Stochastic motion of relativistic particles in the field of a monochromatic wave

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We study the stochastic motion of particles in the field of an electrostatic wave which propagates at an angle to the external magnetic field. We elucidate the conditons under which the interaction between the particles and the wave has effectively a collision character. We obtain for that case a kinetic equation for the particle distribution function and we study it in the stochastic region; we consider on the basis of this equation the problem of the heating of particles by a monochromatic wave.

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1. The interaction between waves and energetic charged particles is one of the fundamental problems of the theory of collective processes in a relativistic plasma. In the case of monochromatic waves propagating in a magnetoactive plasma such an interaction is well known to proceed most effectively with resonant particles for which

$$\omega-k_{\parallel}v_{\parallel}=n\omega_c/\gamma, \tag{1}$$

where $k_{\parallel}$ and $v_{\parallel}$ are the components of the wave vector **k** and the particle velocity **v** along the direction of the external magnetic field $\mathbf{B}_0$, $\omega_c=eB_0/mc$ is the cyclotron frequency ($-e$ and m are the electron charge and mass, c the speed of light) and $\gamma=(1-v^2/c^2)^{-1/2}$ is the Lorentz factor. For finite wave amplitudes the resonances (1) have a finite width and the motion of the particles has a completely different character, depending on whether or not the resonances overlap. For instance, in the case of isolated resonances the particle motion is regular in character, while when the resonances overlap a stochastic particle motion occurs. The idea of resonance overlap, of the stochastic instability of particle motion, the criteria for resonance overlap, and other fundamental ideas in this field were introduced and studied in the basic papers by Zaslavskii and Chirikov.[1,2]

In the last few years the problem of stochasticity has drawn the attention of many people in connection with work on cyclotron heating in mirror traps, on ion heating by radiation in the lower-hybrid resonance band, and on a number of new theoretical studies (see, e.g., Refs. 3 and 4, where there are also references to the above-mentioned experimental work). Apart from the cited laboratory studies we note satellite observations in which powerful electrostatic waves were detected at frequencies close to the frequency of the upper hybrid resonance and propagating almost at right angles to the geomagnetic field. As far as their spectral composition is concerned, these waves are practically monochromatic and as regards to the amplitude of the electrical field they are among the strongest waves in the magnetosphere.[5] Turning to the resonance conditions (1) we see that for small $k_{\parallel}$ only particles with a sufficiently large longitudinal velocity can interact resonantly with the wave. Moreover, overlap of the resonances may occur when the wave amplitude and the particle energy increase. In the case considered it is therefore in contrast to the traditional consideration of the interaction between waves and resonant particles,[6] to take into account the relativistic nature of the particles and the possibility of a transition of the motion into a stochastic regime.

It is natural to assume that the emission observed in Ref. 5 is excited as the result of an instability of the particle distribution function in the magnetosphere. Elimination of this instability may occur as the result of stochastic heating of the particles in a well-defined region of phase space. It is necessary to know the particle distribution function in the stochastic region for a consideration of this problem and for a determination of the saturation amplitude. At the same time in the majority of the studies of stochastic motion the treatment is restricted to a determination of the stochasticity threshold, and the remainder of the analysis is performed numerically,[3,4] since an analytical description of the particle motion at large times is impossible in that case.

In the present paper we pay basically attention to an analytical description of the particle distribution function in the stochastic region. We shall show below that if we assume phase mixing it is sufficient for our aim to solve the equations of motion of the particles for times of the order of the cyclotron period (Sec. 3). In Sec. 4 we derive a kinetic equation for the particle distribution function and on the basis of this we consider the general character of the evolution of the distribution function in the stochastic case. Moreover, we determine the region of applicability of the diffusion equation which is often used in the stochastic region on the basis of phenomenological considerations.[7] In Sec. 5 we construct a WKB solution for the distribution function and we consider the important problem of particle heating in the stochastic region. One should note that the general analysis of the equations of motion, discussed briefly in Sec. 2, differs from that of Ref. 3 only in that the relativistic character of the particles is taken into account.

