Self-induced transparency of anisotropic media

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(Submitted 3 October 1980)
Zh. Eksp. Teor. Fiz. 80, 1746–1756 (May 1981)

An analysis is made of the characteristics of self-induced transparency (SIT) in an optically anisotropic (uniaxial) medium containing impurity centers. It is assumed that (as is true of a large class of laser crystals) the optic axes of the impurity and matrix coincide. Solutions of the field equations in the form of solitons (2π pulses) traveling in an anisotropic crystal are obtained for light frequencies corresponding to impurity resonances and a study is made of the dependence of the characteristics of these pulses on the direction of their velocity **u** and polarization **P**. It is shown that, by analogy with waves in linear crystal optics, SIT pulses traveling in such a medium are either ordinary (**P**⊥**u**) or extraordinary (when the angle between **P** and **u** is not the right angle). A relationship is obtained between the pulse duration and its velocity. It is demonstrated that only in the case of extraordinary pulses does this relationship contain the dependence on the direction of propagation. The feasibility of experimental investigation of the phenomena described is considered.

PACS numbers: 42.65.Gv

§1. INTRODUCTION

A theory of self-induced transparency (SIT) was developed by McCall and Hahn in 1969.[1] They and other investigators (see reviews[2] and monograph[3]) studied propagation of an electromagnetic coherent wave in a resonant isotropic medium, i.e., they considered only the media whose polarization vector **P** has the same direction as the electric field vector **E** of the wave. In these investigations the interaction between molecules was ignored. This approximation is justified for gases and also for crystals in the vicinity of a resonant transition in impurities when the impurity concentration is sufficiently low. However, in the case of crystals the polatization vector is collinear with the electric field vector only for waves traveling along the principal axes of the permittivity tensor. Consequently, it would be interesting to study the characteristics of SIT in anisotropic crystals for an arbitrary direction of wave propagation.

We shall consider SIT in an optically anisotropic (uniaxial) medium containing impurity centers. In this case the permittivity is $\varepsilon_{ij}=\varepsilon_{ij}^{(0)}+4\pi\chi_{ij}$, where $\varepsilon_{ij}^{(0)}$ is the permittivity of the matrix, and χ_{ij} is the polarizability of the impurity centers whose interaction with one another is ignored. We shall assume, as is true of many laser crystals (see, for example, Ref. 4), that the optic axes of the impurity and matrix coincide. It follows that $\chi_{ij}=0$ for $i\neq j$, $\chi_{11}=\chi_{22}\equiv\chi_{\perp}$, $\chi_{33}\equiv\chi_{\parallel}$, and similar relationships apply also to the tensor $\varepsilon_{ij}^{(0)}$. We shall assume also that resonances of the tensor χ_{ij} lie in the transparency range of the matrix[1)] and we shall bear in mind that the resonance frequencies $\omega_{\perp}$ and $\omega_{\parallel}$ of the quantities $\chi_{\perp}(\omega)$ and $\chi_{\parallel}(\omega)$ [$(\chi_{\perp}(\omega_{\perp})=\infty$ and $\chi_{\parallel}(\omega_{\parallel})=\infty$; dissipation is ignored] are always different. We shall assume this difference to be sufficiently large and in discussing SIT in the $\omega\approx\omega_{\perp}$ case, we shall ignore the frequency dependence $\chi_{\parallel}(\omega)$, whereas in discussing SIT with $\omega\approx\omega_{\parallel}$, we shall ignore the frequency dependence $\chi_{\perp}(\omega)$.

In a study of the propagation of an intense light pulse of frequency located in the vicinity of one of these resonances it is necessary to apply (under the above conditions) the nonlinear relationship between the polarization of an impurity and the field under SIT conditions only to that component which corresponds to the polarizations of oscillations of a given resonance. In the case of the other "nonresonant" components of the polarization of the impurity and all the components of the polarization of the matrix we can use the conventional relationships from linear crystal optics.

§2. SELF-INDUCED TRANSPARENCY IN THE REGION OF A RESONANCE OF $\chi_{\parallel}(\omega)$

When the frequency of the electric field ω is close to $\omega_{\parallel}$ only the z components of the polarization of an impurity, i.e., the quantity P_z (the z axis is directed along the optic axis, Fig. 1), depends nonlinearly on the field component E_z. The intensity of the electric field in a medium can be written in the form

$$\mathbf{E}(\mathbf{r},t)=\mathbf{e}(\mathbf{r},t)\exp\left[i(\mathbf{kr}-\omega t+\varphi(\mathbf{r},t))\right],\tag{1}$$

where $\mathbf{e}(\mathbf{r},t)$ and $\varphi(\mathbf{r},t)$ are real functions; similar expressions apply also to the polarization per unit volume of the medium **P** and to the electrical induction **D**.

