Alfvén and thermomagnetic waves in an inhomogeneous plasma

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Various instabilities of Alfvén and thermomagnetic waves in an inhomogeneous plasma in the presence of a magnetic field are considered. Both thermomagnetic and hydromagnetic effects can lead to the excitation of waves. The conditions under which the instabilities grow are found, and the characteristic growth times calculated. It is shown that in a strong magnetic field and for certain relationships between the parameters the Alfvén waves can be rapidly damped by heat conduction in the plasma. The excitation of Alfvén and thermomagnetic waves may be important in laser and cosmic plasmas. Effective damping of waves can occur in the solar corona, where it will be an important factor in the heat balance of the corona.

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Instabilities of various kinds play a very important part in plasma dynamics. For example, when laser radiation interacts with matter strong magnetic fields arise,[1] and these influence the heat transfer, the hydrodynamic expansion of the plasma bunch, and so forth. The instability of thermomagnetic waves[2-4] can play an important part in the excitation of these fields. In addition, the plasma may become turbulent in the unstable regions, which results in turbulent transfer of energy or momentum.

The instabilities of Alfvén and thermomagnetic waves have been investigated by a number of authors. In the absence of a magnetic field, the instability of thermomagnetic waves was studied for the first time in Refs. 2–4, but the characteristic time and condition for the development of instability were not determined correctly in these papers (as was pointed out in Ref. 5). The instabilities of Alfvén and thermomagnetic waves in the presence of a magnetic field and for very varied parameters of the plasma were studied by Gurevich and Gel'mont.[6] However, some of the conclusions of Ref. 6 are inaccurate; in particular, the limits of applicability of some of the expressions are given incorrectly.

In the present paper, we investigate some hydrodynamic instabilities of Alfvén and thermomagnetic waves in an inhomogeneous plasma in the presence of a magnetic field. We determine the conditions under which instabilities can develop. We find the characteristic growth times. These instabilities can play a very important part in processes that take place under both laboratory conditions and in cosmic plasmas. In addition, we show that in a strong magnetic field Alfvén waves can be strongly damped by the anisotropic nature of heat conduction. Such damping may be important for the processes which occur in the solar corona.

1. BASIC EQUATIONS

We consider a fully ionized plasma. We assume that in the unperturbed state in the plasma there are gradients of the density ρ, the temperature T (which will be expressed in energy units in the following formulas) and the pressure p and a homogeneous magnetic field $\mathbf{B}_0$ is present but hydrodynamic motions are absent. The electromagnetic field is assumed to be quasistationary. In this case, the basic equations of the problem are

$$\rho\frac{d\mathbf{V}}{dt}=-\nabla p+\frac{1}{4\pi}[[\nabla\mathbf{B}]\mathbf{B}]+\rho\mathbf{g},\quad \frac{\partial\rho}{\partial t}+\nabla(\rho\mathbf{V})=0, \tag{1}$$

$$[\nabla\mathbf{B}]=\frac{4\pi}{c}\mathbf{j},\quad \nabla\mathbf{B}=0,\quad \frac{\partial\mathbf{B}}{\partial t}=-c[\nabla\mathbf{E}], \tag{2}$$

$$\rho c_p\frac{dT}{dt}-\frac{dp}{dt}=-\nabla\mathbf{q}+\mathbf{j}\left(\mathbf{E}+\left[\frac{\mathbf{V}}{c}\mathbf{B}\right]\right), \tag{3}$$

where $\mathbf{g}$ is the acceleration due to gravity (or some other force which acts on the plasma), and c_p is the specific heat for $p=\text{const}$. We ignore viscosity effects; this is justified if the growth time of the instability is much shorter than the time of viscous dissipation. The expressions for the electric field $\mathbf{E}$ and the heat flux density $\mathbf{q}$ can be written as

$$\mathbf{E}=\hat{\eta}\mathbf{j}-\frac{1}{c}[\mathbf{VB}]+\hat{\Lambda}\nabla T-\frac{\nabla p_e}{en_e}, \tag{4}$$

$$\mathbf{q}=-\hat{\varkappa}\nabla T+T\hat{\Lambda}\mathbf{j}-\frac{5}{2e}T\mathbf{j}. \tag{5}$$