2. We consider an electrostatic wave propagating at an angle to the external magnetic field $\mathbf{B}_0$ which is di-

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rected along the z-axis of a Cartesian system of coordinates. We shall assume that the wave vector **k** lies in the (x,y) plane. The equations of motion for the electrons then take the form

$$\frac{d\mathbf{r}}{dt}=\frac{\mathbf{p}}{m\gamma};\quad \frac{d\mathbf{p}}{dt}=-e\mathbf{E}-\frac{e}{mc\gamma}[\mathbf{p}\times\mathbf{B}_0], \tag{2}$$

where the electrical field of the wave is

$$\mathbf{E}=\left\{-\frac{k_\perp}{k}E_0\cos\xi;\,0;\,-\frac{k_\parallel}{k}E_0\cos\xi\right\},\quad \xi=k_\perp x+k_\parallel z-\omega t.$$

Taking as the units of velocity, mass, and time, respectively, the speed of light, the electron mass, and the reciprocal of the wave frequency, we rewrite (2) in dimensionless variables

$$\frac{d\mathbf{r}}{dt}=\frac{\mathbf{p}}{\gamma};\quad \frac{dp_x}{dt}=k_\perp\varepsilon\cos\xi-\frac{\omega_c}{\gamma}p_y;$$
$$\frac{dp_y}{dt}=\frac{\omega_c}{\gamma}p_x;\quad \frac{dp_z}{dt}=k_\parallel\varepsilon\cos\xi, \tag{3}$$

where

$$\gamma=(1+\mathbf{p}^2)^{1/2} \tag{4}$$

while $\varepsilon=eE_0/mc^2k$ is the dimensionless wave amplitude.

The interaction between the electrons and a monochromatic wave, described by the set (3), can be completely different in character, depending on the values of the parameters ε and ω_c and on the characteristic values of the particle momenta. In a wide range of the parameters it is convenient to charge for the analysis of the set (3) to the canonical variables when there is no wave, using the relations

$$k_\perp x+k_\parallel z=k_\parallel q+k_\perp(2I/\omega_c)^{1/2}\sin\theta;$$
$$p_z=p_\parallel,\quad p_x=(2I\omega_c)^{1/2}\cos\theta,\quad p_y=(2I\omega_c)^{1/2}\sin\theta.$$

In the new variables we can write the set (3) in Hamiltonian form with a Hamiltonian $H(p_\parallel,q;I,\theta;t)$:

$$H=(1+p_\parallel^2+2I\omega_c)^{1/2}-\varepsilon\sin(k_\parallel q+\mu\sin\theta-t)$$
$$=(1+p_\parallel^2+2I\omega_c)^{1/2}-\varepsilon\sum_{n=-\infty}^{\infty}J_n(\mu)\sin(k_\parallel q+n\theta-t), \tag{5}$$

where $\mu=k_\perp(2I/\omega_c)^{1/2}$, and $J_n(\mu)$ is a Bessel function of order n. As the variables q and t occur in (5) only in the combination $k_\parallel q-t$, the quantity

$$H_1=H-p_\parallel/k_\parallel \tag{6}$$

is an integral of the motion.

From (5) we get the conditions for the resonant interaction of particles with the wave

$$\gamma-k_\parallel p_\parallel=n\omega_c. \tag{7}$$

In contrast to the nonrelativistic case, the resonance conditions (7) contain the total energy γ besides the longitudinal momentum. We shall in what follows consider the case when the change in the quantities γ, and $p_\parallel/k_\parallel$ as a result of the interaction with the wave is much larger than ε. We can then use instead of (6) the approximate integral

$$h=\gamma-p_\parallel/k_\parallel\approx\text{const} \tag{8}$$

and write the resonance conditions at fixed h in the form

$$h-k_\parallel p_\parallel(1-k_\parallel^{-2})=n\omega_c. \tag{9}$$

It is well known[8] that when ε is sufficiently small we can retain in the sum in (5) only one slowly varying term. In this approximation, called the approximation of isolated resonances, the equations of motion are integrated by quadrature and this case has been studied very fully (see, e.g., Refs. 3, 6, 7, 9, 10, and the literature cited there). The approximation of isolated resonances is violated when the change in particle momentum during a single cyclotron period is larger than the distance between resonances. In that case the particle motion has a stochastic character.

