Residual time correlations and the $1/\omega$ spectrum for Brownian motion

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(Submitted 16 December 1980)
Zh. Eksp. Teor. Fiz. **80**, 2243–2249 (June 1981)

It is shown that the $1/\omega$ spectrum (natural flicker noise) exists in the low-frequency region in any system of the diffusional type. The nature of the spectrum does not depend on the shape of the body (cube, film, filament). The upper limit of the natural-flicker-noise region is determined by the diffusion coefficient and the minimum characteristic dimension of the body. The lower limit is not determined by the parameters of the system, but by the time of observation. The intensity of natural flicker noise is inversely proportional to the number of particles in the system. The temperature dependence of the noise intensity is not a universal one. It is determined by the temperature dependence of the statistical characteristics of the specific diffusion process. Residual time correlations exist in the time interval corresponding to the flicker noise frequency range. The τ^0 and $1/\omega$ dependences for the residual-correlation and natural-flicker-noise regions are valid when the corresponding diffusion lengths $(D\tau)^{1/2}$ and $(D/\omega)^{1/2}$ are much greater than the minimum dimension of the body. This allows the dimension d of the body to be set equal to zero if the individual objects are grains or domains.

PACS numbers: 05.40. + j

1. INTRODUCTION

The spectrum $1/\omega^\gamma$ with $\gamma \approx 1$ (flicker noise) was discovered more than 50 years ago by Johnson during the investigation of the voltage fluctuations in a vacuum tube with an oxide cathode.[1,2] Subsequent investigations revealed the universality of the $1/\omega^\gamma$ spectrum.[3] A very large number of papers devoted to the analysis of specific models of flicker noise have been published. But there is thus far no convincing explanation for the universality of this phenomenon, and the absence of a lower limit of the spectrum has also not been explained. For example, the $1/\omega^\gamma$ spectrum is observed in semiconductors right down to frequencies of the order of 10^{-6} Hz.

The purpose of the present paper is to show that the $1/\omega$ spectrum is, at low frequencies, characteristic of any system of the diffusional type. An equation of this type (with diffusion coefficient D) describes the diffusion of Brownian particles, the charge carriers in a plasma, electrolytes, and semiconductors, the evolution of the fluctuations of hydrodynamic quantities whose spectrum is concentrated near $\omega = 0$, etc.

We shall call the $1/\omega$ noise generated during "Brownian motion" "natural flicker noise." This designation emphasizes that what we are talking about is equilibrium noise, whose occurrence in macroscopic bodies is due to their atomic-molecular structure. Besides this, there can also exist "technical flicker noise." This type of noise is possible in nonequilibrium systems.

The main results of the paper amount to the following.

1. It is shown that the $1/\omega$ spectrum (natural flicker noise) exists in the low-frequency region during Brownian motion of any kind.

2. The form of the spectrum does not depend on the shape of the body (cube, film, filament).

3. The upper limit of natural flicker noise is determined by the diffusion coefficient and the minimum dimension of the system.

4. In the diffusion model flicker noise does not have a lower frequency limit $\omega_{\min}$ determined by the parameters of the system. The value of $\omega_{\min}$ is determined by the time of observation: $\omega_{\min} \sim \tau_{\text{obs}}^{-1}$.

5. The intensity of natural flicker noise is inversely proportional to the number of particles in the system, in particular, the number of carriers in a semiconductor. This result agrees with the Hooge's empirical formula[4] and the result obtained in Voss and Clarke's paper,[5] in which flicker noise is explained in terms of temperature fluctuations.

6. The temperature dependence of the intensity of natural flicker noise is not a universal one. It is determined by the temperature dependence of the statistical characteristics of the specific diffusion process.

7. Residual time correlations exist in the region of large times, when the inequalities $L^2_{\min}/2D \ll \tau \ll \tau_{\text{obs}}$ are satisfied.

Let us explain this in greater detail.

It is well known (see, for example, Refs. 6 and 7) that for an unbounded system and an unbounded region of wave-number values k $(0 \leqslant k \leqslant \infty)$ the time correlations at large times are proportional to $\tau^{-d/2}$ (d is the dimensionality of the space). The corresponding spectral densities are proportional to $\omega^{d/2-1}$. In the present paper we show that the corresponding diffusion lengths $(2D/\omega)^{1/2}$ and $(2D\tau)^{1/2}$ for the region of natural flicker noise and the region of residual time correlations are much greater than the dimension L of the system. As a result, we can treat the system as a point in determining the dependences on τ and ω. Formally, this corresponds to zero dimensionality. Setting $d=0$ in the above-cited dependences $\tau^{-d/2}$ and $\omega^{d/2-1}$, we arrive at the dependences τ^0 and ω^{-1}, which correspond to residual time correlations and natural flicker noise.