Here, $\mathbf{j}$ is the density of the electric current, and p_e, n_e, and e are, respectively, the pressure, concentration, and modulus of the charge of the electrons. In the expressions (4) and (5), the result of applying any of the tensors, for example, $\hat{\Lambda}$, to the corresponding vector can be represented in the form

$$\hat{\Lambda}\mathbf{j}=\Lambda\mathbf{j}+\Lambda'[\mathbf{jB}]+\Lambda''\mathbf{B}(\mathbf{Bj}).$$

The components of the tensors $\hat{\eta}$, $\hat{\Lambda}$, and $\hat{\varkappa}$ (if the electron thermal conductivity exceeds the radiative thermal conductivity exceeds the radiative thermal conductivity) can be readily obtained from the relations given by Braginskii.[7]

We shall consider only the linear stage in the development of the perturbations. We indicate the unperturbed quantities by the index 0, and the perturbations by the index 1. We shall assume that the wavelength λ of the perturbations is much less than the characteristic scale of variation L of the unperturbed quantities (it is assumed that the scales of variation of ρ_0 and T_0 are of the

*$[\nabla\mathbf{B}]\mathbf{B}] \equiv [\nabla\times\mathbf{B}]\times\mathbf{B}$.

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same order). We consider only magnetic fields satisfying the inequality $x_0=\omega_B\tau_e \ll L/\lambda$, where $\omega_B=eB_0/m_ec$, m_e is the electron mass, and τ_e is the relaxation time of the electrons. Linearizing Eqs. (1)–(5), we obtain a system of homogeneous equations for the small perturbations. We shall investigate the instability in the WKB approximation (for more details, see, for example, Ref. 8).

2. WAVES IN A WEAK MAGNETIC FIELD

We shall say that the magnetic field B_0 is weak if $x_0 \ll 1$. The condition for occurrence of an instability and its growth time depend on the relationship between the characteristic frequencies of the plasma [the sound frequency $\omega_s=kc_s$, the Alfvén frequency $\omega_A=(\mathbf{k}\mathbf{c}_A)$, the thermomagnetic frequency $\omega_T=-c\Lambda_0'(\mathbf{k}\nabla T_0)$, and the "thermal frequency"

$$\omega_\varkappa=[\varkappa_0 k^2+\varkappa_0''(\mathbf{k}\mathbf{B}_0)^2]/\rho_0 c_p,$$

where $\mathbf{c}_A$ and c_s are the Alfvén and sound velocities, and k is the wave vector of the perturbations]. We consider some special cases.

The case $\omega_\varkappa \gg \omega \sim \omega_T \gg \omega_s \gg \omega_A$. For this relationship between the frequencies, the development of the instability is determined solely by the thermomagnetic effects (since the oscillations are very rapid: $\omega \gg \omega_s$). For $1 \gg x_0 \gg \lambda/L$, it is necessary to take into account in the dispersion relation the corrections to ω_T of order $x_0\omega_T$, and for $\lambda/L \gg x_0$ those of order $\omega_T(\lambda/L)$. For $1 \gg x_0 \gg \lambda/L$, an expression for the growth rate was obtained in Ref. 6 [Eq. (4.2)], but it was erroneously assumed that the result is valid for all $x_0<1$. For $x_0 \ll \lambda/L$, it is convenient to reduce the original linearized system of equations to a single equation for $\mathbf{B}_1$. From it, we obtain the complex frequency $\omega^{(B)}$, which determines the variation with time of the perturbations $\mathbf{B}_1$:

$$\omega^{(B)}=\omega_T-i\gamma_1+\frac{ic^2k^2\eta_0}{4\pi},\quad \gamma_1=c\nabla(\Lambda_0'\nabla T_0)-\frac{c\varkappa_0'}{ek^2\varkappa_0}([\mathbf{k}\nabla T_0][\mathbf{k}\nabla\ln\rho_0]). \quad (6)$$

An instability develops for $\gamma_1>c^2k^2\eta_0/4\pi$. In order of magnitude, $\gamma_1 \sim T_0\tau_e/m_eL^2$, and a necessary condition of instability is $\omega_p^2\tau_e^2(T_0/m_ec^2) \gg (kL)^2$. For both $1 \gg x_0 \gg \lambda/L$ and $\lambda/L \gg x_0$ the instability of the thermomagnetic waves is convective.

