Nonlinear acoustic attenuation in superconductors

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The attenuation of longitudinal, finite-amplitude short-wave sound in a pure superconductor is considered. Due to some peculiar properties of the interaction between the longitudinal electric fields induced by the sound and the superconductor excitations, localized resonant excitation states arise in the superconductor. The states are formed through Andreev reflection from the inhomogeneities of the spatial relief of the sound wave. The conditions for localized and the decay mechanism of the localized states, i.e., tunneling of the excitation through the energy gap, are studied. The boundary conditions for the kinetic equation are derived with account taken of the tunneling processes. The attenuation coefficient is calculated. It has a complex amplitude and temperature dependence, and differs from the BCS dependence for any amplitude, however small, near $T = 0$ and $T = T_c$, because the nonlinearity threshold tends to zero.

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The interaction of longitudinal electric fields with superconductors represents one of the central problems of superconductors that is widely discussed at the present time. One of the methods of excitation of an electric field in a superconductor is connected with the transmission through it of a longitudinal sound wave. The acoustic properties of superconductors have been studied in a large number of researches, beginning with the fundamental work of Bardeen, Cooper and Schrieffer;[1] However, they were concerned principally with linear phenomena.[2] At the same time, not all aspects, by far, of the effect of a longitudinal field on a superconductor can be accounted for in a weakly nonequilibrium situation. From this point of view, nonlinear acoustic effects associated with a strong disequilibrium of the electron distribution are of interest. Such a nonlinearity can be divided into two categories, depending on the value of the parameter ql (q is the wave vector of the sound, and l is the impurity free path length). At not too large values of this parameter, the main source of the nonlinearity is the electron heating and effects of stimulation.[3] As one moves into the short-wave region, $ql \gg 1$, due to the release of a resonant group of electrons that interact intensely with the sound wave, the dominant role is assumed by effects of the type of momentum nonlinearity,[4] connected with the strong disequilibrium of the resonant electrons. This disequilibrium manifests itself primarily in the nonlinearity of the acoustic attenuation.

The momentum nonlinearity studied in Ref. 4 is not a specifically superconducting effect. It exists also in the normal metal,[5] while the mechanism of its generation—localization of the electrons by the sound wave—is insensitive to the singularities of the electron motion in the superconducting state. These singularities, however, are important. The fact is that a traveling electric field produced by a sound wave acts on the excitations in the superconductor as some effective transverse perturbation.[6] Such a perturbation, as is well known, is capable of localizing the excitations in the troughs of its spatial relief, due to the mechanism of Andreev reflection,[7] similar to the way in which magnetic surface states are formed in superconductors.[8] A characteristic feature of this effective perturbation is that it is large only in the resonance region, while in the remaining part of phase space it is small in step with the smallness of the ratio s/v_F. Therefore, the sound field has practically no effect on the value of the parameter of superconducting ordering Δ, right up to very large amplitudes. Due to this feature, the dynamic effect of formation of a gapless spectrum in the resonance region becomes predominant in strong fields, together with associated tunnel exchange, that leads to a decay in the localized states, of interactions between the electron and hole branches of the spectrum.

The Andreev localization of the excitations and its nonequilibrium nature brought about by the distribution of the excitations lead, with increase in the amplitude, to a nonlinear decrease in the value of the acoustic attentuation. The threshold of this nonlinearity depends on the temperature, tending to zero at $T=0$. Near T_c, the magnitude of the effect turns out to be comparable with the total attenuation, and not with the small superconducting correction to the damping of the normal metal. Such a strong effect is explained by the radical rearrangement of the trajectories of the excitations, which is preserved up to the moment of onset of the tunnel mechanism of decay of the localized states. The damping reaches the level of the normal metal with disappearance of the localized states.

The present paper is devoted to the study of the effect of the Andreev localization of the excitations on the nonlinear properties of the acoustic attenuation in a superconductor. Our consideration completely ignores the existence of momentum nonlinearity, a procedure correct in those cases in which the considered amplitudes are less than the level of the momentum nonlinearity or if the Andreev and the ordinary electron localizations are centered in different phase regions. In the latter case, both effects are additive. A detailed analysis of the phenomenon of Andreev localization under conditions of momentum nonlinearity lies beyond the framework of this research.

The plan of the paper is the following. In the first section we give the solution of the kinetic equation and classify the trajectories of the excitations. In the second section, the tunneling of the excitations through the gap is analyzed and the corresponding transparency coefficient is calculated. In the third section, the boundary conditions are derived for the distribution functions with account of tunneling processes. The fourth section is devoted to the calculation of the nonlinear attenuation coefficient.

1. SOLUTION OF THE KINETIC EQUATION

We consider the propagation of longitudinal short-wave sound $ql \gg 1$ ($q \parallel x$), in a pure superconductor, under the conditions

$$\omega, \tau^{-1} \ll \Delta; \quad \tau = l/v_F, \quad \omega = sq. \tag{1}$$

In simple models of the "jellium" type with impurities[9] the effect of the sound on the electron system is connected with the longitudinal electric field created by it, Φ. Within the limits of applicability of the inequalities in Eq. (1), the excitation distribution is described by the classical kinetic equation[11,92)]

$$\frac{\partial f_\sigma}{\partial t} + \sigma \frac{\partial \varepsilon}{\partial p_x} \frac{\partial f_\sigma}{\partial x} - \sigma \frac{\partial \varepsilon}{\partial x} \frac{\partial f_\sigma}{\partial p_x} = I_{imp} + I_{ph},$$
$$\varepsilon = ((\xi + \Phi)^2 + \Delta^2)^{1/2}, \quad \sigma = \pm. \tag{2}$$

For simplification of the analysis of the phenomenon of Andreev localization of the excitations that is specific for the superconductor, it is desirable to eliminate the electron-trajectory bending of which leads to the momentum nonlinearity.[4] For this purpose, we use the well known expansion for the electron momentum:

$$p_x \approx p_{Fx} + \xi / v_{Fx}, \quad p_{Fx} = \eta p_F, \tag{3}$$

which is valid under the condition

$$p_x v_x \gg \xi. \tag{4}$$

In connection with the inequality (4), it should be noted that although it is satisfied with a large safety margin for the bulk of the electrons, it requires some care in the resonance region because of the small values of p_x (for example ~ms near T_c).

Along with the expansion (3) in Eqs. (2), it is appropriate also to go over to the variable $\xi + \Phi \to \xi$, which determines the conservation laws in the collision integrals,[9] after which Eq. (2) takes the form

$$\frac{\partial f_\sigma}{\partial t} + \sigma v_x \frac{\xi}{\varepsilon} \frac{\partial f_\sigma}{\partial x} - \frac{\partial \Phi}{\partial t} \frac{\partial f_\sigma}{\partial \xi} = I_{imp} + I_{ph}. \tag{5}$$

The physical premises for constructing the nonlinearized solutions of Eq. (5) are the following. Equation (5) describes the process of transfer of the energy of the wave of the longitudinal electric field to the phonon system. This process takes place in two steps. The electric field interacts directly with the electrons of the superconductor, while at $ql \gg 1$ this interaction is confined to a narrow resonance range. Due to the presence of impurities, isotropization of the resonance disequilibrium takes place. Here, because of the narrowness of the resonance region, the resultant isotropic disequilibrium can turn out to be small, even if the perturbation in the resonance group is comparable with unity. In the second stage, the energy corresponding to the isotropic disequilibrium is slowly ($\tau_{ph} \gg \tau$) transferred to the phonons.