 0038-5646/81/061171-04$02.40

2. NATURAL FLICKER NOISE

Thus, let us, following the foregoing, consider the problem of the low-frequency spectrum for an arbitrary system that can be referred to the class of "diffusional" systems. For concreteness, we choose temperature as the fluctuating quantity. This, naturally, does not impose any limitation on the generality of the results. The diffusion equation in this case is the heat equation. In order to emphasize that this is only an example of the equation of the diffusion type, we shall denote the thermal conductivity coefficient by D.

Let us write the equation for the Fourier transform of the temperature fluctuation

$$\delta T_{\mathbf{k}}(t)=\int \delta T(\mathbf{r},t)\exp\{-i\mathbf{k}\mathbf{r}\}d\mathbf{r} \tag{1}$$

in the form of the Langevin equation (see Ref. 8, Chap. IX):

$$(\partial/\partial t+D\mathbf{k}^2)\delta T_{\mathbf{k}}(t)=y_{\mathbf{k}}(t). \tag{2}$$

The moments of the Langevin source are given by the formulas

$$\langle y\rangle=0, \quad (y(t)y(t'))_{\mathbf{k}}=2D\mathbf{k}^2(\delta T)_{\mathbf{k}}^{2}\delta(t-t'). \tag{3}$$

Let us first assume that the system has one characteristic dimension L (the volume $V\sim L^3$). In studying the spectrum in the region of low frequencies ($\omega<2D/L^2$), it is sufficient to investigate the spectrum of the fluctuations in the temperature averaged over the volume V:

$$\delta T_V(t)=\int \delta T(\mathbf{r},t)\frac{d\mathbf{r}}{V}. \tag{4}$$

Let us consider the expression for the fluctuation spectrum of this function. We find with allowance for (4) that

$$(\delta T_V)_\omega^2=\int(\delta T\,\delta T)_{\omega,\mathbf{r},\mathbf{r}'}\frac{d\mathbf{r}\,d\mathbf{r}'}{V^2}=\int(\delta T\,\delta T)_{\omega,\mathbf{r}-\mathbf{r}'}\frac{d(\mathbf{r}-\mathbf{r}')}{V}. \tag{5}$$

In going over to the last expression, we used the condition that the fluctuation distribution within the volume V should be spatially homogeneous. This is justified, since for $(D/\omega)^{3/2}\gg V$ (the low-frequency region) we can consider an ensemble in which the distribution of the positions of an individual system within the boundaries of the volume $(D/\omega)^{3/2}$ is an equiprobable one. Let us Fourier transform the last expression in (5) with respect to $\mathbf{r}-\mathbf{r}'$ (below we set $\mathbf{r}-\mathbf{r}'=\mathbf{r}$). As a result, we obtain the equality

$$(\delta T_V)_\omega^2=\int(\delta T)_{\omega,\mathbf{k}}^2 e^{i\mathbf{k}\mathbf{r}}\frac{d\mathbf{r}}{V}\frac{d\mathbf{k}}{(2\pi)^3}. \tag{6}$$

Let us substitute here the expression, obtained with the aid of the Langevin equation (2), for the space-time spectral density:

$$(\delta T)_{\omega,\mathbf{k}}^2=\frac{2D\mathbf{k}^2}{\omega^2+D^2\mathbf{k}^4}(\delta T(t))_{\mathbf{k}}^2. \tag{7}$$

As a result we obtain the following expression for the sought spectrum:

$$(\delta T_V)_\omega^2=\int\frac{2D\mathbf{k}^2}{\omega^2+D^2\mathbf{k}^4}(\delta T)_{\mathbf{k}}^2 e^{i\mathbf{k}\mathbf{r}}\frac{d\mathbf{k}}{(2\pi)^3}\frac{d\mathbf{r}}{V}. \tag{8}$$

The frequency integral

$$\int(\delta T_V)_\omega^2\frac{d\omega}{2\pi}=\langle(\delta T_V(t))^2\rangle=\int\langle\delta T\,\delta T\rangle_{\mathbf{r}}\frac{d\mathbf{r}}{V}=\frac{(\delta T)_{\mathbf{k}=0}^2}{V}. \tag{9}$$

The equal-time correlator $(\delta T)_{\mathbf{k}}^2$ in the formula (8) does not depend on k in a broad range of k values ($0\leqslant k\leqslant V_{\mathrm{ph}}^{-1/3}$, where V_{ph} is a physically infinitesimal volume: $V_{\mathrm{ph}}\ll V$), and can be taken out from under the integral sign at the value $\mathbf{k}=0$. It then follows from (8) and (9) that

$$(\delta T_V)_\omega^2=\int\frac{2D\mathbf{k}^2}{\omega^2+D^2\mathbf{k}^4}e^{i\mathbf{k}\mathbf{r}}d\mathbf{r}\frac{d\mathbf{k}}{(2\pi)^3}\langle(\delta T_V)^2\rangle. \tag{10}$$