Of particular interest are thermomagnetic waves with $\mathbf{k} \parallel \nabla T_0$ propagating in a plasma with $\mathbf{B}_0=0$. It can be seen from Eqs. (1)–(7) that in this case there are oscillations of only the magnetic field $\mathbf{B}_1$ in the wave, the perturbations of all the remaining quantities being zero. The possible existence of such waves was pointed out for the first time in Ref. 9, but their stability was not investigated. In a plasma with $\nabla(\Lambda_0'\nabla T_0)>0$ it is possible to have not only the propagation of these waves but also their excitation. It is readily seen that the magnetic field in such a "magnetic" wave satisfies the equation

$$\frac{\partial \mathbf{B}_1}{\partial t}+\frac{c^2}{4\pi}[\nabla[\eta_0\nabla\mathbf{B}_1]]+c[\nabla[\Lambda_0'\nabla T_0,\mathbf{B}_1]]=0.$$

For the growth rate, we obtain

$$\mathrm{Re}\, i\omega=c\nabla(\Lambda_0'\nabla T_0)-c^2k^2\eta_0/4\pi.$$

Thus, in a plasma with $\nabla(\Lambda_0'\nabla T_0)>0$ there can be amplification of oscillations of the magnetic field alone, the oscillations of the other quantities being negligibly small.

Note that for $\gamma_1<0$ but $|\gamma_1|>c^2k^2\eta_0/4\pi$ there will be damping of the thermomagnetic waves by thermomagnetic effects. In a hot and low-density plasma, this damping may be much more rapid than the damping due to ohmic or viscous dissipation.

The case $\omega_s \gg \omega \sim \omega_T \gg \omega_A$. For $1 \gg x_0 \gg \lambda/L$, we must retain in the dispersion relation the corrections to ω_T of order $x_0\omega_T$ (an expression for ω was obtained in Ref. 6, but, as in the preceeding case, the authors assumed that their result is valid for all $x_0<1$). For $\lambda/L \gg x_0$, it is necessary to take into account the corrections to ω_T of order $(\lambda/L)\omega_T$. Then for the frequency with which the magnetic field varies, we obtain

$$\omega^{(B)}=\omega_T-i\gamma_2+\frac{ic^2k^2\eta_0}{4\pi},\quad \gamma_2=c\nabla(\Lambda_0'\nabla T_0)-\frac{c\varkappa_0'}{ek^2\varkappa_0}([\mathbf{k}\nabla T_0][\mathbf{k}\nabla\ln p_0]). \quad (7)$$

For an instability to occur, it is necessary that $\gamma_2>c^2k^2\eta_0/4\pi$. In order of magnitude, $\gamma_2 \sim \gamma_1 \sim T_0\tau_e/m_eL^2$. For $\mathbf{k} \parallel \nabla T_0$, and $\mathbf{B}_0=0$, as in the preceeding case, oscillations of only the magnetic field occur in the thermomagnetic wave. If $|\gamma_2|>c^2k^2\eta_0/4\pi$, but $\gamma_2<0$, then, as when $\omega_T \gg \omega_s \gg \omega_A$, the thermomagnetic waves can be rapidly damped by thermomagnetic effects.