The acoustic attenuation length in the superconductor is large, as is well known, in comparison with the length of the sound wave, just as is the characteristic length of generation of the harmonics. Therefore, the field Φ in not too long samples can be assumed to be harmonic:

$$\Phi(x, t) = \Phi_0 \cos \chi, \quad \chi = qx - \omega t. \tag{6}$$

Under these conditions, a stationary state is established which, in correspondence with what has been said above, is conveniently represented in the form

$$f_\sigma(\xi, \eta, x, t) = f_\sigma^0(\varepsilon) + f_\sigma^1(\xi, \eta, \chi), \quad \overline{\langle f_\sigma^1 \rangle} = 0, \tag{7}$$

where $\langle \cdots \rangle$ denotes averaging over the angles and the bar denotes averaging over the wave coordinate χ.

Relative to the degree of nonequilibrium condition, we shall assume that

$$\langle f_\sigma^1 \rangle \ll f_\sigma^0,$$
$$\delta f_\sigma^0 = f_\sigma^0 - \frac{1}{2}\left(1 + \operatorname{th} \frac{\sigma \varepsilon}{2T}\right) \ll f_\sigma^0, \tag{8}$$

here f_σ^1 is not necessarily small everywhere:

$$\max f_\sigma^1 \lesssim f_\sigma^0. \tag{9}$$

The inequalities (8) allow us to linearize the phonon collision integral, and also to neglect the arrival terms in the collision integrals. As a result, the equation for f_σ^1 takes the form

$$\left(\sigma \frac{\xi}{\varepsilon} - \beta\right) \frac{\partial f_\sigma^1}{\partial \chi} - \beta \Phi_\chi' \frac{\partial f_\sigma^1}{\partial \xi} + \frac{\beta}{\omega \tau} \left| \frac{\xi}{\varepsilon} \right| f_\sigma^1 = \beta \Phi_\chi' \frac{d f_\sigma^0}{d \xi} \tag{10}$$

$$\beta = s / \eta v_F.$$

Carrying out the averaging of Eq. (5) with account of the condition (8), we obtain the following equation for the determination of the isotropic increment δf_σ^0:

$$\overline{\left\langle \dot{\Phi} \frac{\partial f_\sigma^1}{\partial \xi} \right\rangle} = \frac{\hat{1}}{\tau_{ph}} \delta f_\sigma^0. \tag{11}$$

An estimate of the smallness of the heating effects described by (11) can be obtained by substituting there the function f_σ^1 calculated in linear approximation:[2)]

$$\left(\frac{\Phi}{T}\right)^2 \frac{s}{v_F} \omega \tau_{ph} \ll 1. \tag{12}$$

From a comparison of (12) with the nonlinearity levels (54) and (59) (see below), it is seen that under satisfaction of the condition

$$ql \gg \tau_{ph}/\tau \tag{13}$$

the heating effects are negligible. Starting out from this, the function f_σ^0 we shall assume to be in equilibrium in what follows.

Proceeding to the solution of Eq. (10), we note the symmetries possessed by its solution

$$f_\sigma^1(-\beta) = -f_{-\sigma}^1(\beta), \tag{14a}$$
$$f_{-\sigma}^1(-\xi, \chi) = -f_\sigma^1(\xi, \chi + \pi). \tag{14b}$$

The symmetry (14a) allows us to limit ourselves to positive β in what follows.

The solution of the kinetic equation (10) is constructed by the method of characteristics. The characteristic system of Eq. (10) has the form

$$\frac{d\chi}{dt} = \sigma \frac{\xi}{\varepsilon} - \beta \equiv v, \quad \frac{d\xi}{dt} = -\beta \frac{d\Phi}{d\chi}, \tag{15}$$

where t is the time of motion along the characteristic. Equations (15) posses the integral

$$E=\sigma\varepsilon-\beta\xi+\beta\Phi=K(\xi)+\beta\Phi, \tag{16}$$

which has the meaning of the total energy of motion along the characteristics and consists of the sum of the kinetic $K(\xi)$ and potential $\beta\Phi$ energies. The kinetic energy $K(\xi)$ consists of two branches which are separated at $\beta<1$ by the anisotropic energy gap (Fig. 1a):

$$|K|>\Delta(1-\beta^2)^{1/2}. \tag{17}$$

At $\beta>1$ the energy gap is absent (Fig. 1b). The appearance of an anisotropy in the dispersion law has a kinematic origin, associated with the fact that in writing down the conservation law (16), a transition was made to a moving set of coordinates. In what follows, we shall need the inverse function $\xi(K)$:

$$\xi_\alpha=\frac{1}{1-\beta^2}[\beta K+\alpha(K^2-\Delta^2(1-\beta^2))^{1/2}],$$
$$\alpha=\pm,\quad \alpha=\operatorname{sign}K\cdot\operatorname{sign}v,\quad \beta<1. \tag{18}$$

The potential energy $\beta\Phi$ in (16) stems exclusively from the traveling character of the electric field accompanying the sound wave and represents an effective perturbation of the transverse type.

We now considered the classification of the phase trajectories of the dynamical system (15).

1. $\beta<1$ (Fig. 2).

a) $|E|>\Delta(1-\beta^2)^{1/2}+\beta\Phi_0$ are infinite trajectories;

b) $|\Delta(1-\beta^2)^{1/2}-\beta\Phi_0|<|E|<\Delta(1-\beta^2)^{1/2}+\beta\Phi_0$ are finite trajectories, and the equation for the determination of the turning points is:

$$E-\sigma\Delta(1-\beta^2)^{1/2}=\beta\Phi(\chi_{1,2});$$

c) $|E|<\Delta(1-\beta^2)^{1/2}-\beta\Phi_0$, $\beta\Phi_0<\Delta(1-\beta^2)^{1/2}$ is a forbidden region (Fig. 2a):

d) $|E|<\beta\Phi_0-\Delta(1-\beta^2)^{1/2}$, $\beta\Phi_0>\Delta(1-\beta^2)^{1/2}$ are tunnel trajectories (Fig. 2b); in this region, the motion has a complicated character, connected with the possibility of tunnel coupling between states belonging to different branches of $K(\xi)$ and separated by a spatial energy barrier. In this case, the quantum spectrum of the system is gapless.

2. $\beta>1$ (Fig. 3)—infinite trajectories; the motion takes place from right to left. At $\beta\Phi_0\gg\Delta(\beta^2-1)^{1/2}$, as will be shown in the following point, tunnel coupling also arises

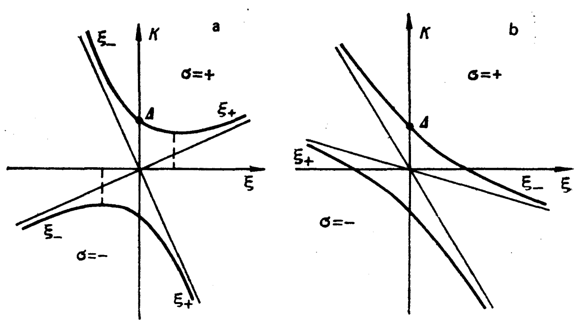

FIG. 1.

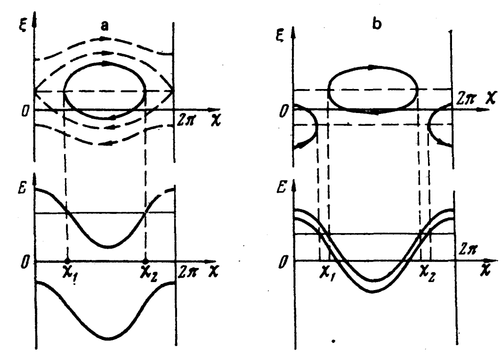

FIG. 2.

between branches of the spectrum Fig. 1b—splitting of the characteristics. The splitting points are determined from the equation $E=\beta\Phi(\chi_{1,2})$.