We shall consider a wave with a wave vector **k** making an angle θ with the z axis. In view of the isotropy of the medium in the xy plane, we can assume that the vector **k** lies in the plane yz without any loss of generality. Then,

$$\mathbf{kr}=k_z z+k_y y=k(z\cos\theta+y\sin\theta)\equiv k\zeta,\tag{2}$$

where $k=|\mathbf{k}|$ and $\zeta=z\cos\theta+y\sin\theta$. We shall consider only the case of plane waves, i.e., we shall assume that the quantities $\mathbf{e}(\mathbf{r},t)$ and $\varphi(\mathbf{r},t)$ are functions of time t and of a single spatial variable ζ.

In this sense the investigated waves are similar to one-dimensional solutions for SIT in isotropic problems.

The wave equation for the field **E** has obviously the form

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\mathbf{E}=-4\pi\left[\frac{1}{c^2}\frac{\partial^2\mathbf{P}}{\partial t^2}-\nabla(\nabla\mathbf{P})\right],\tag{3}$$

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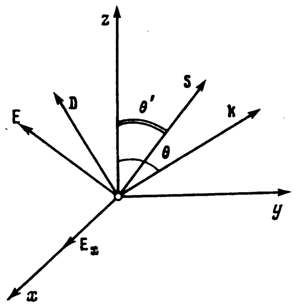

FIG. 1.

where we have applied the condition div $\mathbf{D} \equiv \mathrm{div}\,(\mathbf{E}+4\pi\mathbf{P}) = 0$. As mentioned earlier, all the quantities in a plane wave described by Eq. (1) or Eq. (2) are independent of the coordinate x. Therefore, a system of three equations (3) for the field components E_x, E_y, and E_z splits into a system of two coupled nonlinear equations for the components E_y and E_z:

$$\left.\begin{aligned}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)E_y&=-4\pi\left[\frac{1}{c^2}\frac{\partial^2 P_y}{\partial t^2}-\frac{\partial}{\partial y}\left(\frac{\partial P_y}{\partial y}+\frac{\partial P_z}{\partial z}\right)\right],\\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)E_z&=-4\pi\left[\frac{1}{c^2}\frac{\partial^2 P_z}{\partial t^2}-\frac{\partial}{\partial z}\left(\frac{\partial P_y}{\partial y}+\frac{\partial P_z}{\partial z}\right)\right]\end{aligned}\right\} \tag{3a}$$

and an independent linear equation for the component E_x:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)E_x=-\frac{4\pi}{c^2}\frac{\partial^2 P_x}{\partial t^2}. \tag{3b}$$

Since we have assumed that the resonance frequency $\omega_\perp$ is far from $\omega_\parallel$, the relationship between the polarization components $P_{x,y}$ and the field components $E_{x,y}$ can be regarded (as mentioned earlier) as linear, i.e.,

$$P_{x,y}=\frac{\varepsilon_\perp-1}{4\pi}E_{x,y} \tag{4}$$

and we can ignore the dependence of $\varepsilon_\perp$ on ω. The equation for E_x is linear [see Eqs. (3b) and (4)]. Therefore, its solution has the usual form

$$E_x=\mathrm{const}\cdot e^{i(k\zeta-\omega t)}, \tag{5}$$

where ω and k are linked by the dispersion relationship

$$k^2c^2/\omega^2=n_1^2=\varepsilon_\perp, \tag{6}$$

corresponding to an ordinary linear wave in a uniaxial crystal.[6]

We shall now consider a system of two equations (3a) for the field components E_y and E_z. We can find the coupling between these components employing the condition that the field $\mathbf{D}$ is transverse:

$$\mathrm{div}\,\mathbf{D}=\mathrm{div}\,\{\mathbf{d}(\zeta,t)\exp[i(k\zeta-\omega t+\varphi(\zeta,t))]\}=0.$$

Since

$$\frac{\partial}{\partial z}=\cos\theta\frac{\partial}{\partial\zeta},\quad \frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial\zeta}, \tag{7}$$

the condition $\mathrm{div}\,\mathbf{D}=0$ can be written in the form

$$\frac{\partial d_\zeta}{\partial\zeta}+i\left(k+\frac{\partial\varphi}{\partial\zeta}\right)d_\zeta=0, \tag{8}$$

where $d_\zeta=d_z\cos\theta+d_y\sin\theta$ is the projection of the vector $\mathbf{d}$ onto the direction of $\mathbf{k}$.

