpha\cos^2\varphi]^n. \tag{36}$$

The values of this polynomical for $\alpha=0$ and $\pi/2$ are given in Table I. In the expression (35), a_0 is the Bohr radius, and the amplitude E_0 and frequency ω of the field are expresssed in atomic units.

Note the weak (proportional to E^{-1}) and identical for all n dependence of the cross sections (35) on the electron energy. The frequency dependence of the cross sections is very strong, although somewhat weaker than in the case of low frequencies. It can be seen from Table I that the cross sections $\sigma_n(\alpha)$ in the case of field polarization along the incident beam are appreciably larger, especially at large n, than the corresponding cross sections for field polarization perpendicular to the incident beam. This is due to the circumstance that the emission is maximal in the case of backward scattering, when the particle approaches closest to the scattering center. If the vector $\mathbf{E}_0$ is directed along the incident beam, then in the case of backward scattering the particle's trajectory is parallel to the vector $\mathbf{E}_0$ the whole time. However, as can be seen from (10), it is the product $\mathbf{E}_0\cdot\mathbf{v}(t)$ that determines the cross section σ_n.

TABLE I.

n	$G_n(0)$	$G_n(\pi/2)$	n	$G_n(0)$	$G_n(\pi/2)$
1	7.59	1.898	5	23.8	0.084
2	8.53	0.675	6	35.9	0.047
3	11.3	0.307	7	55.1	0.027
4	16.1	0.156	8	85.4	0.019

The expression (35) is similar to the classical Kramers formula for spontaneous emission.[16] To verify the accuracy of (35), as well as that of the more general expression (22), the corresponding quantum-mechanical calculations must be made. In the case of spontaneous emission, the classical theory, and, in particular, the simple Kramers formula, gave good results in a very wide range of frequencies.[16]

APPENDIX

DERIVATION OF EQUATION (4)

We note first that in the Hamiltonian (2) we can eliminate the last term if we go over to new creation and annihilation operators and reduce the two last terms in (2) to diagonal form. This changes the constants of the third and fourth terms of the Hamiltonian (2), but in the classical limit, in which we are now interested, these constants go over into their previous values when the population numbers n and the quantization volume V tend to infinity but with a constant ratio. The Hamiltonian then acquires a constant that is unimportant for what follows and is equal to the oscillation energy of the particle in the field of the wave. Thus, in the classical limit we can ignore the last term in the Hamiltonian (2) (see also Ref. 9).

The Hamiltonian (2) conserves the projection of the angular momentum of the motion onto the direction of the wave polarization. We choose this direction along the z axis. We shall seek the wave function corresponding to the Hamiltonian (2) in the form of an expansion in the wave functions $|n\rangle$ of the field of the wave and spherical functions[17]:

$$\Psi(r,b,b^+)=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{F_{nl}^M(r)}{r}Y_{lM}(\vartheta,\varphi)|n\rangle. \tag{A.1}$$

Substituting (A.1) in the Schrödinger equation with the Hamiltonian (2) and using the orthogonality of not only the spherical functions but also the functions $|n\rangle$, we obtain the following system of coupled equations for $F_{nl}^M(r)$:

$$\left[\frac{d^2}{dr^2}+k_{nl}^2(r)\right]F_{nl}^M=i\frac{2e}{\hbar}\left(\frac{2\pi\hbar}{V\omega}\right)^{1/2}\left\{\left[\frac{l^2-M^2}{(2l+1)(2l-1)}\right]^{1/2}\left(\frac{d}{dr}-\frac{l}{r}\right)\right.$$
$$\times[n^{1/2}F_{n+1,l-1}^M+(n+1)^{1/2}F_{n-1,l-1}^M]+\left[\frac{(l+1)^2-M^2}{(2l+1)(2l+3)}\right]^{1/2}$$
$$\left.\times\left(\frac{d}{dr}+\frac{l+1}{r}\right)[n^{1/2}F_{n+1,l+1}^M+(n+1)^{1/2}F_{n-1,l+1}^M]\right\}, \tag{A.2}$$

$$k_{nl}^2(r)=\frac{2m}{\hbar^2}\left[E-n\hbar\omega-V(r)-\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]. \tag{A.3}$$