In those cases in which tunnel processes can be neglected, the characteristic curves are continuous and the solution of the kinetic equation has the form

$$f_\sigma^1=-\int_{-\infty}^{t}df_\sigma^0(t')\exp\left(\frac{\beta}{\omega\tau}\int_t^{t'}|v(t'')+\beta|dt''\right). \tag{19}$$

For construction of the solution in the region of tunnel trajectories it is necessary to formulate the boundary condition at the turning points. As Δ decreases, the fraction of tunnel trajectories increases, reflecting the process of decay of the localized states and the transition to a normal metal, where the phenomenon of Andreev localization is obviously lacking. From the viewpoint of this transition, the analysis of tunnel processes is of fundamental interest.

2. TUNNELING THROUGH THE GAP

In the general quantum case, the quasiparticle states in a superconductor in a longitudinal electric field Φ are described by the wave equation

$$i\dot{\psi}=\hat{H}\psi,\quad \hat{H}=\left(-iv_z\frac{\partial}{\partial x}+\Phi\right)\sigma_z+\Delta\sigma_x. \tag{20}$$

In a system of coordinates accompanying the sound wave, the Hamiltonian (20) has stationary states which, with account of the transformation corresponding to the transition $\xi+\Phi\to\xi$, is conveniently written in the form

$$\psi(x,t)=\exp\left\{-iEt+\frac{i\beta}{\omega}\int^{\chi}d\chi'\Phi(\chi')\right\}\varphi(\chi),$$
$$\hat{H}\varphi=E\varphi,\quad \hat{H}=-\frac{i\omega}{\beta}(\sigma_z-\beta)\frac{\partial}{\partial\chi}+\beta\Phi+\Delta\sigma_x. \tag{21}$$

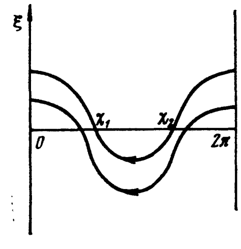

FIG. 3.

The solutions of this equation have a close formal analogy with the states of a Dirac particle located in a static magnetic field.

At $\Delta=0$, the system (21) splits into two independent equations of first order, from which the field can be eliminated by a phase transformation. Thus, in a normal metal, in the approximation (3), the effect of the longitudinal field on the electron spectrum is absent.

We now consider the quasiclassical solutions of Eq. (21). We shall seek them in the form

$$\varphi=u\exp\left\{\frac{i\beta}{\omega}\int\xi\,d\chi\right\}, \tag{22}$$

then $u(\chi)$ is determined from equation

$$\{(\sigma_z-\beta)\xi+\Delta\sigma_x+\beta\Phi-E\}u=\frac{i\omega}{\beta}(\sigma_z-\beta)\frac{du}{d\chi}. \tag{23}$$

The basis of the quasiclassical expansion is the assumption that the change of $u(\chi)$ is slow in comparison with the scale of oscillations of the exponential (22). Far from the turning points, the corresponding inequality has the form

$$\frac{1}{u}\frac{du}{d\chi}\ll\frac{\beta\xi}{\omega}. \tag{24}$$

At $T\sim\Delta$, for resonance excitations with $\beta\sim\min(1,\omega\tau)$, this condition is equivalent to the basic inequalities (1).

Taking what has been said above into account, for the construction of the solution (23) in zeroth approximation, the right-hand part of the equation must be set equal to zero. At $\Delta=0$, the condition for solvability of the truncated equation has the form

$$(\pm1-\beta)\xi+\beta\Phi=0, \tag{25}$$

whence it is seen that the excitations in a normal metal move along the asymptotes of the graphs of Fig. 1, while the point of intersection of the asymptotes for them is not singular (the Andreev reflection is absent).

At $\Delta\neq0$ the condition of solvability is identical with formula (16):

$$(K+\beta\xi)^2=\xi^2+\Delta^2,\quad K=E-\beta\Phi. \tag{26}$$

The solution of Eq. (21) in zeroth approximation has the form

$$u^0=A\begin{pmatrix}((1+\beta+v)/2)^{1/2}\\ \operatorname{sign}(K+\beta\xi)((1-\beta-v)/2)^{1/2}\end{pmatrix},\quad (u^0,u^0)=A^2(\chi). \tag{27}$$

The spatial dependence of the scalar normalization constant $A(\chi)$ is found from the condition of solvability of the equation of the next approximation for u^1:

$$\left(u^0,(\sigma_z-\beta)\frac{du^0}{d\chi}\right)=0,$$

whence

$$A=\mathrm{const}\cdot|v|^{-1/2}. \tag{28}$$

The resultant solution satisfies the current conservation law which follows from (21):

$$\frac{dj}{d\chi}=\frac{d}{d\chi}(\varphi,(\sigma_z-\beta)\varphi)=0. \tag{29}$$

Substituting Eqs. (22), (27) and (28) in (29), we have

$$j(\chi)\sim\operatorname{sign}v. \tag{30}$$

At $\beta<1$, the quasiparticle trajectories, as noted above, have turning points near which the quasiclassical approximation becomes unsuitable. Interest attaches to the situation of Fig. 2b, when at one and the same energy E there are turning points corresponding to both branches of the kinetic energy and a possibility of tunneling arises. For the calculation of the tunneling coefficient in the region of finite transparency, we need to consider the case of two closely situated turning points, since in the opposite case the quasiclassical probability of tunneling is exponentially small. We expand the potential at the point χ_0: $\beta\Phi(\chi_0)=E$ (for definiteness, let $0<\chi_0<\pi$); we then have from the condition of turning of Fig. 1b

$$\chi_1-\chi_0=\pm\varkappa,\quad \varkappa=\Delta(1-\beta^2)^{1/2}(\beta^2\Phi_0^2-E^2)^{-1/2}\ll1. \tag{31}$$

Completing the analogous expansion in Eq. (21), we represent its solution in the form

$$\varphi=\exp\left(\frac{i\beta z^2}{2\lambda}\right)\begin{pmatrix}u/(1-\beta)^{1/2}\\ \dfrac{i\lambda u'+zu}{(1+\beta)^{1/2}}\end{pmatrix},\qquad z=\frac{\chi-\chi_0}{\varkappa},\quad \lambda=\frac{\omega(\beta^2\Phi_0^2-E^2)^{1/2}}{\beta\Delta^2}, \tag{32}$$

then $u(z)$ satisfies the equation

$$\lambda^2u''+(z^2-1-i\lambda)u=0. \tag{33}$$

The solutions of Eq. (33) are expressed in terms of the parabolic cylinder function D_ν.[10] Using the fundamental system of solutions of Eq. (33), we construct a pair of linearly independent functions

$$\varphi^\pm=\exp\left(\frac{i\beta z^2}{2\lambda}\right)\begin{pmatrix}(1-\beta)^{-1/2}D_{-1+i/2\lambda}\left(\pm z\dfrac{1+i}{\lambda^{1/2}}\right)\\ \dfrac{\pm\lambda^{1/2}(1-i)}{(1+\beta)^{1/2}}D_{i/2\lambda}\left(\pm z\dfrac{1+i}{\lambda^{1/2}}\right)\end{pmatrix}. \tag{34}$$

Linear combinations of the functions (34) complete the entire set of solutions of Eq. (21), including those corresponding to the scattering states. The latter are defined such that at $+\infty$ $(-\infty)$ there is only a departing wave. For the construction of such states, we express the asymptotes of the solutions (34) in the region of overlap $1\ll z\ll1/\varkappa$ in terms of the quasiclassical waves functions (22). At $z>0$ (<0) the departing wave, by virtue of (30) and (18), corresponds to the solution $\varphi_\alpha=\varphi_+$. Selecting suitable linear combinations, we obtain the following expressions for the scattering states:

$$|1\rangle=\begin{cases}\varphi_-+\dfrac{2\operatorname{sh}(\pi/2\lambda)}{(2\pi)^{1/2}}\Gamma\left(-\dfrac{i}{2\lambda}\right)\left(\dfrac{e^{i\pi/4}}{(2\lambda)^{1/2}}\right)^{1+i/\lambda}\varphi_+, & z<0\\ e^{-\pi/2\lambda}\varphi_+, & z>0\end{cases} \tag{35}$$

$$|2\rangle=\begin{cases}-e^{-\pi/2\lambda}\varphi_+, & z<0\\ \varphi_-+(2\pi)^{1/2}e^{-\pi/2\lambda}\Gamma^{-1}\left(-\dfrac{i}{2\lambda}\right)\left(\dfrac{e^{i\pi/4}}{(2\lambda)^{1/2}}\right)^{-1-i/\lambda}\varphi_+, & z>0\end{cases} \tag{36}$$

Constructing an expression for the current (29) with the help of (35) and (36), we obtain the coefficients of transmission and reflection:

$$D=\frac{j_{\mathrm{tran}}}{j_{\mathrm{inc}}}=e^{-\pi/\lambda},\quad R=\frac{j_{\mathrm{refl}}}{j_{\mathrm{inc}}}=1-e^{-\pi/\lambda},\quad D+R=1. \tag{37}$$

The form of the coefficients D and R (37) is preserved at the point χ_2.