The case $\omega_s \gg \omega \sim \omega_A \gg \omega_T$, $\omega_\varkappa \gg \omega_A$. In this case, the hydrodynamic motions can be important in both the induction equation (2) and the heat transfer equation (3). Thus, the rate of change of the amplitude of the magnetic field in the wave due to the thermomagnetic effects is of order $x_0\omega_T$, while the rate of change of the amplitude $\mathbf{B}_1$ due to the hydrodynamic motions is $\sim\omega_A(\lambda/L)$. Thus, for $(\omega_A/\omega_T)(\lambda/L) \gg x_0$ the main contribution to the excitation of waves is made by the hydrodynamic motions. This circumstance was not taken into account in the calculation by Gurevich and Gel'mont,[6] and so their result is correct only for $x_0 \gg (\lambda/L)(\omega_A/\omega_T)$. But if $(\lambda/L)(\omega_A/\omega_T) \gg x_0$, then a weak convective instability of Alfvén waves arises in the plasma. The growth rate of this instability is of order c_A/L.

3. WAVES IN A STRONG MAGNETIC FIELD

We shall say that the magnetic fields are strong if $x_0 \gg 1$. Suppose

$$\omega_T'^2=-[ck\Lambda_0''(\mathbf{B}_0\nabla T_0)(1-i\alpha)]^2,\quad \alpha=\frac{c\eta_0'(\mathbf{k}\mathbf{B}_0)}{4\pi\Lambda_0''(\mathbf{B}_0\nabla T_0)}.$$

We consider the instabilities for different relationships between ω_s, ω_A, ω, and ω_T'.

The case $\omega_\varkappa \gg \omega \sim \omega_T' \gg \omega_s \gg \omega_A$. In this case (as in the corresponding case for weak fields), the hydrodynamic motions in the plasma influence neither the heat transfer nor the excitation of the magnetic fields. We shall assume that the thermomagnetic terms in the induction equation are much greater than the terms describing the ohmic dissipation of the magnetic field:

$$\frac{ck}{eB_0}|\nabla T_0| \gg \frac{c^2k^2\eta_0}{4\pi}, \quad x_0\left(\frac{c_s^2}{c_A^2}\right) \gg \frac{L}{\lambda} . \tag{8}$$

For the frequency, we obtain

$$\omega = \pm\omega_T'. \tag{9}$$

In Ref. 6, the instability was investigated only for the case when the thermomagnetic currents in the plasma exceed the Hall currents, i.e., $\alpha \ll 1$ (or $c_s^2/c_A^2 \gg L/\lambda$). It was noted in Ref. 6 that the instability is absolute. However, as can be seen from (9), the instability also occurs for $\alpha \gtrsim 1$. For its development, the much less stringent condition (8) is necessary. Since $\eta_0' \approx 1/ecn_e$ for $x_0 \gg 1$, we obtain when $\alpha \gg 1$

$$\omega = \pm\frac{ck(\mathbf{kB}_0)}{4\pi en_e} \pm ick\Lambda_0''(\mathbf{B}_0\nabla T_0)$$

and the instability will be convective. The growth rate is maximal for parallel or antiparallel vectors $\mathbf{B}_0$ and ∇T_0 and in order of magnitude is equal to the growth rate for $\alpha < 1$.

The case $\omega_s \gg \omega \sim \omega_T' \gg \omega_A$, $\omega_x \gg \omega_T'$. In this case, the expression for the complex frequency is exactly the same as (9).

The case $\omega_s \gg \omega \sim \omega_A \gg \omega_T'$, $\omega_x \gg \omega_A$. Perturbations of the velocity will be important for excitation of the magnetic field. By virtue of the thermomagnetic effects and the Hall currents, the rate of change of the magnetic field is of order

$$ck^2B_0^2\Lambda_0''(1-i\alpha)T_1 \sim \omega_T'B_0(kL)T_1/T_0,$$

and the rate of change due to the curvature of the lines of force by the motions of the medium is $\sim kB_0V_1 \sim \omega B_0(T_1/T_0)$. Thus, for $(L/\lambda)(\omega_T'/\omega_A) \gg 1$ it is not necessary to take into account the hydrodynamic effects, but in the case of the opposite inequality they will be more important than the thermomagnetic effects. Assuming that (8) is satisfied, we obtain for $(L/\lambda)(\omega_T'/\omega_A) \gg 1$ the following expression for the frequency:

$$\omega = \omega_A \pm \omega_T'/2 \tag{10}$$

and the instability in this case is convective.