For large V ($V\to\infty$), the function $\delta(\mathbf{k})$ arises in the expression (10) after the integration with respect to r; therefore, the spectrum $(\delta T_V)_\omega^2$ is equal to zero at all frequencies $\omega\neq 0$. The situation changes significantly at frequencies $\omega\ll 2D/L^2$, when the diffusion length $(2D/\omega)^{1/2}$ becomes significantly greater than the dimension of the system, and, in consequence, the passage to the limit $V\to\infty$ becomes impossible. The separation of the normal mode with $\mathbf{k}=0$ therefore becomes impossible, and we must consider an ensemble of systems for which the k values of the normal mode are defined statistically. We can then perform averaging over the wave numbers near $\mathbf{k}=0$.

For the averaging let us take the Gaussian distribution with a dispersion given by the following combination of two length parameters: the dimension L of the system and the diffusion length $(2D/\omega)^{1/2}$:

$$L_\omega^{-2}=(L^2+2D/\omega)^{-1}. \tag{11}$$

The operation of averaging over $\mathbf{k}$ corresponds in the formula (10) to the substitution

$$\frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\mathbf{r}}d\mathbf{r}\to\left(\frac{L_\omega^2}{2\pi}\right)^{3/2}\exp\left\{-\frac{L_\omega^2\mathbf{k}^2}{2}\right\}. \tag{12}$$

Instead of the Gaussian distribution, we can also use other distributions peaked at $\mathbf{k}=0$, and satisfying the normalization condition

$$\int f(L_\omega\mathbf{k})L_\omega^3\,d\mathbf{k}=1. \tag{13}$$

Taking account of the substitution (12), we obtain the following expression for the spectrum:

$$(\delta T_V)_\omega^2=\left(\frac{L_\omega^2}{2\pi}\right)^{3/2}\int\frac{2D\mathbf{k}^2}{\omega^2+D^2\mathbf{k}^4}\exp\left\{-\frac{L_\omega^2\mathbf{k}^2}{2}\right\}d\mathbf{k}\,\langle(\delta T_V)^2\rangle. \tag{14}$$

From this we have for the region of low frequencies ($\omega\ll 2D/L^2$) a spectrum $\propto 1/\omega$—natural flicker noise:

$$(\delta T_V)_\omega^2=C_{\mathrm{ph}}\frac{\langle(\delta T_V)^2\rangle}{|\omega|}\propto\frac{1}{|\omega|N},\quad \omega_{max}=2D/L^2. \tag{15}$$

If δT_V is a temperature fluctuation, then the equal-time correlator $\langle(\delta T_V)^2\rangle$ is given by the expression (see Ref. 9, §112)

$$\langle(\delta T_V)^2\rangle=\frac{kT^2}{C_p}\propto\frac{1}{N}. \tag{16}$$

Here C_p is the heat capacity of the system. If, on the other hand, δT_V is, for example, a concentration fluctuation $\delta n(\mathbf{r},t)$ averaged over the volume V, i.e., if $\delta T_V\to\delta n_V$, then

$$\langle(\delta n_V)^2\rangle=\frac{n}{V}\left(1+n\int g_2(\mathbf{r})d\mathbf{r}\right)\propto\frac{1}{N}. \tag{17}$$

Here g_2 is a two-particle correlation function, e.g., for the carriers. The temperature dependence of the flicker-noise intensity is then significantly different. The volume V can then be related to the correlation volume. In the formula (15) C_{ph} is a constant of the order of unity. In the case in which the Gaussian distribution is used

$$C_{ph}=\frac{4}{\sqrt{2\pi}}\int_0^\infty e^{-x^2/2}\frac{x^4\,dx}{1+x^4}. \qquad (18)$$

The spectrum (15) does not possess a lower limit determined by the parameters of the system under consideration. The lower spectral limit is determined by the observation time, i.e.,

$$\omega_{min}\sim 1/\tau_{obs} \qquad (19)$$

The lower limit can be explicitly taken into account in the formula (15) by making the substitution

$$|\omega|\to(\omega^2+\tau_{obs}^{-2})^{1/2}. \qquad (20)$$

Flicker noise can be concentrated not only near zero frequency, but also near distinct frequencies.[10] If ω_0 is one of them, then we can show that the spectrum is again given by the formula (15) if we make in it the substitution

$$|\omega|\to[(\omega-\omega_0)^2+\tau_{obs}^{-2}]^{1/2}. \qquad (21)$$

It follows from the formulas (15)–(17) that the flicker-noise spectrum $\propto N^{-1}$. For this reason, the noise is fairly intense either in systems of small dimensions (films, filaments), or in systems made up of grains or domains. The dependence N^{-1} corresponds to Hooge's empirical formula.[4] This type of dependence on the particle number was obtained by Clarke and Voss[5] in their investigation of flicker noise in films. But the frequency dependence $1/\omega$ was, in effect, postulated in this investigation.