At $\beta>1$, there are no turning points on the phase trajectories; however, as analysis shows, in the vicinity of the point χ_0,

$$|\chi-\chi_0|\sim|\varkappa|\ll 1$$

trajectories that correspond to different α pass close to one another and a tunneling coupling arises between them. Calculation of the transmission coefficient in this case is completely analogous to that considered above. The solitary difference is that in the splitting state both departing waves are located to the left of χ_0. The final expressions for the splitting states are the following:

$$|1\rangle=\begin{cases}\left(\dfrac{e^{-i\varkappa}}{(2\lambda)^{1/2}}\right)^{-i/2\lambda}\varphi_- -\dfrac{(2\pi)^{1/2}e^{-\pi/2\lambda}}{2\lambda}\Gamma^{-1}\left(\dfrac{i}{2\lambda}\right)\left(\dfrac{e^{-i\pi/4}}{(2\lambda)^{1/2}}\right)^{-3+i/\lambda}\varphi_+, & z<0\\ \varphi_+, & z>0\end{cases}, \tag{38}$$

$$|2\rangle=\begin{cases}(2\pi)^{-1/2}\Gamma\left(\dfrac{i}{2\lambda}\right)(2\lambda)^{-1/2+i/2\lambda}(1-e^{-3\pi/4\lambda})\varphi_- +e^{-\pi/2\lambda}\varphi_+, & z<0\\ -\varphi_-, & z>0\end{cases}. \tag{39}$$

The coefficients D and R coincide with the expressions (37).

The tunneling process considered above is a nonlinear effect, the probability of which is close to unity at

$$\Phi_0\gg\Phi_T=\Delta^2/\omega. \tag{40}$$

As is seen from this expression, high transparency can be achieved at the expense of a reduction of Δ or by increasing the pumping level. The latter means that by in principle increasing the pumping we can, at any $T<T_c$, cause the resonance states to move just as in a normal metal, as a result of which the acoustic attenuation reaches that of the normal metal. Actually, this phenomenon can be observed only near T_c.

In the calculation of the transparency coefficient D above, we used the inequality $\varkappa\ll 1$ (31). We now show that in the tunneling region (40) it is certainly satisfied. As follows from the estimates of Sec. 4, in the tunneling region we have the combination

$$(1-\beta^2)^{1/2}/\beta\sim\max(1/\omega\tau,\ (1/\omega\tau)^{1/2}).$$

Substituting in (31) this estimate and $\Phi_0\sim\Phi_T$, we obtain

$$\varkappa\sim\max(1/\tau\Delta,\ (\omega/\tau\Delta^2)^{1/2})\ll 1.$$

The condition (31) allows us to assume the phase trajectories in Fig. 2b and Fig. 3 to be intersecting. Moreover, a comparison of (31) with formulas (16)–(18) shows that in the region of high transmission, the gap in the kinetic energy spectrum cannot be taken into account.

3. BOUNDARY CONDITIONS FOR THE DISTRIBUTION FUNCTIONS

The results obtained in the preceding section allow us to establish the form of the boundary conditions for the distribution functions. For this purpose, we need to extend the coupling formulas obtained for the wave functions to include mixed states described by a density matrix. We consider initially the formulas (35) and (36) ($\beta<1$, $0<\chi_0<\pi$). Starting out from the complete set of scattering states $|E,\nu\rangle$, we can define the scattering matrix $\hat\gamma$ by the expression

$$\hat\gamma=\int dE\sum_{\nu=1}^{2}|E,\nu\rangle\langle E,\nu|F_\nu(E), \tag{41}$$

where $F_\nu(E)$ is the excitation distribution function over the scattering states.

Substituting formulas (35) and (36) in (41), we have, with quasiclassical accuracy,

$$\begin{aligned}\hat\gamma(\chi,\chi')&=\int dE\left[\frac{1}{|v_+|}\hat E_+(DF_1+RF_2)+\frac{1}{|v_-|}\hat E_-F_2\right],\quad \chi\gg\chi_0,\\ \hat\gamma(\chi,\chi')&=\int dE\left[\frac{1}{|v_+|}\hat E_+(RF_1+DF_2)+\frac{1}{|v_-|}\hat E_-F_1\right],\quad \chi\ll\chi_0,\end{aligned} \tag{42}$$

where

$$\hat E_\alpha=\frac{1}{2}\left(1+\frac{\sigma_z\xi_\alpha+\sigma_x\Delta}{K-\beta\xi_\alpha}\right)\exp\left[\frac{i\beta\xi}{\omega}(\chi-\chi')\right].$$

On the other hand, the expression for the density matrix $\hat\gamma$ in terms of the distribution functions f_σ, which enter into the kinetic expression (10), has the following form in the mixed Wigner representation:[11]

$$\hat\gamma(\xi,\chi)=\sum_\sigma\frac{1}{2}\left(1+\sigma\frac{\sigma_z\xi+\sigma_x\Delta}{\varepsilon}\right)f_\sigma(\xi,\chi). \tag{43}$$

Transforming to the coordinate representation in (43)

$$\hat\gamma(\chi,\chi')=\int d\xi\exp\left[\frac{i\beta\xi}{\omega}(\chi-\chi')\right]\hat\gamma\left(\xi,\frac{\chi+\chi'}{2}\right)$$

and making the change of variables $\xi,\sigma\to E,\alpha$, we get after comparison of the resultant expression with (42), the following boundary condition for the distribution function $f_\alpha(E,\chi)$ at the point χ_1 (Fig. 2b):

$$f_+(\chi_1\pm0)=Rf_-(\chi_1\pm0)+Df_-(\chi_1\mp0). \tag{44a}$$

At the turning point $\pi<\chi_2<2\pi$ the boundary condition has the form

$$f_-(\chi_2\pm0)=Rf_+(\chi_2\pm0)+Df_+(\chi_2\mp0). \tag{44b}$$

Repeating the analogous discussions as applied to formulas (38) and (39), we obtain the following boundary condition at the splitting points (Fig. 3):

$$f_\alpha(\chi_{1,2}-0)=Rf_\alpha(\chi_{1,2}+0)+Df_{-\alpha}(\chi_{1,2}+0). \tag{45}$$

4. NONLINEAR ATTENUATION

The nonlinear acoustic-attenuation coefficient is determined by the average value of the energy transferred to the electron system per unit time,

$$W=-\mathrm{Sp}\,\overline{\hat{\dot H}\hat\gamma},$$

referred to the energy of the sound wave. Turning to the explicit forms of the Hamiltonian (20) and of the density matrix (43), we obtain from this definition the following expression for the attenuation coefficient:

$$\Gamma=\frac{\omega}{\Phi_0^2}\int d\xi\sum\sigma\frac{\xi}{\varepsilon}\overline{\Phi_\chi'\langle f_\sigma^1\rangle}. \tag{46}$$

It is convenient to transform this expression, going over to the variables E and α and interchanging the order of integration over the energy and of the averaging over the wave coordinate. Using the equations of the characteristics (15) and the symmetry property (14a), we rewrite (46) in the form

$$\Gamma=-\frac{\omega s}{\Phi_0^2 v_F}\int_0^\infty\frac{d\beta}{\beta^2}\int dE\oint\frac{d\varepsilon(t)}{2\pi\beta}f_\alpha^1(t),$$

$$\varepsilon=\pm\frac{\Delta}{(1-(v+\beta)^2)^{1/2}},\quad v=\alpha\frac{(1-\beta^2)(K^2-\Delta^2(1-\beta^2))^{1/2}}{K+\alpha\beta(K^2-\Delta^2(1-\beta^2))^{1/2}},\quad K=E-\beta\Phi, \tag{47}$$

where the line integral along the characteristics accomplishes averaging over the entire allowed interval of motion within the limits of a single period.