To derive a quantitative criterion for the overlap of resonances (sometimes called the Chirikov criterion) we turn to the equation for $p_\parallel$ which follows from (5):

$$dp_\parallel/dt=\varepsilon k_\parallel\cos(k_\parallel q+\mu\sin\theta-t). \tag{10}$$

It is clear that the change in momentum $\Delta p_\parallel$ is rather large when the quantity $k_\parallel q+\mu\sin\theta-t$ has a stationary-phase point determined by the equation

$$\gamma-k_\parallel p_\parallel-\mu\omega_c\cos\theta=0. \tag{11}$$

We note that in dimensional variables Eq. (11) is equivalent to the condition $\omega-\mathbf{k}\cdot\mathbf{v}=0$ where **v** is the particle velocity, so that the interaction between the particles and the wave is most effective if the phase velocity of the wave is less than the speed of light.

We shall assume in what follows that the condition

$$\mu^{1/2}\gg 1 \tag{12}$$

is satisfied, meaning that the time for the particles to pass through the stationary phase point $\gamma/\mu^{1/2}\omega_c$, which is equal to the effective time of the change in the particle momentum, is much shorter than the cyclotron period. In that case the longitudinal particle momentum changes by an amount [see (10)].

$$\Delta p_\parallel\sim\varepsilon k_\parallel\gamma/\mu^{1/2}\omega_c. \tag{13}$$

Recognizing that according to (9) the distance between resonances is

$$\Delta p_n=\omega_c/k_\parallel(1-k_\parallel^{-2}),$$

we get from (13) the condition for overlap of the resonances $\Delta p_\parallel\gtrsim p_n$ in the form

$$\varepsilon>\mu^{1/2}\omega_c^2/\gamma(1-k_\parallel^2), \tag{14}$$

where we assume for the sake of argument that $k_\parallel^2<1$. In deriving (14) and in what follows we use the approximate integral (8). Comparing (8) with the exact integral (6) we check that this is justified when

$$\gamma/\mu^{1/2}\omega_c\gg 1. \tag{15}$$

It is clear that inequalities (12), (14), and (15) are compatible only when $\omega_c\ll 1$.

When the indicated conditions are satisfied the action of the wave upon a particle has thus the character of impacts which occur twice during a cyclotron period. These collisions are characterized by a mean free path time $\pi\gamma/\omega_c$, a collision time

$$\tau_{\text{cr}}\sim\gamma/\mu^{1/2}\omega_c\ll\pi\gamma/\omega_c$$

and a change in momentum which is of the order of (13).

3. For the analysis of the evolution of the particle distribution function in the stochastic region we shall need an exact calculation of the change in the particle momentum during a single impact. To do this we turn to the set (3) and write the equations for the phase ξ and the quantity $\xi/2$ which we denote by u:

$$u={}^1/_2(k_\perp p_x/\gamma+k_\parallel p_z/\gamma-1).$$

From (3) we have

$$\frac{d\xi}{dt}=2u;\quad \frac{du}{dt}=\frac{\varepsilon\cos\xi}{2\gamma}\left[k^2-\frac{(k_\perp p_x+k_\parallel p_z)^2}{\gamma^2}\right]-\frac{k_\perp\omega_c}{2\gamma^2}p_y, \tag{16}$$

where $k^2=k_\parallel^2+k_\perp^2$. The set (16) is not closed, but in contrast to (5) it does not contain variable quantities in the phase. This makes it possible to simplify Eq. (16) appreciably near the stationary-phase point, where the quantities γ and p_z have a maximum change, by replacing all the factors by their values at the resonance point $u=0$. We have

$$\frac{d\xi}{dt}=2u;\quad \frac{du}{dt}=\frac{\varepsilon\cos\xi}{2\gamma_r}(k^2-1)-\frac{k_\perp\omega_c}{2\gamma_r^2}p_{yr}, \tag{17}$$

where the index r indicates the values of quantities at the resonance point, while p_{yr} can be expressed in terms of γ_r and h through Eqs. (8) and (4) and the condition $u=0$:

$$p_{yr}^2=(\gamma_r^2-1)-k_\parallel^2(\gamma_r-h)^2-\{[\gamma_r-k_\parallel^2(\gamma_r-h)]/k_\perp\}^2.$$

With the accuracy used here we get from (3)

$$d\gamma/dt=\varepsilon\cos\xi. \tag{18}$$

From (17) and (18) we get the relation

$$u-\frac{k^2-1}{2\gamma_r}\gamma+\int\frac{k_\perp\omega_c}{2\gamma_r^2}p_{yr}dt=\text{const}, \tag{19}$$

which connects the changes in the quantities γ and u as a result of the "collision".