The only real solution of Eq. (8) is $d_\zeta(\zeta,t)=0$. It follows from the condition $d_\zeta=0$ that $d_y/d_z=-\cot\theta$ and hence we can express the quantities E_y and P_y in terms of the z component of the electric field and polarization:

$$E_y=-\frac{\mathrm{ctg}\,\theta}{\varepsilon_\perp}(E_z+4\pi P_z),\quad P_y=-\frac{\varepsilon_\perp-1}{4\pi}\frac{\mathrm{ctg}\,\theta}{\varepsilon_\perp}(E_z+4\pi P_z). \tag{9}$$

When the relationships in Eq. (9) are substituted into Eq. (3a), we obtain the following equation containing only the functions E_z and P_z:

$$\frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}-\left(1-\frac{\varepsilon_\perp-1}{\varepsilon_\perp}\cos^2\theta\right)\frac{\partial^2 E_z}{\partial\zeta^2}=-4\pi\left(\frac{1}{c^2}\frac{\partial^2 P_z}{\partial t^2}-\frac{\cos^2\theta}{\varepsilon_\perp}\frac{\partial^2 P_z}{\partial\zeta^2}\right). \tag{10}$$

It should be stressed that we have not assumed any specific dependence of the polarization component P_z on the electric field component E_z and, consequently, Eq. (10) holds for any medium with uniaxial anisotropy and with a linear relationship between the polarization and field components given by Eq. (4) in the xy plane. If, for example, we assume that P_z and E_z are also related linearly

$$P_z=\frac{\varepsilon_\parallel-1}{4\pi}E_z,$$

we find that in this case Eq. (10) has a solution in the form of a plane wave of constant amplitude, and the wave vector $\mathbf{k}$ and frequency ω are linked by the dispersion relationship for an extraordinary wave[6]

$$\frac{k^2c^2}{\omega^2}=n^2=\frac{\varepsilon_\perp\varepsilon_\parallel}{\varepsilon_\perp+(\varepsilon_\parallel-\varepsilon_\perp)\cos^2\theta}. \tag{11}$$

Following now the theory of SIT of Refs. 1–3, we shall utilize the nonlinear relationship between P_z and E_z and assume that the field E_z interacts with an impurity which is a two-level system for which the dipole moment μ of a transition, found including a correction for the effective field (see, for example, Ref. 7), is directed along the z axis. If in this case we ignore all the relaxation processes and consider only the case of exact resonance (i.e., if we assume that the field frequency is $\omega=\omega_\parallel$), we obtain the following expression for the total polarization P_z created by the field $E_z(\zeta,t)=e_z(\zeta,t)e^{i(k\zeta-\omega t)}$:

$$\left.\begin{aligned}&P_z(\zeta,t)=ip_z(\zeta,t)e^{i(k\zeta-\omega t)}+\frac{\varepsilon_\parallel^{(0)}-1}{4\pi}E_z,\\ &p_z={}^1/_2\mu\rho\sin\Psi(\zeta,t),\quad \Psi(\zeta,t)=\mu\int_{-\infty}^{t}e_z(\zeta,t')\,dt',\end{aligned}\right\} \tag{12}$$

where ρ is the density of two-level systems.

The expression (12) is a special case of the solution of the Bloch equations.[1-3] In describing real situations we naturally have to allow for a possible detuning of the field frequency ω from the exact resonance as well as for the inhomogeneous broadening of the transition line, occurrence of phase modulation ($\varphi \neq 0$), etc. All these effects have been discussed in detail in the theory of SIT.[1-3] It has been found that allowance for them results in some change in the shape and parameters of a pulse propagating in a medium. Since we shall be interested in the effects of anisotropy, we shall study them employing the simplest nonlinear relationship (12) which permits us to study the most important new features of SIT in the case under discussion.

We shall seek solutions of Eq. (10) satisfying, as in Eq. (12), the condition of slowness

$$\left.\begin{array}{ll} |\partial e_z/\partial t| \ll \omega e_z, & |\partial e_z/\partial \zeta| \ll k e_z, \\ |\partial \varphi/\partial t| \ll \omega, & |\partial \varphi/\partial \zeta| \ll k. \end{array}\right\}$$

In this case Eq. (10) for a field of general form (1) can be reduced to a system of equations for the field amplitude $e_z(\zeta, t)$ and phase $\varphi(\zeta, t)$. We can easily show that this system of equations has the form:

$$\frac{\partial e_z}{\partial t}+\frac{c^2k}{\omega}\frac{\varepsilon_\perp+(\varepsilon_\parallel^{(0)}-\varepsilon_\perp)\cos^2\theta}{\varepsilon_\perp\varepsilon_\parallel^{(0)}}\frac{\partial e_z}{\partial \zeta}=-\frac{2\pi\omega}{\varepsilon_\parallel^{(0)}}\left(1-\frac{k^2c^2}{\omega^2}\frac{\cos^2\theta}{\varepsilon_\perp}\right)p,$$

$$\frac{\partial \varphi}{\partial t}+\frac{c^2k}{\omega}\frac{\varepsilon_\perp+(\varepsilon_\parallel^{(0)}-\varepsilon_\perp)\cos^2\theta}{\varepsilon_\perp\varepsilon_\parallel^{(0)}}\frac{\partial \varphi}{\partial \zeta}=\frac{\omega}{2}\left(1-\frac{k^2c^2}{\omega^2}\frac{\varepsilon_\perp+(\varepsilon_\parallel^{(0)}-\varepsilon_\perp)\cos^2\theta}{\varepsilon_\perp\varepsilon_\parallel^{(0)}}\right). \tag{10a}$$