For continuous solutions, the formula (47), after substituting expression (19) in it, allows a further transformation: a change in the order of integration along the characteristics, which makes use of the property of invariance of the averaging operation relative to displacements. Formula (47) in this case takes the form

$$\Gamma=\frac{\omega s}{\Phi_0^2 v_F}\int_0^\infty\frac{d\beta}{\beta^2}\int dE\oint\frac{df^0(t)}{2\pi\beta}\int_t^\infty d\varepsilon(t')\exp\left(-\frac{\beta}{\omega\tau}\int_t^{t'}|v+\beta|dt''\right). \tag{48}$$

We make use of formula (48) for calculation of the linear damping coefficient. In the linear region, the velocity of motion is not perturbed: v=const, and after substitution of the variable

$$d\varepsilon=-\beta(1+\beta/v)d\Phi$$

the integrals along the characteristics are easily calculated:

$$\Gamma=\frac{\omega s}{4v_F}\int_0^\infty\frac{d\beta}{\beta^2}\int_{-\infty}^\infty dE\sum_\alpha\frac{d}{d\varepsilon}\left(\text{th}\frac{\varepsilon}{2T}\right)\left|1+\frac{\beta}{v}\right|\frac{1/\omega\tau}{[v/\beta(v+\beta)]^2+(1/\omega\tau)^2}. \tag{49}$$

Subsequent calculation is connected with a transition to integration over ε, after which the integrals in formulas (49) are separated and reduced to standard ones. As a result, the well known BCS formula is obtained[1]

$$\Gamma=\Gamma_n\left(1-\text{th}\frac{\Delta}{2T}\right),\quad\Gamma_n=\frac{\pi}{2}\frac{\omega s}{v_F}. \tag{50}$$

Formula (50) describes the attenuation in two physically different frequency regions: in the collisionless region (large $\omega\tau$) and in the region with the resonance strongly smeared out by the collisions (small $\omega\tau$). This characteristic property of linear acoustic attenuation disappears in the nonlinear region. The reason for this is that, depending on the structure of the resonance region, the nonlinearity mechansims turn out to be completely different. For explanation of these mechansims, we analyze formulas (48) and (49).

The structure of the resonance region is determined by the denominator in the expression (49), which is obtained as a result of double integration of the kinetic exponential in (48) over $\Phi(\chi)$. The resonance values of β and v are so chosen as to guarantee a rate of change of this exponential equal to the period of the sound wave.

a) At large $\omega\tau$, the resonance velocities are small:

$$v\ll\beta,\quad v\sim\beta^2/\omega\tau. \tag{51}$$

With account of (51), the formulas for ε and v (47) at $T\lesssim\Delta$ take the form

$$\varepsilon=\pm\frac{\Delta}{(1-\beta^2)^{1/2}},\quad v=\alpha(1-\beta^2)^{3/4}\left[\frac{2}{\Delta}(|K|-\Delta(1-\beta^2)^{1/2})\right]^{1/2}. \tag{52}$$

The resonance β, as follows from (49), are concentrated in the regions

$$\beta\sim(T/\Delta)^{1/2},\quad T\ll\Delta;\quad 1-\beta\sim\Delta^2/T^2,\quad\Delta\ll T.$$

It is then seen that, by strengthening the inequality (51) near T_c,

$$v\ll1-\beta,\quad\Delta\ll T, \tag{53}$$

the approximations (52) can be extended into the region $\Delta\ll T$.

As is seen from (52), the perturbation does not affect the energy ε, but only the velocity v, which, after transition in (48) to integration over χ, appears only in the argument of the kinetic exponential. Therefore acoustic pumping affects only this argument, causing a nonlinear broadening of the resonance region. The threshold of the nonlinearity obviously corresponds with such a value of the perturbation $\beta\Phi$ which spans the collision width of the resonance, $|K|-\Delta(1-\beta^2)^{1/2}\sim T^2\max(1,T/\Delta)/\Delta(\omega\tau)^2$:

$$\Phi_0\gtrsim\Phi_1,\quad\Phi_1=\frac{T}{(\omega\tau)^2}\max\left(\left(\frac{T}{\Delta}\right)^{1/2},\frac{T^2}{\Delta^2}\right). \tag{54}$$

The relations (51) and (53) compatible with the estimates for the resonant β restrict the high-frequency region by the condition

$$\omega\tau\gg\max((T/\Delta)^{1/2},T^2/\Delta^2). \tag{55}$$

In the nonlinear region, the inequalities (51) and (53) impose a definite restriction also on its amplitude. Substituting in their estimate of v determined by the nonlinear resonance width $|K|-\Delta(1-\beta^2)^{1/2}\sim\beta\Phi$, we obtain the condition

$$\Phi_0\ll\Phi_2,\quad\Phi_2=T\min((\Delta/T)^{1/2},\Delta^2/T^2)\gg\Phi_1. \tag{56}$$

b) At small $\omega\tau$ the characteristics β are small:

$$\beta\ll v,\quad\beta\sim\omega\tau. \tag{57}$$

In this case, the velocity v, after transition in (48) to integration over χ, drops out completely of the argument of the kinetic exponential. Therefore, the mechanism of nonlinear broadening of the resonance ceases to be effective. The onset of the nonlinearity is now connected simply with the appreciable perturbation of the kinetic energy of the resonance excitations: $\beta\Phi\sim K-\Delta$. Taking it into account that, as a consequence of (57),

$$\varepsilon=K=\pm\Delta/(1-v^2)^{1/2}, \tag{58}$$

while the resonant velocities lie in the region $v\sim\min(1,(T/\Delta)^{1/2})$, we obtain the following estimate for the level of the low-frequency nonlinearity:

$$\Phi_0\gtrsim\Phi_3,\quad\Phi_3=T/\omega\tau. \tag{59}$$

The frequency limitation for this nonlinearity has the form

$$\omega\tau\ll\min(1,(T/\Delta)^{1/2}). \tag{60}$$

c) The inequalities (54) and (60) span the entire frequency range, with the exception of the region of intermediate frequencies near T_c:

$$1\ll\omega\tau\ll T^2/\Delta^2. \tag{61}$$

Analysis of this region is more complicated, since here we must construct an expansion in terms of two independent small parameters: Δ/T and $1/\omega\tau$. We limit ourselves to the zeroth approximation in the parameter Δ/T, since, as will be seen from what follows, the nonlinear effect is contained even in this approximation. The corresponding approximations and estimates have the form

$$v=\alpha\,\mathrm{sign}\,K-\beta,\quad \varepsilon=\frac{K}{1-\alpha\beta\,\mathrm{sign}\,K},\qquad v\sim1-\beta\sim1/\omega\tau,\quad v\ll\beta. \tag{62}$$

As is seen from (62), the velocity in this case remains constant almost everywhere; therefore the mechanism of nonlinearity here is analogous to the low-frequency mechansim described in the previous section. The level of the nonlinearity that follows from the estimate (62) with account of the resonance condition α sign $K>0$ is identical with (59).