Apart from the notation, Eqs. (17) are the same as the equations of motion of electrons in the field of a Lanqmuir wave in an inhomogeneous plasma, which were solved in Ref. 11. Introducing the notation

$$\alpha=k_\perp\omega_c p_{yr}/2\gamma_r^2;\qquad \beta=\varepsilon(k^2-1)/2\gamma_r,$$

we get, according to Ref. 11, when $\beta/|\alpha|<1$:

$$u-u_0+\sum_{n=1}^{\infty}\left(\frac{2\pi|\alpha|}{n}\right)^{1/2}J_n\left(\frac{n\beta}{|\alpha|}\right)\left\{\cos\frac{n\mathscr{E}}{|\alpha|}\left[C\left(\frac{n^{1/2}u}{|\alpha|^{1/2}}\right)-C\left(\frac{n^{1/2}u_0}{|\alpha|^{1/2}}\right)\right]\right.$$
$$\left.+\sin\frac{n\mathscr{E}}{|\alpha|}\left[S\left(\frac{n^{1/2}u}{|\alpha|^{1/2}}\right)-S\left(\frac{n^{1/2}u_0}{|\alpha|^{1/2}}\right)\right]\right\}+\int_{t_0}^{t}\alpha\,dt'=0, \tag{20}$$

where $\mathscr{E}=u^2+\alpha\xi-\beta\sin\xi$ is the conserved effective "total energy" of the particle, $C(\zeta)$ and $S(\zeta)$ are the Fresnel cosine and sine integrals, and the index 0 labels the initial values of the various quantities. The condition $\beta/|\alpha|<1$ indicates the absence of particles captured in phase close to resonance. When $\beta/|\alpha|\ll 1$ we can in the sum in (20) restrict ourselves to the first term and put $J_1\approx\beta/2|\alpha|$. Bearing in mind that at the resonance point $u=0$ and that up to and after resonance $|u|\gg 1$ and $|u_0|\gg 1$, and using the expressions for the Fresnel integrals: $C(0)=S(0)=0$, $C(\zeta)=S(\zeta)=\operatorname{sign}\zeta/2$ as $|\zeta|\to\infty$, we get from (19) and (20)

$$\gamma=\gamma_r+{}^1/_2a(\gamma_r)\sin\Phi,\quad \gamma_0=\gamma_r-{}^1/_2a(\gamma_r)\sin\Phi \tag{21}$$

$(\beta/|\alpha|\ll 1)$, where

$$\Phi=\frac{\mathscr{E}}{|\alpha|}+\frac{\pi}{4},\quad a(\gamma_r)=\frac{2\gamma_r}{k^2-1}\left(\frac{\pi}{|\alpha|}\right)^{1/2}\beta.$$

Equations (21) determine in parametric form the Φ-dependence of the quantities γ and γ_0. The region of applicability of (20) and (21) is determined from the condition that the coefficients in Eq. (16) for u in the resonance region differ little from the quantities β and α, respectively. Elementary estimates lead to the following limitations:

$$\varepsilon/\mu^{1/2}\omega_c\sim a(\gamma_r)/\gamma_r\ll 1;\quad \gamma/\omega_c\ll\mu^{1/2}. \tag{22}$$

4. We now turn to a description of the particle distribution function in the stochasticity region. We shall consider particles with a well defined value of the integral of motion h of (8) and we denote the boundary of the stochasticity region by γ_1. Under condition (12) the particles and the wave interact via collisions while in the stochasticity region the particles lose memory of their phase within a time of the order of a cyclotron period. For fixed h the "coarse-grained" distribution function then depends only on γ and the time and can be determined from the particle conservation law:

$$\frac{\partial\delta N}{\partial t}=\int dN'\mathscr{P}(\Omega'\to\Omega)\delta\Omega-\int\delta N\mathscr{P}(\Omega\to\Omega')d\Omega', \tag{23}$$