It follows from the second equation of this system that the assumption of absence of phase modulation $\varphi = 0$, used to derive Eq. (12), is justified only when the frequency ω and the wave vector **k** satisfy the following dispersion relationship:

$$\frac{k^2c^2}{\omega^2}=n_2^2=\frac{\varepsilon_\perp\varepsilon_\parallel^{(0)}}{\varepsilon_\perp+(\varepsilon_\parallel^{(0)}-\varepsilon_\perp)\cos^2\theta}. \tag{11a}$$

This expression gives the dispersion law for the carrier frequency in Eq. (12), is a special case of Eq. (11), (11), and reduces to the latter if $\chi_\parallel = 0$. Its origin is self-evident since the polarization of the impurity molecules along the z axis given by Eq. (12) is, under SIT conditions, shifted in phase away from the field by $\pi/2$ and, consequently, it makes no contribution to the dispersion relationship, although it does give rise to a dependence of the field amplitude e_z on the spatial and temporal variables.

The substitution of Eq. (12) into Eq. (10a) gives the following nonlinear equation for e_z:

$$\frac{\partial e_z}{\partial t}+\frac{c}{n_2}\frac{\partial e_z}{\partial \zeta}=-\frac{\pi\omega}{\varepsilon_\parallel^{(0)}}\mu\rho\left(1-\frac{n_2^2}{\varepsilon_\perp}\cos^2\theta\right)\sin\Psi, \tag{13}$$

where n_2 is defined by Eq. (11a). Equation (13) reduces to the sine-Gordon equation for the function Ψ, for which we can solve[8] the Cauchy problem and investigate the solution for any initial condition. However, we shall consider here only the influence of the anisotropy on the parameters of a self-similar soliton solution of the sine-Gordon equation (which is known as a 2π pulse).

We shall introduce a variable $\tau = t - \zeta/u$, where u is the velocity of such a pulse. We then find from Eq. (13) that

$$\frac{d^2\Psi}{d\tau^2}=\frac{1}{\tau_p^2}\sin\Psi,\quad \frac{1}{\tau_p^2}=\frac{\pi\omega\mu^2\rho}{\varepsilon_\parallel^{(0)}(c/un_2-1)}\left(1-\frac{n_2^2}{\varepsilon_\perp}\cos^2\theta\right). \tag{14}$$

Thus, in our case the self-similar solution is in the form of a soliton of duration τ_p depending not only on the velocity u, as in the case of isotropic media, but also on the direction of the wave vector relative to the optic axis. If $\theta = \pi/2$ (propagation across the optic axis), the expression for τ_p is identical with the corresponding expression for an isotropic medium characterized by $\varepsilon = \varepsilon_\parallel^{(0)}$:

$$\tau_p^2=\tau_p^{(0)2}=\frac{\varepsilon_\parallel^{(0)}(c/u(\varepsilon_\parallel^{(0)})^{1/2}-1)}{\pi\omega\mu^2\rho}$$

When the angle θ is reduced, the soliton duration for a fixed velocity u can both increase or decrease, depending on the relationship between the parameters $c/u(\varepsilon_\parallel^{(0)})^{1/2}$ and $\varepsilon_\perp/\varepsilon_\parallel^{(0)}$. Expanding the expression for τ_p based on Eq. (14) in powers of $\cos^2\theta \ll 1$, we obtain

$$\tau_p^2\approx\tau_p^{(0)2}\left[1+\frac{\varepsilon_\parallel^{(0)}}{\varepsilon_\perp}\left(\frac{c}{u(\varepsilon_\parallel^{(0)})^{1/2}}-1\right)\left(1+\frac{1}{2}\frac{1-\varepsilon_\perp/\varepsilon_\parallel^{(0)}}{1-u(\varepsilon_\parallel^{(0)})^{1/2}/c}\right)\cos^2\theta\right].$$

It is therefore clear that for a fixed value of the soliton velocity u, its duration τ_p is a nonmonotonic function of θ, decreasing on reduction in θ from the value $\theta = \pi/2$ and then rising in the limit $\theta \to 0$ when the inequality $u(\varepsilon_\parallel^{(0)})^{1/2}/c > 3/2 - \varepsilon_\perp/2\varepsilon_\parallel^{(0)}$ is obeyed. However, when $u(\varepsilon_\parallel^{(0)})^{1/2}/c < 3/2 - \varepsilon_\perp/2\varepsilon_\parallel^{(0)}$, the dependence of the soliton duration τ_p on the angle θ is monotonic. Figure 2 shows the dependence of τ_p on θ for certain values of the parameters $u(\varepsilon_\parallel^{(0)})^{1/2}/c$ and $\varepsilon_\perp/\varepsilon_\parallel^{(0)}$.