We now compare the obtained level of the nonlinearity with the level of tunneling (40). In the high-frequency region, not only is the inequality $\Phi_1\ll\Phi_T$ satisfied, but also $\Phi_2\ll\Phi_T$ [a consequence of (1)]. Therefore, in the entire high-frequency range, the tunneling processes can be neglected. A similar situation exists also at very low frequencies, far from T_c: $\Phi_3\ll\Phi_T$, $\Delta^2\tau\gg T$. Near T_c: $\Delta^2\tau\ll T$ the amplitude decay of the localized states sets in earlier then the low-frequency nonlinearity $\Phi_3\gg\Phi_T$.

Basing ourselves on the approximations that have been made, we proceed directly to a calculation of the coefficient of nonlinear attenuation.

a) High frequencies

In this region β is always less than unity; therefore, one needs to take into account only two types of trajectories: the infinite 1a and the finite 1b. Substituting (51)–(53) in the expression (48), we reduce it to the form

$$\Gamma=\frac{2\omega s}{\Phi_0^2v_F}\int_0^1\frac{d\beta\beta}{4T}\,\mathrm{ch}^{-2}\left(\frac{\Delta}{2T(1-\beta^2)^{1/2}}\right)\int_{\Delta(1-\beta^2)^{1/2}-\beta\Phi_0}^{\infty}dE\sum_\alpha\times\oint\frac{d\Phi(t)}{2\pi|v|}\int_t^\infty\frac{d\Phi(t')}{v(t')}\exp\left(-\frac{\beta^2}{\omega\tau}(t'-t)\right). \tag{63}$$

In writing down the formulas (63), we have taken into account the symmetry (14b), as a consequence of which the energies lying above and below the gap make the same contribution to the attenuation.

The next step consists in the transformation of the inner integral over the characteristic, which takes into account the periodic character of the motion

$$\int_t^\infty\exp\left(-\frac{\beta^2}{\omega\tau}(t'-t)\right)=\sum_{n=0}^\infty\exp\left(-\frac{\beta^2nT}{\omega\tau}\right)\int_t^{t+T}\exp\left(-\frac{\beta^2}{\omega\tau}(t'-t)\right), \tag{64}$$

where T is the period corresponding to motion along the infinite trajectory from $\chi=0$ to $\chi=2\pi$ and also the finite from χ_1 to χ_2 and back. A change in the integral (64) to the variable χ' gives the following formulas after several transformations:

infinite trajectories:

$$\int_t^\infty=\left(1-\exp\left[-\frac{\alpha\beta^2T}{\omega\tau}\right]\right)^{-1}\int_\chi^{\chi+2\pi}\frac{d\Phi(\chi')}{v(\chi')}\exp\left[-\frac{\alpha\beta^2}{\omega\tau}|t'-t|\right]; \tag{65}$$

finite trajectories:

$$\int_t^\infty=\frac{1}{2}\,\mathrm{ch}^{-1}\left(\frac{\beta^2T}{4\omega\tau}\right)\left[\int_\chi^{\chi_2}\frac{d\Phi(\chi')}{v(\chi')}\exp\left(-\frac{\alpha\beta^2}{\omega\tau}\left(|t'-t|-\frac{T}{4}\right)\right)-\int_{\chi_1}^{\chi}\frac{d\Phi(\chi')}{v(\chi')}\exp\left(-\frac{\alpha\beta^2}{\omega\tau}\left(|t'-t|+\frac{T}{4}\right)\right)\right]. \tag{66}$$

In the strong nonlinearity limit, the arguments of the exponentials in formulas (65) and (66) are small: $\beta^2/\nu\omega t\sim\Phi_1/\Phi_0\ll1$. Substituting (65) and (66) in (63) and expanding the exponentials, and after taking all possible parameters outside the integral signs, we obtain the result

$$\Gamma=\Gamma_n(\Delta/\Phi_0(\omega\tau)^2)^{1/3}Cf(\Delta/T), \tag{67}$$

where the temperature function $f(\Delta/T)$ is defined by the integral

$$f(x)=x\int_1^\infty dt(t^2-1)^{1/6}\,\mathrm{ch}^{-2}\frac{tx}{2}=\begin{cases}6{,}2x^{-1/6}e^{-x}, & x\gg1\\ 4{,}1x^{-4/3}, & x\ll1\end{cases},$$

and the numerical coefficient C consists of the sum of the contributions of the infinite and finite motions:

$$C=C_{inf}+C_{fin};$$

$$C_{inf}=\frac{1}{2^{4/3}\pi^2}\int_1^\infty dh\int_0^{2\pi}\frac{d\cos\chi}{(h-\cos\chi)^{1/3}}\int_\chi^{\chi+2\pi}\frac{d\cos\chi'}{(h-\cos\chi')^{1/3}}\left(\frac{\gamma(\chi',\chi)^2}{\gamma(2\pi,0)}-\gamma(\chi',\chi)\right),$$

$$C_{fin}=\frac{1}{2^{4/3}\pi^2}\int_{-1}^1dh\int_{\chi_1}^{\pi}\frac{d\cos\chi}{(h-\cos\chi)^{1/3}}\int_{\chi_1}^{\chi}\frac{d\cos\chi'}{(h-\cos\chi')^{1/3}}\gamma(\chi',\chi),$$

$$\gamma(\chi',\chi)=\int_\chi^{\chi'}\frac{d\chi''}{(h-\cos\chi'')^{1/3}},\quad \cos\chi_1=h.$$

b) Low frequencies

1. $\Phi_0\ll\Phi_T, D=0$. We substitute the relations (57) and (58) in the formula (48) and, using the symmetry (14b), we reduce it to the form

$$\Gamma=-\frac{2\omega s}{\Phi_0^2v_F}\int_0^\infty\frac{d\beta}{\beta^2}\int_{\Delta-\beta\Phi_0}^\infty dE\sum_\alpha\mathrm{sign}\,v\oint\frac{df^0(\varphi)}{2\pi}\int_\varphi^\infty d\Phi(\varphi')\exp\left[-\frac{\beta}{\omega\tau}(\varphi'-\varphi)\right],\qquad \frac{d\varphi}{d\chi}=\mathrm{sign}\,v. \tag{68}$$

As follows from (57), the basic contribution to the damping is made by small $\beta\ll1$; therefore, in the calculation of (68), we must take into account only trajectories of two types: 1a and 1b. In the region of infinite motion, the integrals over φ' and E are calculated in elementary fashion:

$$\Gamma_{inf}=2\frac{\omega s}{v_F}\int_0^\infty\frac{d\beta}{1+\beta^2}\int_0^{2\pi}\frac{d\chi}{2\pi}\sin^2\chi\left(1-\mathrm{th}\frac{\Delta+\beta\omega\tau\Phi_0(1-\cos\chi)}{2T}\right),\quad \beta/\omega\tau\to\beta. \tag{69}$$

In the region of finite trajectories, the integral over φ' is transformed analogously to the high-frequency case:

$$\int_\varphi^\infty=\frac{1}{2}\,\mathrm{sh}^{-1}\frac{\beta\Pi}{2\omega\tau}\left[\int_{\chi_1}^{\chi}d\Phi\exp\left(-\frac{\alpha\beta}{\omega\tau}\left(\chi'-\chi+\frac{\Pi}{2}\right)\right)+\int_\chi^{\chi_2}d\Phi\exp\left(-\frac{\alpha\beta}{\omega\tau}\left(\chi'-\chi-\frac{\Pi}{2}\right)\right)\right],\quad \Pi=\chi_2-\chi_1, \tag{70}$$

and is also easily calculated. Substitution of the result in (68) leads to the expression

$$\Gamma_{fin}=4\frac{\omega s}{v_F}\int_0^\infty\frac{d\beta}{1+\beta^2}\int_{-1}^{1}dh\int_{\chi_1}^{\chi_2}\frac{df^0}{2\pi}\left(\sin\chi+\sin\chi_1\frac{\operatorname{sh}\beta(\chi-\pi)}{\operatorname{sh}(\beta\Pi/2)}\right),\quad E=\Delta+\beta\Phi_0 h. \tag{71}$$