where δN is the particle density in a phase volume element $\delta\Omega$

$$\delta\Omega=\pi k_\parallel(mc)^3d\gamma^2dh,$$

$\mathscr{P}(\Omega\to\Omega')d\Omega'$ is the probability that per unit time a particle goes from a state Ω into a phase volume element $d\Omega'$ and the quantity δN is connected with the distribution function $f(t,\gamma,h)$ through the relation $\delta N=f(t,\gamma,h)\delta\Omega$. As h is an integral of motion, $\mathscr{P}(\Omega\to\Omega')\propto\delta(h-h')$. Using this and dividing (23) by $\delta\Omega$ we get the equation for the distribution function in the form

$$\frac{\partial f(t,\gamma)}{\partial t}=\int d\gamma'^2f(t,\gamma')P(\gamma'\to\gamma)-\int f(t,\gamma)P(\gamma\to\gamma')d\gamma'^2,$$
$$P(\gamma'\to\gamma)=\pi k_\parallel(mc)^3\mathscr{P}(\Omega'\to\Omega)/\delta(h'-h). \tag{24}$$

To evaluate the transition probability $P(\gamma'\to\gamma)$ we turn to the solution of (21) and put in it $\gamma_0=\gamma'$, and obtain

$$\gamma-\gamma'=a(\gamma_r)\sin\Phi,\quad \gamma+\gamma'=2\gamma_r. \tag{25}$$

Bearing in mind that transitions from the state γ' occur with a frequency $\omega_c/\pi\gamma'$ and that all values of Φ in a 2π interval are equally probable, we have for the transition probability

$$P(\gamma'\to\gamma)d\gamma^2=\frac{\omega_c}{\pi\gamma'}\frac{|d\Phi_1|+|d\Phi_2|}{2\pi}, \tag{26}$$

where $d\Phi_1$ and $d\Phi_2$ are differentials corresponding to $d\gamma^2$.

Differentiating the first Eq. (25) and using condition (22) we obtain

$$d\gamma^2/2\gamma=a(\gamma_r)\cos\Phi\,d\Phi. \tag{27}$$

From (27) and (25) it follows that

$$|d\Phi_1|=|d\Phi_2|=d\gamma^2/2\gamma[a^2-(\gamma'-\gamma)^2]^{1/2}. \tag{28}$$

Substituting (26) and (28) in (24) we get finally

$$\frac{\partial f}{\partial t}=\frac{\omega_c}{\pi^2\gamma}\int\limits_{\gamma-|a|}^{\gamma+|a|}\frac{f(\gamma')-f(\gamma)}{[a^2-(\gamma'-\gamma)^2]^{1/2}}d\gamma'. \quad (29)$$

When inequalities (22) are satisfied we can consider the quantity a^2 in (29) to be a function of γ (rather than of γ_r), and at the chosen accuracy we can assume it to be constant when γ changes in an interval $2|a|$.

Equation (29) is applicable in the region $\gamma>\gamma_1>\bar{\gamma}$ [see explanation of Eq. (45)]. It is thus necessary to impose upon the distribution function, apart from the initial conditions, a condition guaranteeing conservation of the particle density N_e in that region. From (29) we have

$$\frac{dN_e}{dt}\infty\frac{d}{dt}\int\limits_{\gamma_1}^{\infty}f\gamma d\gamma=\frac{\omega_c}{\pi^2}\int\limits_{\gamma_1}^{\infty}d\gamma\int\limits_{\gamma-|a|}^{\gamma+|a|}\frac{f(\gamma')-f(\gamma)}{[a^2-(\gamma'-\gamma)^2]^{1/2}}d\gamma'. \quad (30)$$

Splitting the integral in (30) into two and changing the order of integration in the first one we get

$$\frac{dN_e}{dt}\infty\frac{\omega_c}{\pi^2}\left\{\int\limits_{\gamma_1-|a|}^{\gamma_1+|a|}f(\gamma')d\gamma'\left[\frac{\pi}{2}+\arcsin\frac{\gamma'-\gamma_1}{|a|}\right]\right.$$

$$\left.+\pi\int\limits_{\gamma_1+|a|}^{\infty}f(\gamma')d\gamma'-\pi\int\limits_{\gamma_1}^{\infty}f(\gamma)d\gamma\right\}. \quad (31)$$