Substituting the expressions for E_z and P_z found from Eq. (14)

$$\left.\begin{array}{l} E_z=\dfrac{2}{\mu\tau_p}\,\mathrm{ch}^{-1}\dfrac{\tau}{\tau_p}\exp[i(k\zeta-\omega t)], \\ P_z=-i\mu\rho\dfrac{\mathrm{th}(\tau/\tau_p)}{\mathrm{ch}(\tau/\tau_p)}\exp[i(k\zeta-\omega t)]+\dfrac{\varepsilon_\parallel^{(0)}-1}{4\pi}E_z \end{array}\right\} \tag{15}$$

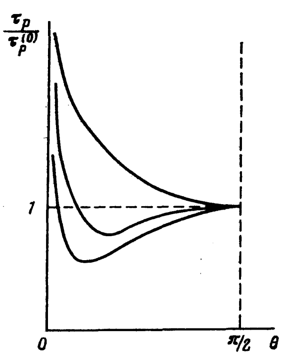

FIG. 2. Dependences of the relative duration of a soliton on the angle θ for different values of the parameters $u(\varepsilon_\parallel^{(0)})^{1/2}/c$ and $\varepsilon_\perp/\varepsilon_\parallel^{(0)}$ in the case of a resonance of $\chi_\parallel(\omega)$.

into Eq. (9), we obtain the final expression for the field component E_y:

$$E_y=-\frac{\operatorname{ctg}\theta}{\varepsilon_\perp}E_z\left(\varepsilon_\parallel^{(0)}-2\pi i\mu^2\rho\tau_p\operatorname{th}\frac{\tau}{\tau_p}\right)=E_y^{(1)}+iE_y^{(2)}. \qquad (16)$$

This component E_y has a part $E_y^{(1)}$ with the same phase as the component E_z and a part $E_y^{(2)}$ shifted in phase by $\pi/2$. Therefore, the direction of the vector $\mathbf{E}$ in the yz plane is not constant but oscillates at the field frequency about a certain average value whose direction governs the direction of the Poynting vector S in the wave.[6] The angle θ' between the Poynting vector S and the optic axis of the medium is given by

$$\operatorname{tg}\theta'/\operatorname{tg}\theta=\varepsilon_\perp/\varepsilon_\parallel^{(0)}, \qquad (17a)$$

which in our case replaces the familiar expression

$$\operatorname{tg}\theta'/\operatorname{tg}\theta=\varepsilon_\perp/\varepsilon_\parallel \qquad (17b)$$

from linear crystal optics.

Our solutions for an ordinary pulse (5) and an extraordinary pulse (15), (16) in a uniaxial medium in the region of resonance of $\chi_\parallel(\omega)$ are independent. This means that a pulse with an arbitrary polarization may be expanded into ordinary and extraordinary pulses, which propagate in a medium indepedently of one another and normally along different directions and different velocities.

Under these conditions (i.e., when $\omega=\omega_\parallel$) an extraordinary pulse should be stationary (stable) and this theoretical prediction should be borne in mind in designing suitable experiments. We shall now consider SIT under conditions such that the field frequency is $\omega\approx\omega_\perp$, i.e., it is close to the resonance frequency of the transverse polarizability of the impurity molecules $\chi_\perp(\omega)$.

§3. SELF-INDUCED TRANSPARENCY IN THE REGION OF A RESONANCE OF $\chi_\perp(\omega)$

In this case the nonlinear relationship between the polarization of the medium and the field applies only in the xy plane, whereas the component of the polarization vector along the z axis is related linearly to the field component E_z

$$P_z=\frac{\varepsilon_\parallel-1}{4\pi}E_z. \qquad (18)$$

The optical isotropy in the xy plane applies to crystals with different unit cells. In particular, a unit cell may contain an impurity molecule, whose symmetry group is such that the components μ_x and μ_y of the dipole moment of a transition at a frequency $\omega_\perp$ transform in accordance with the same irreducible representation of the group. We shall call this case degenerate. In this case the direction of the polarization $\mathbf{P}_\perp$ in a plane is always the same as the direction of the field $\mathbf{E}_\perp$ in the same plane.