The first term in (69) gives the damping of the normal metal; the second is cancelled by the first term upon addition to (71). As a result, the nonlinear amplitude dependence of the attenuation coefficient is contained only in the second term of the expression (71). Adding (69) and (71) we obtain after a number of transformations the final result:

$$\frac{\Gamma}{\Gamma_n}=1-\left(\frac{2}{\pi}\right)^2\int_0^\infty\frac{d\beta\,\beta}{1+\beta^2}\int_0^\pi d\chi\frac{\sin^2\chi}{\operatorname{sh}\beta\chi}\int_0^\chi d\chi'\operatorname{ch}\beta\chi'\operatorname{th}\frac{\Delta+\beta\omega T\Phi_0(\cos\chi'-\cos\chi)}{2T} \tag{72}$$

The asymptotes of the expression (72) have the following form at small and large amplitudes:

$$\frac{\Gamma}{\Gamma_n}=1-\operatorname{th}\frac{\Delta}{2T}-0{,}2\frac{\Phi_0}{\Phi_3}\operatorname{ch}^{-2}\frac{\Delta}{2T},\quad \Phi_0\ll\Phi_3,$$
$$\frac{\Gamma}{\Gamma_n}=\frac{\Phi_3}{\Phi_0}\left(0{,}6\ln(1+e^{-\Delta/T})\ln\frac{\Phi_0}{\Phi_3}-f\left(\frac{\Delta}{T}\right)\right),\quad \Phi_0\gg\Phi_3, \tag{73}$$

$$f(x)=2.5\ln(1+e^{-x})+0.8\int_0^\infty\frac{dy\ln y}{1+e^{x+y}}=\begin{cases}2e^{-x}, & x\gg1\\ 1.6, & x\ll1\end{cases}.$$

2. $\Phi_0\gg\Phi_T, R\ll1$. In the region of high transparency, the contribution of the infinite trajectories to the damping remains as before—formula (69), but in correspondence with what was said at the end of Sec. 2, Δ should be omitted from it. The finite trajectories are now absent while the energy interval $|E|<\beta\Phi_0$ is filled with tunnel trajectories of type 1d.

To obtain for the kinetic equation (10) a solution that takes into account the tunneling processes, we proceed as follows. Within the limits of the continuity interval, the solution of Eq. (10) satisfies the relation

$$f_{\sigma\alpha}^1(\varphi)=-\int_{\varphi_0}^{\varphi'}df^0(\varphi')\exp\left[\frac{\beta}{\omega\tau}(\varphi'-\varphi)\right]+\exp\left[\frac{\beta}{\omega\tau}(\varphi_0-\varphi)\right]f_{\sigma\alpha}^1(\varphi_0). \tag{74}$$

Here it is convenient to introduce the band index σ explicitly, since, as a consequence of the tunneling acts the distribution functions of the different bands turn out to be coupled. We make use of the relation (74) and express the distribution function at a certain point in terms of its value $c_{\sigma\alpha}=f_{\sigma\alpha}^1(\varphi_{\sigma\alpha})$ at the nearest turning point $\varphi_{\sigma\alpha}<\varphi$. At different values of σ and α, these points correspond to the following wave coordinates (Fig. 2b):

$$\varphi_{++}\to\chi_1,\quad \varphi_{+-}\to\chi_2,\quad \varphi_{-+}\to\chi_1,\quad \varphi_{--}\to-\chi_1(\chi_2).$$

The four quantities $c_{\sigma\alpha}$ are connected by four equations which are a combination of the boundary conditions (44) and the relations (74) with account of the periodicity:

$$c_{\sigma,\alpha}-R\exp[-\beta\Pi_\sigma/\omega\tau]c_{\sigma,-\alpha}-D\exp[-\beta\Pi_{-\sigma}/\omega\tau]c_{-\sigma,-\alpha}=-RI_\sigma-DI_{-\sigma}, \tag{75}$$

$$I_+=\int_{\chi_1}^{\chi_2}df^0\exp\left[-\frac{\beta}{\omega\tau}(\chi_2-\chi')\right],\quad I_-=-\int_{-\chi_1}^{\chi_1}df^0\exp\left[-\frac{\beta}{\omega\tau}(\chi'+\chi_1)\right],$$
$$\Pi_+=\chi_2-\chi_1,\quad \Pi_-=2\chi_1.$$

The symmetry condition

$$c_{\sigma,\alpha}=c_{\sigma,-\alpha}=c_\sigma$$

follows from this system of equations; this is a natural consequence, in view of the symmetry of the spectrum branches $\alpha=\pm$.

Substitution of the functions $f_{\sigma\alpha}^1(\varphi)$, which are expressed in terms of the quantity $c_{\sigma\alpha}$, in formula (47), after integration over the wave coordinate and a number of transformations, with account of the infinite contribution, reduces it to the form

$$\frac{\Gamma}{\Gamma_n}=1-\left(\frac{2}{\pi}\right)^2\int_0^\infty\frac{d\beta\,\beta}{1+\beta^2}\int_{-1}^{1}dh\sin\chi_1\frac{R}{R-D}(c_+-c_-). \tag{76}$$

At $D=0$, the formula (76) duplicates the result (72) of the previous section, in which $\Delta=0$. In the opposite limit $R=0$ the damping in the superconductor is identical with the damping in the normal metal. We emphasize that this result is valid at any temperature.

To obtain the first correction in $R\ll1$, it suffices to solve the equation (75), setting $R=0$ in it. Substitution of the solution in formula (76) gives

$$\frac{\Gamma}{\Gamma_n}=1-\frac{8\Delta^2}{\pi\omega\Phi_0}\int_0^\infty\frac{d\beta\,\beta}{1+\beta^2}\int_0^\pi d\chi\sin\chi\frac{\operatorname{sh}\beta(\pi-x)}{\operatorname{sh}\beta\pi}$$
$$\times\int_0^\chi d\chi'\operatorname{ch}\beta\chi'\operatorname{th}\frac{\beta\omega\tau\Phi_0(\cos\chi'-\cos\chi)}{2T}. \tag{77}$$

In the limit of small and large amplitudes, the asymptotes of (77) have the form

$$\frac{\Gamma-\Gamma_n}{\Gamma_n}=\begin{cases}-0{,}8\Delta^2\tau/T, & \Phi_T\ll\Phi_0\ll\Phi_3\\ -3{,}1\Delta^2/\omega\Phi_0, & \Phi_T,\Phi_3\ll\Phi_0\end{cases}. \tag{78}$$

The resultant expressions indicate the presence of a singularity in the behavior of the acoustic attenuation on going through T_c. This singularity is of greatest interest in the region of small amplitudes. According to the BCS formula (50), the transition to the normal metal is brought about by the parameter Δ/T, while Γ tends to Γ_n in proportion to $(T_c-T)^{+1/2}$. This result, however, is valid, strictly speaking, only at $\Phi_0=0$. At any small, but finite amplitude (within the limits of the assumed approximations), the transition to the normal metal is due, as is seen from (73) and (78), not to the parameter Δ/T but to the tunneling parameter $\Delta^2/\omega\Phi_0$. The temperature dependence $\Gamma(T)$ then acquires near T_c we have a section that is linear in T_c-T.

c) Intermediate frequencies

1. $\Phi_0\ll\Phi_T, D=0$. In this frequency range, the resonances β lie near $\beta=1$ (62); therefore, along with trajectories of type 1a and 1b ($\beta<1$), we must also take into account the infinite trajectories of type 2 ($\beta>1$). This circumstance, and also the absence of symmetry in α, complicates the calculation of the damping, although in principle it repeats the scheme of the previous section. Substituting the relations (62) in formula (48), we have

$$\Gamma=2\frac{\omega s}{\Phi_0^2 v_F}\int_0^\infty\frac{d\beta}{\beta^2}\int_{-\beta\Phi_0}^\infty dE\sum_\alpha\operatorname{sign}v$$
$$\times\oint\frac{df^0}{2\pi\beta}\int_\varphi^\infty d\varepsilon_\alpha(\varphi')\exp\left(-\frac{\beta}{\omega\tau}\int_\varphi^{\varphi'}|Q_\alpha|d\varphi''\right),$$
$$Q_\alpha=\frac{1}{\alpha\operatorname{sign}K-\beta} \tag{79}$$

Simple calculation of the contribution of the infinite trajectories with energies $E>\beta\Phi_0$ leads to a formula identical with (69). Here, just as in the linear region, only terms with α sign $K>0$, corresponding to resonance velocities, are important.