3. FLICKER NOISE IN FILMS AND FILAMENTS

Above we assumed that the system has one characteristic length parameter. Then the volume $V\sim L^3$. Let us now consider a film of thickness d and width and length L. Thus, $V=dL^2$. Let us find the upper limit of the flicker-noise spectrum under the condition that

$$d^2\ll 2D/|\omega|\ll L^2, \quad L\to\infty. \qquad (22)$$

Instead of (12), we then obtain the distribution

$$\left(\frac{D}{\pi\omega}\right)^{1/2}\exp\left\{-\frac{Dk_x^2}{2}\right\}\delta(k_y)\delta(k_z). \qquad (23)$$

Using this distribution in the formula (14), we again obtain the flicker-noise spectrum (15). Now, however,

$$\omega_{max}\approx 2D/d^2. \qquad (24)$$

Thus, the upper frequency limit of the flicker-noise region is determined by the minimum geometric dimension of the sample. For a filament d in the formula (24) is the diameter of the filament.

4. THE LANGEVIN EQUATION FOR FLICKER NOISE

The flicker noise spectrum can be computed with the aid of the corresponding Langevin equation. To derive this equation, let us make in the equation for the Fourier component

$$(-i\omega+D\mathbf{k}^2)\delta T(\omega,\mathbf{k})=y(\omega,\mathbf{k}), \qquad (25)$$

which follows from the Langevin equation (2), the following substitutions, which reflect the averaging over the wave numbers for the low-frequency region:

$$k^2\to|\omega|/D, \quad (y)_{\omega k}{}^2\to 2|\omega|C_{ph}\langle(\delta T_V)^2\rangle. \qquad (26)$$

It follows from the formulas (25) and (26) that the dissipative term in the Langevin equation now depends not on the wave number, but on the frequency, namely,

$$\gamma_\omega=|\omega|. \qquad (27)$$

The intensity of the noise source is also proportional to $|\omega|$. The expression for the spectrum can be written in the form

$$(\delta T_V)_\omega{}^2=\frac{(y)_\omega{}^2}{\omega^2+\gamma_\omega{}^2}, \quad (y)_\omega{}^2=2\gamma_\omega C_{ph}\langle(\delta T_{ph})^2\rangle. \qquad (28)$$

It coincides with (15). The frequency integral

$$\int(\delta T_V)_\omega{}^2\frac{d\omega}{2\pi}=\langle(\delta T_V)^2\rangle_{ph}=\frac{C_{ph}}{2\pi}\ln(\omega_{max}\tau_{obs})\langle(\delta T_V)^2\rangle \qquad (29)$$

gives the mean square fluctuation for the flicker-noise region. The divergence of the expression as $\tau_{obs}\to\infty$ only indicates that we shall ultimately fall outside the limits of the "diffusion model." This does not, of course, invalidate the model under consideration. This restriction is somewhat similar to the restriction of the model of white noise.

5. RESIDUAL TIME CORRELATIONS

The expression, corresponding to the formula (10), for the time correlation of the fluctuations $\delta T_V(t)$ has the form

$$(\delta T_V)_\tau{}^2=\left(\frac{L_\tau}{2\pi}\right)^{1/2}\int\exp\left\{-D\mathbf{k}^2\tau-\frac{L_\tau^2\mathbf{k}^2}{2}\right\}d\mathbf{k}\,\langle(\delta T_V)^2\rangle. \qquad (30)$$

Here the dispersion of the wave-number distribution is

$$L_\tau{}^{-2}=(L^2+2D\tau)^{-1}. \qquad (31)$$

It follows from this formula that for the time interval

$$L^2/2D\ll\tau\ll\tau_{obs} \qquad (32)$$

the function $\langle(\delta T)^2\rangle_\tau$ does not depend on τ (the residual-time-correlation region).

The described method of finding the low-frequency fluctuation spectra may be useful in the investigation of the flicker noise generated during the equilibrium and nonequilibrium phase transitions described in chapters 4, 5, 11, and 12 of Ref. 11. For the theory of flicker noise in semiconductors, plasmas, and electrolytes, the hydrodynamic theory of a weakly ionized plasma developed in Ref. 12 may be found to be useful.

I take the opportunity to thank I. A. Shishmarev for a discussion of the problems considered here.

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Translated by A. K. Agyei