We can also have a different situation when a transition of frequency $\omega_\perp$ in each of the impurity molecules is nondegenerate but a unit cell of a crystal contains two molecules. In this case a substituted molecule may have two orientations, since the dipole moments of a transition in an impurity may be oriented along the x and y directions, respectively. These two alternatives are indistinguishable in linear electrodynamics: in both cases the form of the tensor ε_{ij} is the same. However, in our case when the polarization is related nonlinearly to the field, these two situations are basically different and should be considered separately.

a. We shall begin with the degenerate case. We shall consider the propagation of an electromagnetic wave with a wave vector $\mathbf{k}$ which has the spherical coordinates φ, θ (Fig. 3). We shall adopt a coordinate system $\xi\eta\zeta$ with the ζ axis directed along $\mathbf{k}$. This can be done by rotating the initial coordinate system about the z axis through an angle φ and then about the η axis through an angle θ. The vectors in the old and new systems are related by

$$E_i=A_{ik}E_k',$$

where A_{ik} is the rotation matrix[9]; similar relationships apply also for P and D.

The wave equation for the vector $\mathbf{E}$ still has the form (3). From Eq. (3) we can obtain equations for the field components

$$E_\eta=-\sin\varphi E_x+\cos\varphi E_y,$$
$$E_\zeta=\cos\theta(\cos\varphi E_x+\sin\varphi E_y)-\sin\theta E_z=\cos\theta E_\perp-\sin\theta E_z,$$

which have the following form

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial\zeta^2}\right)E_\eta=-\frac{4\pi}{c^2}\frac{\partial^2 P_\eta}{\partial t^2}, \qquad (19)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial\zeta^2}\right)E_\zeta=-\frac{4\pi}{c^2}\frac{\partial^2 P_\zeta}{\partial t^2}, \qquad (20)$$

where $E_\perp$ is the projection onto the xy plane of the component of the field directed along $\mathbf{k}$.

We shall now use, as in Sec. 2, the condition that the field D is transverse. In the case under discussion this condition has the form

$$D_\zeta=\sin\theta D_\perp+\cos\theta D_z=0,$$

so that allowing for Eq. (18), we obtain

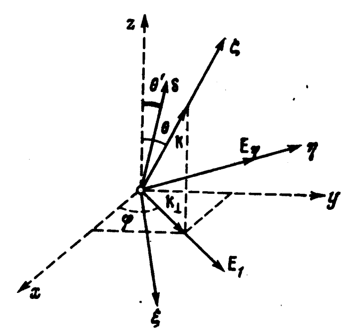

FIG. 3.

$$E_z = -\frac{\operatorname{tg}\theta}{\varepsilon_\parallel}(E_1 + 4\pi P_1), \quad P_z = -\frac{\varepsilon_\parallel - 1}{4\pi}\frac{\operatorname{tg}\theta}{\varepsilon_\parallel}(E_1 + 4\pi P_1). \tag{21}$$

Substituting Eq. (21) into Eq. (20), we finally obtain the following equation for the component E_1:

$$\frac{1}{c^2}\frac{\partial^2 E_1}{\partial t^2} - \left(\cos^2\theta + \frac{\sin^2\theta}{\varepsilon_\parallel}\right)\frac{\partial^2 E_1}{\partial \zeta^2} = -4\pi\left(\frac{1}{c^2}\frac{\partial^2 P_1}{\partial t^2} - \frac{\sin^2\theta}{\varepsilon_\parallel}\frac{\partial^2 P_1}{\partial \zeta^2}\right). \tag{22}$$

As in the preceding section, it is assumed in the derivation of Eqs. (19), (20), and (22) that (exactly as in the case of plane waves) all the functions depend on just one spatial variable ζ and on time t.

It should be noted that Eqs. (19) and (22) are not independent. Since the polarizations P_η and P_1 lie in the xy plane, each of them is governed by the total projection of $\mathbf{E}_\perp$ onto this plane. Therefore, in order to investigate the propagation of a wave at an arbitrary initial polarization and allow for the nonlinear relationship between the polarization and the field, it is necessary to solve the system of equations (19), (22) which clearly requires numerical methods even after going over to slow variables. In view of this, we shall consider the propagation of waves only in two special cases: 1) when the field $\mathbf{E}$ is polarized along the η axis ($E = E_\eta$, $E_1 = P_1 = 0$); 2) $E_\eta = 0$. In the former case we can use the relationship between the polarization P_η and the field E_η in the form of Eq. (12) and we clearly obtain from Eq. (19) the usual 2π pulse of McCall and Hahn with the dispersion $\omega = ck/(\varepsilon_\perp^{(0)})^{1/2}$ and the duration τ_p independent of the angles. Thus, this wave is an analog of an ordinary wave in linear crystal optics.