For simplification of the further calculations, it is advantageous to use this circumstance, immediately replacing the exponentials of Q_- by unity. Taking this fact into account, we calculate the contribution of the finite trajectories $E<\beta\Phi_0$, $\beta<1$. A transformation similar to formula (70) gives, in the case $\alpha=+$,

$$\int_{\varphi}^{\infty}=-\frac{1}{2}\operatorname{sh}^{-1}\frac{\beta\Pi Q_+}{2\omega\tau}\Big[\int_{\chi_1}^{\chi}d\varepsilon_+\exp\Big(-\frac{\beta Q_+}{\omega\tau}\Big(\chi'-\chi+\frac{\Pi}{2}\Big)\Big)$$

$$+\int_{\chi}^{\chi_2}d\varepsilon_+\exp\Big(-\frac{\beta Q_+}{\omega\tau}\Big(\chi'-\chi-\frac{\Pi}{2}\Big)\Big)-\exp\Big(-\frac{\beta Q_+}{\omega\tau}\Big(\chi_2-\chi-\frac{\Pi}{2}\Big)\Big)\int_{\chi_1}^{\chi_2}d\varepsilon_-\Big]. \quad (80)$$

The last integral in (80) is equal to zero: $\varepsilon_-(\chi_2)-\varepsilon_-(\chi_1)=0$. In the case $\alpha=-$, the transformation takes the form

$$\int_{\varphi}^{\infty}=\varepsilon_-(\chi)-\varepsilon_-(\chi_1)+\frac{1}{2}\operatorname{sh}^{-1}\frac{\beta\Pi Q_+}{2\omega\tau}\int_{\chi_1}^{\chi}d\varepsilon_+\exp\Big(-\frac{\beta Q_+}{\omega\tau}\Big(\chi'-\chi_1-\frac{\Pi}{2}\Big)\Big).$$

If this expression is substituted in (79) and the resultant expression averaged over the wave coordinate, the expression vanishes:

$$\int_{\chi_1}^{\chi_2}df^0=\int_{\chi_1}^{\chi_2}df^0\,\varepsilon_-=0.$$

Substitution of (80) in (79) leads to an expression differing from (71) only in the factor $\frac{1}{2}$. The same result is obtained at $\beta>1$. As a result, the total attenuation is identical with that found in the previous section [formula (72), $\Delta=0$].

2. $\Phi_0\gg\Phi_T$, $R\ll1$. Just as in the region of low transparency, the calculation of the attenuation in this case is a complicated variant of the calculation in the low-frequency range $\omega\tau\ll1$. Omitting the very cumbersome calculations, which take into account tunnel transitions both at the turning points ($\beta<1$) and at the points of splitting of the characteristics ($\beta>1$), we formulate the final result: the nonlinear correction to Γ_n is identical, with accuracy to within the factor $\frac{1}{2}$, with the analogous correction in formula (77). All that was said above relative to the singularity of the attenuation at $T=T_c$ in the low-frequency region holds for the present case without change.

5. CONCLUSION

In conclusion, we bring together all the results and sketch the general picutre of nonlinear attenuation of

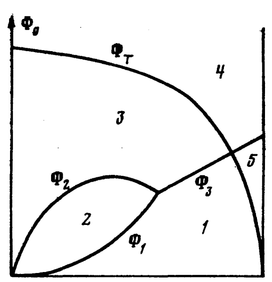

FIG. 4.

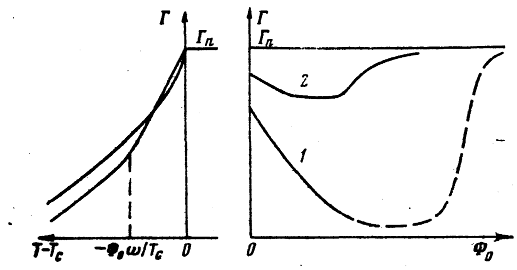

FIG. 5.

longitudinal sound in a superconductor, due to Andreev localization of the excitations of the sound wave.

We introduce the temperature-dependent characteristic frequency

$$\omega_0(T)=\tau^{-1}\max((T/\Delta)^{1/2},\quad T^2/\Delta^2),$$

in the vicinity of which the nonlinear attenuation undergoes a frequency dispersion due to the change in the nonlinearity mechanism. Depending on the relation between ω and ω_0, the threshold of the nonlinearity is determined by one of the following formulas: $\Phi_3=T/\omega\tau$, $\omega\ll\omega_0$ (59); $\Phi_1=(T/\omega\tau)(\omega_0/\omega)$, $\omega\gg\omega_0$ (54). The temperature dependences of $\Phi_1(T)$ and $\Phi_3(T)$ are given in Fig. 4. In this same drawing, we have the temperature dependences of two other characteristic amplitudes: the upper boundary of the high-frequency nonlinearity $\Phi_2=T/\omega_0\tau$ (56) and the tunneling level $\Phi_T=\Delta^2/\omega$ (40). These curves divide the plane Φ_0-T into a number of regions:

1. Region of linear BCS damping: $\Gamma_0=\Gamma_n(1-\operatorname{th}(\Delta/2T))$ (50);
2. Region of high-frequency nonlinearity: $\Gamma/\Gamma_0\sim(\Phi_1/\Phi_0)^{1/2}$ (67);
3. Region of low- and intermediate-frequency nonlinearity

$$\Gamma/\Gamma_0\sim(\Phi_3/\Phi_0)\ln(\Phi_0/\Phi_3). \quad (73)$$

4. Region of high transparency: $1-\Gamma/\Gamma_n\sim\Phi_T/\Phi_0$ (78);
5. Region of low intensities, in which the BCS formula is incorrect; the damping here is determined by the formula $1-\Gamma/\Gamma_n\sim\Delta^2\tau/T_c$. (78).

The temperature dependences of the nonlinear damping and BCS damping near T_c are shown in Fig. 5. The change in the course of the nonlinear curve corresponds to a transition from the region 1(3) of stability of the Andreev localization in the region 5(4) of tunnel decay at low (high) pumping (see Fig. 4). Here (Fig. 5) are shown the amplitude dependences of the damping in the region of stability of the localized states ($\Delta^2\tau\gg T_c$—curve 1) and in the region of their decay ($\Delta^2\tau\ll T_c$—curve 2). The given dependences agree qualitatively with the recently published experimental data of Fil' and coworkers[12] on nonlinear acoustic damping in pure superconducting gallium.

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[2] The applicability of (2) in the general case is also limited by the inequality $qv_F \ll \Delta$, for resonance excitations, however, the corresponding inequality is significantly weaker [see formula (24)] and in typical cases reduces to (1).

[3] A similar inequality is obtained in Ref. 4. To neglect heating effects in the superconducting corrections near T_c, we require a refinement of (12)[3]; however, this estimate is sufficient for the effects considered below.

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