Denoting for simplicity of the further calculations the set of quantum numbers n and l by n, we can write the system (A.2) in the form

$$\left[\frac{d^2}{dr^2}+k_n^2(r)\right]F_n=\frac{2m}{\hbar^2}\sum_{n'}V_{nn'}F_{n'}. \tag{A.4}$$

We now apply the quasiclassical approximation to the system (A.4). For this, we seek the function $F_n(r)$ in the form[18]

$$F_n(r)=\frac{1}{2ik_n^{1/2}}\left\{a_n^+\exp\left[i\left(S_n+\frac{\pi}{4}\right)\right]-a_n^-\exp\left[-i\left(S_n+\frac{\pi}{4}\right)\right]\right\};\tag{A.5}$$

$$S_n=\int_{r_n}^{r}k_n(r)\,dr, \tag{A.6}$$

where r_n is the turning point, and a_n^+ and a_n^- are arbitrary functions that vary slowly with r. For constant a_n^+ and a_n^-, the functions (A.5) are the quasiclassical solutions of the system (A.4) in the absence of the right-hand sides.

We impose on the functions a_n^+ and a_n^- the condition[19]

$$\exp\left[i\left(S_n+\frac{\pi}{4}\right)\right]\frac{da_n^+}{dr}-\exp\left[-i\left(S_n+\frac{\pi}{4}\right)\right]\frac{da_n^-}{dr}=0. \quad (A.7)$$

We now substitute (A.5) in Eq. (A.4) and use the condition (A.7) and the ordinary conditions of applicability of the quasiclassical treatment[19]:

$$k_n^{-1/2}\frac{d^2}{dr^2}k_n^{-1/2}\ll 1, \quad (A.8)$$

$$S_n-S_{n'}\ll S_n+S_{n'}. \quad (A.9)$$

As a result, the system (A.4) in the quasiclassical approximation reduces to the system of first-order equations[18]

$$i\frac{da_n^+}{dr}=\frac{m}{\hbar^2}\sum_{n'}\frac{V_n^{n'}}{(k_nk_{n'})^{1/2}}\exp[i(S_{n'}-S_n)]a_{n'}^+ \quad (A.10)$$

and the complex-conjugate system for the functions a_n^-. The physical meaning of the functions a_n^+ and a_n^- and the boundary conditions for them have already been discussed in Ref. 18.

Although the transition from Eqs. (A.4) to (A.10) has been made subject to the conditions (A.8) and (A.9), in reality the choice of the functions $F_n(r)$ in the form (A.5) presupposes implicitly that the nondiagonal matrix elements $V_n^{n'}$ have little influence on the relative motion in all the channels, i.e., we also assume that

$$\hbar^2k_n^2(r)/2m\gg|V_n^{n'}|. \quad (A.11)$$

Physically, this condition means that in the region important for the bremsstrahlung the kinetic energy of the particle must be greater than the energy of its interaction with the external field. If we also assume that in this region the kinetic energy of the particle is also greater than the photon energy,

$$\hbar^2k_n^2(r)/2m\gg\hbar\omega, \quad (A.12)$$

then the quantities $k_n(r)$ will differ only slightly, and from the expressions (A.3) and (A.6) it follows approximately[14] that

$$S_{nl}-S_{n'l'}\approx(n'-n)\omega t(r)+(l'-l)\varphi(r), \quad (A.13)$$

where $t(r)$ and $\varphi(r)$ are the time and angle related to r by the well-known classical integrals.[13]

If we go over from the variable r to the variable t, use (A.13), and the fact that large angular momenta l correspond to the quasiclassical approximation, we finally find that the system of strong-coupling equations (A.12) goes over into the following system of impact parameter equations:

$$i\hbar\frac{da_n}{dt}=-e\left(\frac{2\pi\hbar}{V\omega}\right)^{1/2}v_z(t)\,[n^{1/2}e^{-i\omega t}a_{n+1}+(n+1)^{1/2}e^{i\omega t}a_{n-1}]. \quad (A.14)$$

Here, the subscript l for the functions a_{nl} has been omitted, since Eqs. (A.14) do not contain this parameter at all.

Note also that for variation of t from $-\infty$ to 0 the system (A.14) is a system for the functions a_n^-, and for variation of t from 0 to ∞ it is a system for the functions a_n^+. It can now be seen that Eqs. (A.14) follow directly from Eq. (4) if the wave function χ is sought in the form of an expansion in the states $|n\rangle$ of the radiation field.

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