In the latter case there is a dependence on the direction of the wave vector. Then, the solution also has the form of a 2π pulse with the dispersion law

$$\frac{k^2c^2}{\omega^2} = n_2^2 = \frac{\varepsilon_\perp^{(0)}\varepsilon_\parallel}{\varepsilon_\perp^{(0)} + (\varepsilon_\parallel - \varepsilon_\perp^{(0)})\cos^2\theta} \tag{23}$$

and the duration

$$\frac{1}{\tau_p^2} = \frac{\pi\omega\mu^2\rho}{\varepsilon_\perp^{(0)}(c/un_2 - 1)}\left(1 - \frac{n_2^2}{\varepsilon_\parallel}\sin^2\theta\right). \tag{24}$$

The expression (23) is in the form of the dispersion relationship for the linear case of Eq. (11) when $\chi_\perp = 0$. This agreement and the form of the dispersion for an ordinary wave are due to, as pointed out in Sec. 2, the absence from the expression (12) for the impurity polarization of a term which has the same phase as the field E_η or E_1, respectively. The field with the components E_1 and E_z is obviously an analog of an extraordinary linear wave.

When a wave propagates along the optic axis (angle $\theta = 0$) we have $n_2^2 = \varepsilon_\perp^{(0)}$ and the formula for the duration τ_p is of the same nature as the formula for the duration of a soliton in the isotropic case. However, if $\theta \ll 1$, the expression (24) can be expanded as a series

$$\tau_p^2 = \tau_p^{(0)^2}\left[1 + \frac{\varepsilon_\perp^{(0)}}{\varepsilon_\parallel}\left(1 + \frac{1}{2}\frac{1 - \varepsilon_\parallel/\varepsilon_\perp^{(0)}}{1 - u(\varepsilon_\perp^{0})^{1/2}/c}\right)\theta^2\right],$$

$$\tau_p^{(0)^2} = \varepsilon_\perp^{(0)}\left(\frac{c}{u(\varepsilon_\perp^{(0)})^{1/2}} - 1\right)\Big/\pi\omega\mu^2\rho$$

and, consequently, the dependence of τ_p on the angle θ for a fixed velocity of a soliton u once again is nonmonotonic for $u(\varepsilon_\perp^{(0)})^{1/2}/c > 3/2 - \varepsilon_\parallel/2\varepsilon_\perp^{(0)}$. If $\theta \to \pi/2$, the duration of a soliton increases and for $\theta = \pi/2$ (propagation across the optic axis) we have an extraordinary wave in the form of a linear wave whose polarization is directed along the z axis. Figure 4 shows the dependences of the soliton duration τ_p on the angle θ for some values of the parameters $u(\varepsilon_\perp^{(0)})^{1/2}/c$ and $\varepsilon_\parallel/\varepsilon_\perp^{(0)}$ in the case under discussion.

Substituting the solutions for E_1 and P_1 into Eq. (21) and averaging in the same way as in Sec. 2 over the field period, we find that instead of Eq. (17a) we have an analogous expression

$$\operatorname{tg}\theta'/\operatorname{tg}\theta = \varepsilon_\perp^{(0)}/\varepsilon_\parallel. \tag{25}$$

b. We shall now consider a crystal whose unit cell contains two molecules, when a substitutional impurity has a nondegenerate transition frequency $\omega_\perp$, and the dipole moment of the transition in the impurity replacing the first or second molecule of the matrix is directed along the x and y axes, respectively. As mentioned at the beginning of this section, in the linear theory this case is indistinguishable from a degenerate transition. However, if the polarizations P_x and P_y are nonlinear functions of the corresponding field components E_x and E_y [see, for example, Eq. (12)], the polarization $\mathbf{P}_\perp$ in the xy plane is not collinear with the direction of the field $E_\perp$ in the same plane, in contrast to the case of a degenerate transition. This is a consequence of the trivial circumstance that the ratio of two nonlinear functions is not proportional to the ratio of their arguments. In this case, we can obtain Eqs. (19) and (22) for the field components E_η and E_1. In the case of a degenerate transition we can obtain independent solutions of Eqs. (19) and (22) by specifying a given polarization of the electric field in the medium, but there is no need to restrict the direction of the wave vector $\mathbf{k}$. In view of the noncollinearity of the vectors $\mathbf{P}_\perp$ and $\mathbf{E}_\perp$ for an arbitrary direction of the vector $\mathbf{k}$, this condition is necessary but not sufficient. For example, if the electric field is polarized along the η axis (polarization of an ordinary wave), the polariza-

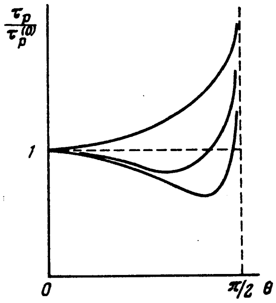

FIG. 4. Dependences of the relative duration of a soliton on the angle θ for different values of the parameters $u(\varepsilon_\perp^{(0)})^{1/2}/c$. and $\varepsilon_\parallel/\varepsilon_\perp^{(0)}$ in the case of a resonance of $\chi_\perp(\omega)$.

tion vector **P** of a unit volume of the medium has other nonzero components. Clearly, this rule is not obeyed in those cases when the wave vector is directed along the principal symmetry axes of a crystal and, as shown below, along the directions with the azimuthal angle $\varphi=\pi/4$.