From the condition that $f(\gamma)$ be an even function of γ_1 in the region $\gamma_1-|a|<\gamma<\gamma_1+|a|$ it follows that the sum (31) vanishes. We now change from the variable γ to a new variable

$$w=\gamma-\gamma_1$$

and we shall consider Eq. (29) in the whole region of w-values, supplementing for $w<0$ the definition of the collision frequency $\omega_c/\pi\gamma$ and the transition amplitude a by the relations

$$\frac{\omega_c}{\pi\gamma}\to\frac{\omega_c}{\pi(\gamma_1+|w|)};\quad a(\gamma)\to a(\gamma_1+|w|).$$

We then get instead of (29) the equation

$$\frac{\partial f}{\partial t}=\frac{\omega_c}{\pi^2(\gamma_1+|w|)}\int\limits_{w-|a|}^{w+|a|}\frac{f(w')-f(w)}{[a^2-(w'-w)^2]^{1/2}}dw'. \quad (32)$$

It is clear that the solutions of Eq. (32) which are even in w automatically guarantee conservation of the particles in the stochastic region.

As the coefficients of Eq. (32) are time-independent we can look for its solution in the form

$$f_\lambda(t,w)=A(\lambda,w)\exp[-\lambda^2t+i\psi(\lambda,w)], \quad (33)$$

where $A(\lambda,w)$ is a function which varies slowly (as compared to ψ). Substituting (33) in (32) and bearing in mind that the integration in (32) is taken over a small region $2|a|$, we can neglect the difference between $A(w')$ and $A(w)$ and expand $\psi(w')$ close to $\psi(w)$. Dividing then both sides of the equation by f_λ, we arrive at the "local" dispersion relation

$$\lambda^2=\frac{\omega_c}{\pi(\gamma_1+|w|)}[1-J_0(\varkappa a)], \quad (34)$$

where J_0 is a Bessel function with index zero and

$$\varkappa(\lambda,w)=\partial\psi(\lambda,w)/\partial w$$

is the local "wave" number. It follows from (34) that all higher harmonics of the distribution function, corresponding to small values of $J_0(\varkappa a)$, are rapidly damped exponentially with a characteristic time $\tau\sim(\gamma_1+|w|)/\omega_c$. At the same time it is clear from its derivation that Eq. (32) is applicable for times $t>\tau$. For such times we can restrict ourselves to small values of $\varkappa$ and expand $J_0(\varkappa a)$ in (34) near zero, which gives

$$\lambda^2=\frac{\omega_c a^2}{4\pi(\gamma_1+|w|)}\varkappa^2(\lambda,w). \quad (35)$$

The dispersion equation (35) corresponds to an expansion of the difference $f(w')-f(w)$ in (32) up to terms of second order in $(w'-w)$, i.e., to a change from the integral equation (32) to a diffusion type equation

$$\frac{\partial f}{\partial t}=\frac{\omega_c a^2}{4\pi(\gamma_1+|w|)}\frac{\partial^2 f}{\partial w^2}. \quad (36)$$

Thus, the distribution function evolves in the stochastic region in two stages. In the first stage, during a time of the order of the cyclotron period, a "smoothing" of the higher harmonics of the initial distribution occurs. In the second stage the evolution of the distribution function has a diffusion character.

5. We now construct a WKB solution of Eq. (36), using the lucid premises of the theory of adiabatic invariants. To do this we write $f(t,w)$ in the form

$$f=\varphi(\lambda,w)\exp(-\lambda^2t). \quad (37)$$

Substituting (37) in (36) we get an equation for φ:

$$\frac{d^2\varphi}{dw^2}+\varkappa^2(\lambda,w)\varphi=0;\quad \varkappa^2=\frac{4\pi(\gamma_1+|w|)}{\omega_c a^2}\lambda^2. \quad (38)$$

Equation (38) describes the oscillations of a pendulum with a slowly varying frequency. It is well known[12] that in that case the ratio of the energy of the oscillations to the frequency is the adiabatic invariant, i. e.,

$$A^2\varkappa^2/|\varkappa|\approx\text{const},$$

where A is the amplitude of the oscillations. The fundamental solution of Eq. (36) is then

$$f_\lambda=\varphi(\lambda,w)\exp(-\lambda^2t);\ \varphi(\lambda,w)=\frac{1}{|\varkappa(\lambda,w)|^{1/2}}\exp\left[i\int\limits_0^w\varkappa(\lambda,w')dw'\right], \quad (39)$$