Now suppose

$$(L/\lambda)(\omega_T'/\omega_A) \sim |1-i\alpha|\omega_x/x_0\omega_A \ll 1.$$

In what follows, we shall consider only the case when the change in the magnetic field due to the hydrodynamic motions takes place more rapidly than the change due to the Hall currents, i.e.,

$$kB_0V_1 \sim \omega B_0\left(\frac{\lambda}{L}\right)\left(\frac{B_1}{B_0}\right) \gg \frac{c^2k^2\eta_0x_0}{4\pi}B_1 \quad \text{or} \quad x_0\left(\frac{c_s^2}{c_A^2}\right)\left(\frac{\omega_A}{\omega_x}\right) \gg \frac{L}{\lambda}. \tag{11}$$

If (11) holds and, in addition, $(c_s^2/c_A^2)\times(\omega_x/\omega_A) \gg (L/\lambda)$, then a weak convective instability can arise in the plasma. The growth rate of this instability is of order $\omega_A(\lambda/L)$. Perturbations in the plasma for which

$$(c_s^2/c_A^2)(\omega_x/\omega_A) \ll (L/\lambda) \tag{12}$$

will be damped. For the damping rate of the Alfvén waves, we obtain

$$\mathrm{Re}\, i\omega = -\frac{1}{5}\omega_A\left(\frac{c_A^2}{c_s^2}\right)\left(\frac{\omega_A}{\omega_x}\right)(1-\cos^2\varphi), \quad \cos^2\varphi = \frac{(\mathbf{kB}_0)^2}{k^2B_0^2}. \tag{13}$$

The damping is due to the hydromagnetic effects and the thermal conductivity of the plasma and can be more rapid than the damping due to ohmic dissipation or viscosity [characteristic times of order $(x_0^2/\omega_x)(c_s^2/c_A^2)$ and $(m_i/m_e)^{1/2}\omega_x^{-1}$), respectively]. Note that the damping rate does not depend on λ.

The case $\omega_x \gg \omega \sim \omega_A \gg \omega_s \gg \omega_T'$. In this case, instability of Alfvén waves does not occur. The magnetoacoustic perturbations are damped with the damping rate

$$\mathrm{Re}\, i\omega = -\frac{1}{5}\frac{\omega_s^2}{\omega_x}(1-\cos^2\varphi) - \frac{c^2k^2\eta_0}{8\pi}. \tag{14}$$

If $(\omega_s/\omega_x)(c_s/c_A)x_0 \gg 1$, then the damping due to the thermal conductivity of the plasma can be more rapid than the damping due to the ohmic dissipation. The ratio ω_x/ω_s^2 is of the order of the time of energy exchange $m_i\tau_e/m_e$ between the electrons and ions. Since we have assumed $T_e \approx T_i$, we are justified in considering only processes with characteristic times longer than $m_i\tau_e/m_e$. Therefore, the result (14) is justified only for waves with $1-\cos^2\varphi \ll 1$. The reason for this damping is the nonadiabatically of the oscillations. Small perturbations of the pressure in the wave lead to perturbations of the temperature, and because of the nonadiabaticity $T_1/T_0 \sim (i\omega/\omega_x)(p_1/p_0)$. Since $p_1/p_0 \sim B_1/B_0$ in a magnetoacoustic wave, $T_1/T_0 \sim (i\omega/\omega_x)B_1/B_0$. The perturbations of the magnetic field are determined solely by the hydrodynamic motions: $i\omega B_1 \sim ikV_1B_0$. The velocity perturbations are made up of two parts: the perturbations due to the magnetic field, $V_1' \sim c_AB_1/B_0$, and the perturbations due to the pressure, $V_1'' \sim c_s(\omega_s/\omega)T_1/T_0$, and these lead to a deviation of the frequency from $\omega_0 = kc_A$ by $\Delta\omega$ and the occurrence of damping. In order of magnitude

$$i\Delta\omega B_1 \sim ikB_0c_s(\omega_s/\omega)T_1/T_0 \sim \omega_s^2B_1/\omega_x.$$

Thus, $\mathrm{Re}\, i\omega \sim \omega_s^2/\omega_x$. Note that in this case too the damping rate does not depend on the wavelength, whereas the damping rate of the ohmic (or viscous) damping is proportional to λ^2. Therefore, for sufficiently short wavelengths the damping we have considered will be unimportant.