It is convenient to continue our analysis in the coordinate system xyz. In all the cases under discussion the system of equations for the field components E_x and E_y splits and it is quite easy to analyze their solutions.

As before, the linear polarization P_z can be eliminated and then a plane wave in the case when the slowness condition is satisfied is described by the following system of equations:

$$a_{\alpha\beta}\frac{d^2\Psi_\beta}{d\tau^2}=-\frac{\pi\omega\mu^2\rho}{\varepsilon_\perp^{(0)}}b_{\alpha\beta}\sin\Psi_\beta, \qquad (26)$$

where $a_{\alpha\beta}$ is a matrix with the components

$$a_{11}=\frac{c}{u(\varepsilon_\perp{}^0)^{1/2}}n\left(\frac{\sin^2\theta\cos^2\varphi}{\varepsilon_\parallel}+\sin^2\varphi+\cos^2\theta\right)-1,$$

$$a_{12}=-\frac{c}{u(\varepsilon_\perp{}^0)^{1/2}}n\frac{\varepsilon_\parallel-1}{\varepsilon_\parallel}\sin^2\theta\sin\varphi\cos\varphi,$$

$$a_{21}=-\frac{c}{u(\varepsilon_\perp{}^0)^{1/2}}n\frac{\varepsilon_\parallel-1}{\varepsilon_\parallel}\sin^2\theta\sin\varphi\cos\varphi,$$

$$a_{22}=\frac{c}{u(\varepsilon_\perp{}^0)^{1/2}}n\left(\sin^2\theta\cos^2\varphi+\frac{\sin^2\theta\sin^2\varphi}{\varepsilon_\parallel}+\cos^2\theta\right)-1,$$

$$b_{\alpha\beta}=\begin{bmatrix}1-\frac{n^2}{\varepsilon_\parallel}\sin^2\theta\cos^2\varphi & -\frac{n^2}{\varepsilon_\parallel}\sin^2\theta\sin\varphi\cos\varphi\\ -\frac{n^2}{\varepsilon_\parallel}\sin^2\theta\sin\varphi\cos\varphi & 1-\frac{n^2}{\varepsilon_\parallel}\sin^2\theta\sin^2\varphi\end{bmatrix}.$$

The expression for the refractive index n of ordinary and extraordinary waves is identical with the corresponding expression for the case of a degenerate transition and we also have

$$\Psi_\alpha=\mu\int_{-\infty}^{t}e_\alpha(t')\,dt',\quad \alpha,\beta=x,y.$$

It is clear from Eq. (26) that only when a wave propagates along the xyz axes or along the direction with the azimuthal angle $\varphi=\pi/4$ do the equations (26) split and a further analysis is similar to the case of a degenerate transition in the xy plane. In the case of an arbitrary direction of propagation of a plane wave we have been unable to find an analytic solution of the system (26) and a numerical analysis is clearly needed. It should be noted that throughout this paper we have eliminated the linear part of the polarization from the wave equation in the case of a plane wave; this restriction is not essential and it can be introduced by applying the condition $\operatorname{div}\mathbf{D}=0$ for an arbitrary field.

We shall end here with an analysis of the influence of the anisotropy of an optically uniaxial matrix on the characteristics of self-induced transparency in the region of impurity resonances and we shall stress that it is also desirable to analyze the characteristics of self-induced transparency in matrices with other symmetries and structures.

When an allowance is made for nonlinear optical effects (i.e., in other words, when an allowance is made for nonlinear polarizabilities) the symmetry of the medium becomes lower than the symmetry of the tensor ε_{ij} of linear crystal optics. However, under SIT conditions this lowering of the symmetry is manifested extremally. In fact, in the case of crystals with several molecules per unit cell—as is clear, for example, from the system (26)—the shape of an SIT pulse for an arbitrary direction of propagation is governed by the values of the functions Ψ_α, where $\alpha=1,2,\ldots,\nu$ and where ν is the number of molecules per unit cell; each of the functions Ψ_α is governed by the orientation and spectral properties of the molecule α. This means that an investigation of SIT in complex crystals may be of interest not only from the point of view of applications (generation of pulses of different durations and shapes) but also in the form of a characteristic site-selection spectroscopy which supplements linear spectroscopy methods (see, for example, Ref. 10).

Use of anisotropic crystals extends greatly the range of investigations and further analysis of SIT in anisotropic media is undoubtedly urgent. Its importance will increase further when suitable experiments are carried out.

[1)]Characteristics of SIT are discussed in Ref. 5 in the case when the impurity and matrix resonances are close.

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Translated by A. Tybulewicz