and the general solution of Eq. (36) takes the form

$$f(t,w)=\int\limits_{-\infty}^{\infty}\frac{c(\lambda)}{|\varkappa(\lambda,w)|^{1/2}}\exp\left[-\lambda^2t+i\int\limits_0^w\varkappa(\lambda,w')dw'\right]d\lambda, \quad (40)$$

where the function $c(\lambda)$ is determined from the initial conditions

$$f(t=0,w)=F(w)=\int\limits_{-\infty}^{\infty}\frac{c(\lambda)}{|\varkappa(\lambda,w)|^{1/2}}\exp\left[i\int\limits_0^w\varkappa(\lambda,w')dw'\right]d\lambda. \quad (41)$$

We arrive at the problem of expanding an arbitrary function $F(w)$ in the given set of WKB functions $\varphi(\lambda,w)$ of (39); this problem is solved as follows. Introducing the quantity

$$s(w)=\int\limits_0^w\sigma(w')dw';\quad \sigma(w)=\frac{\varkappa(\lambda,w)}{\lambda}=\left(\frac{4\pi(\gamma_1+|w|)}{\omega_c a^2}\right)^{1/2}, \quad (42)$$

which is a one-to-one function of w, we rewrite (41) in the form

$$F(s)=\frac{1}{\sigma^{1/2}(s)}\int\limits_{-\infty}^{\infty}\frac{c(\lambda)}{|\lambda|^{1/2}}e^{i\lambda s}d\lambda,$$

which is the same as the standard form of the Fourier expansion so that

$$\frac{c(\lambda)}{|\lambda|^{1/2}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)\sigma^{1/2}(s)e^{-i\lambda s}ds$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(w)\sigma^{1/2}(w)e^{-i\lambda s(w)}\frac{ds}{dw}dw. \qquad (43)$$

Equations (41) and (43) solve the problem of the expansion of a given function $F(w)$ in the WKB functions $\varphi(\lambda, w)$ of (39). (In the Appendix we consider the generalization to the case of an arbitrary set of WKB functions.) After substituting (43) in (40) and integrating over λ the WKB solution of Eq. (36) takes the form

$$f=(4\pi\sigma(w)t)^{-1/2}\int_{-\infty}^{\infty}dw'F(w')\sigma^{3/2}(w')\exp\{-[s(w)-s(w')]^2/4t\}\ .$$

In conclusion we evaluate the quantity $d\Pi_c/dt$—the rate of change of the particle energy density in the stochastic region. The latter is an important characteristic of the process of interaction between the wave and the particles as it determines the absorbed wave power, the particle heating, and in the case of instability the saturation amplitude of the wave. Using the fact that

$$\delta\Pi_c=\pi k_\parallel(mc)^3f(\gamma, h)\gamma d\gamma^2dh,$$

we have from (29)

$$\frac{d\Pi_c}{dt}=-\frac{2k_\parallel(mc)^3\omega_c}{\pi}\int_{-\infty}^{\infty}dh\int_{\gamma_1}^{\infty}\gamma d\gamma\int_{\gamma-|a|}^{\gamma+|a|}\frac{f(\gamma')-f(\gamma)}{[a^2-(\gamma'-\gamma)^2]^{1/2}}d\gamma'.$$

Proceeding similarly as when deriving (31) and changing to the variable w we get after simple transformations

$$\frac{d\Pi_c}{dt}=-\frac{2k_\parallel(mc)^3\omega_c}{\pi}\int_{-\infty}^{\infty}dh\int_0^{|a|}f(\gamma_1+w, h)\left\{2w\arcsin\frac{w}{|a|}+2(a^2-w^2)^{1/2}-\pi w\right\}dw. \qquad (44)$$

The quantity in braces in (44) is positive so that, regardless of the form of the distribution function, the particles are heated in the stochastic region—a fact known earlier from a number of papers.[3,4] We emphasize that in the given case this result is obtained on the basis of the general equation (29). Taking the distribution function from under the integral sign for $w=0$ and integrating over w we get finally

$$\frac{d\Pi_c}{dt}=\frac{k_\parallel(mc)^3\omega_c}{2}\int_{-\infty}^{\infty}dhf(\gamma_1, h)a^2(\gamma_1, h).$$

The particle heating process is thus determined by the value of the distribution function at the boundary of the stochasticity region:

$$f(t, \gamma_1, h)=\frac{1}{2[\pi\sigma(0)t]^{1/2}}\int_{-\infty}^{\infty}F(w, h)\sigma^{3/2}(w)\exp[-s^2(w)/4t]dw. \qquad (45)$$

It follows, in particular, from (45) that for sufficiently large t (and especially when $t>\pi\gamma_1\bar{\gamma}^2/\omega_c a^2$, where $\bar{\gamma}$ is the characteristic energy with which the drop in the distribution function starts) $d\Pi_c/dt\propto t^{-1/2}$.