4. DISCUSSION OF THE RESULTS

The development of small perturbations in unstable regions leads to the development of small-scale turbulence. This turbulence can influence the transport processes in the plasma. The viscosity, thermal conductivity, and electrical conductivity of the medium can differ appreciably depending on the presence or absence of turbulence. In addition, the excitation of thermomagnetic waves may lead to the generation in the plasma of strong magnetic fields during very short times (see, for example, Ref. 3). The anomalous damping investigated in Sec. 3 can have a strong influence on the processes that take place in the corona of the Sun and other stars. The plasma concentration in the corona is $\sim 10^{10}$ cm^{-3}, the temperature is of order 10^5–10^6 °K (in the numerical estimates, we shall assume $T = 3\times10^5$ °K), and the magnetic field is ~1 G. For these parameters, the plasma is strongly magnetized:

$x_0 \sim 2\times 10^3$. The sound velocity is $\sim 4\times 10^6$ cm/sec and the Alfvén velocity $\sim 3\times 10^6$ cm/sec. In the coronal plasma, $\omega_\varkappa > \omega_s > \omega_A > \omega_T'$ (we assume that the characteristic inhomogeneity scale is $L\sim 10^8$ cm). In addition, $(\lambda/L)(c_s^2/c_A^2)(\omega_\varkappa/\omega_A)<1$ and the inequality (12) holds. Therefore, in the solar corona the Alfvèn waves must be damped with the damping rate (13). The thermal frequency $\omega_\varkappa$ is in order of magnitude $c_e^2\tau_e k^2$, so that the characteristic time during which an Alfvén wave is damped is

$$(m_i/m_e)(c_s/c_A)^4\tau_e.$$

It exceeds by only $(c_s/c_A)^4$ times the time of energy transfer between the electrons and the ions. Therefore, the damping of Alfvèn waves due to the thermal conductivity of the plasma may make an appreciable contribution to the heat balance of stellar coronas.

[1]J. A. Stamper, K. Paradopoulos, R. N. Sudan. S. O. Dean. E. A. McLean, and J. M. Dawson, Phys. Rev. Lett. **26**, 1012 (1971).

[2]D. A. Tidman and R. A. Shanny, Phys. Fluids **17**, 1207 (1974).

[3]L. A. Bol'shov, Yu. A. Dreizin, and A. M. Dykhne, Pis'ma Zh. Eksp. Teor. Fiz. **19**, 288 (1974) [JETP Lett. **19**, 168 (1974)].

[4]B. A. Al'terkop, E. V. Mishin, and A. A. Rukhadze, Pis'ma Zh. Eksp. Teor. Fiz. **19**, 291 (1974) [JETP Lett. **19**, 66 (1974)].

[5]A. Z. Dolginov and V. A. Urpin, Zh. Eksp. Teor. Fiz. **77**, 1921 (1979) [Sov. Phys. JETP **50**, 912 (1979)].

[6]L. É. Gurevich and B. L. Gel'mont, Zh. Eksp. Teor. Fiz. **46**, 884 (1964) [Sov. Phys. JETP **19**, 604 (1964)].

[7]S. I. Braginskiĭ, in: Voprosy teorii plazmy (Ed. M. A. Leontovich), Vol. 1, Atomizdat, Moscow (1963); English translation: Reviews of Plasma Physics, Vol. 1, Consultants Bureau, New York (1965), p. 205.

[8]A. B. Mikhaĭlovskiĭ, Teoriya plazmennykh neustoĭchivosteĭ (Theory of Plasma Instabilities) Vol. 2, Atomizdat, Moscow (1977), p. 19.(1966) [Sov. Phys. JETP **24**, 123 (1967)].

[9]L. É. Gurevich and B. L. Gel'mont, Zh. Eksp. Teor. Fiz. **51**, 183 (1966) [Sov. Phys. JETP **24**, 124 (1967)].

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