In the present paper we have considered a concrete case of stochastic motion of relativistic particles while they interact with an electrostatic monochromatic wave which propagates at an angle to the external magnetic field. However, the approach expounded here, which led to Eq. (29), may turn out to be useful for an analysis of stochasticity also in other cases when the motion of the particles in the phase plane has the character of jumps arising as the result of some effective collisions.

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APPENDIX

The problem of expanding an arbitrary function $F(w)$ in a given set of WKB functions arises when solving spatially inhomogeneous problems with initial conditions. The expansion obtained in Sec. 5 in terms of the functions $\varphi(\lambda, w)$ of (39) is exact, since it reduces to a Fourier expansion in a new independent variable. Here we consider a more general situation and obtain the appropriate approximate formulae. First we give the set of WKB functions. We specify the relation

$$\chi(\lambda, \varkappa, w)=0,$$

which defines a function $\varkappa(\lambda, w)$ such that

$$\partial\varkappa/\partial\lambda>0. \qquad (A.1)$$

The single-parameter family of functions $\varphi(\lambda, w)$ is defined as follows:

$$\varphi(\lambda, w)=Ae^{i\psi};\quad A(\lambda, w)=\left|\frac{\partial\chi[\lambda, \varkappa(\lambda, w), w]}{\partial\varkappa}\right|^{-1/2};\quad \psi=\int_0^w\varkappa(\lambda, w')dw'$$

and we shall assume that the inequality

$$\frac{1}{A}\frac{\partial A}{\partial w}\ll\varkappa \qquad (A.2)$$

is satisfied. We now prove that the following expansion holds:

$$F(w)=\frac{1}{2\pi}\iint d\lambda dw'F(w')\frac{A(\lambda, w)}{A(\lambda, w')}\frac{\partial^2\psi(\lambda, w')}{\partial\lambda\,\partial w'}\times\exp[i\psi(\lambda, w)-i\psi(\lambda, w')]. \qquad (A.3)$$

It is clear that the main contribution to the integral (A.3) comes from a region of w' close to w. We therefore put in (A.2) $A(\lambda, w')=A(\lambda, w)$,

$$\psi(\lambda, w)-\psi(\lambda, w')=\frac{\partial\psi(\lambda, w')}{\partial w'}(w-w').$$

It then follows from (A.3) that

$$\frac{1}{2\pi}\iint d\lambda dw'F(w')\frac{\partial^2\psi}{\partial\lambda\partial w'}\exp\left[i\frac{\partial\psi}{\partial w'}(w-w')\right]$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dw'F(w')\int_{-\infty}^{\infty}d\left(\frac{\partial\psi}{\partial w'}\right)\exp\left[i\frac{\partial\psi}{\partial w'}(w-w')\right]$$

$$=\int_{-\infty}^{\infty}dw'F(w')\delta(w-w')=F(w), \qquad (A.4)$$

which proves in fact the expansion (A.3). In deriving (A.4) we used the fact that when condition (A.1) is satisfied the quantity $\partial\psi/\partial w$ changes in the interval $(-\infty, \infty)$ when λ changes in that interval. We can rewrite the expansion (A.3), which now has been proved, in the form of the relations

$$F(w)=\int_{-\infty}^{\infty} d\lambda c(\lambda) A(\lambda, w) \exp[i\psi(\lambda, w)],$$

$$c(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dw F(w) \frac{1}{A(\lambda, w)} \frac{\partial^2\psi}{\partial\lambda\partial w} \exp[-i\psi(\lambda, w)], \tag{A.5}$$

which solve our problem. Equation (A.5) generalizes the formula for the Fourier integral transformation to the case when the above-defined WKB functions $\varphi(\lambda, w)$ serve as the basic system.

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