Current-voltage characteristic of superconductors with large-size defects

A. I. Larkin and Yu. N. Ovchinnikov

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences

(Submitted 16 October 1980)
Zh. Eksp. Teor. Fiz. **80**, 2334–2345 (June 1981)

Large defects deform the vortex lattice in a supersonductor. The pinning force is determined by the dimension of the region in which the vortex layer is tangent to the defect surface. The vortex deformation and the pinning force decrease as the vortices move. A region of negative differential resistance appears on the current-voltage characteristic.

PACS numbers: 74.60.Ge

1. INTRODUCTION

The current-voltage characteristic of a hard superconductor depends substantially on the sizes of the defects. For small defects, in magnetic fields not too close to H_{c2}, the current-voltage characteristic has the universal form[1]

$$j=j_c F(\sigma E/j_c). \qquad \text{(a)}$$

The critical current is small in this case. Therefore a small number of large defects has a strong influence on the critical current and on the shape of the current-voltage characteristic. We consider below defects of sufficiently large size, for which the Labusch criterion is satisfied.[2,3] These defects can be clusters of dislocations or of impurities, or else regions of precipitation of a new phase. If the superconducting properties of the defect differ strongly from the properties of the matrix, then such a defect can lead to plastic deformation of the vortex lattice. In this paper we consider defects that produce in the vortex lattice a weak deformation that can be described within the framework of elasticity theory. The dimension of the defects is assumed to exceed significantly the period of the vortex lattice.

2. EQUATIONS OF MOTION OF THE VORTEX LATTICE

The vortex lattice is acted upon by four forces: elastic, viscous, force of Lorentz interaction with the transport current, and force of interaction with the defects. The sum of all these forces is zero[4,5]

$$(C_{11}-C_{66})\frac{\partial}{\partial\rho}\left(\frac{\partial \mathbf{u}}{\partial\rho}\right)+\left(C_{66}\frac{\partial^2}{\partial\rho^2}+C_{44}\frac{\partial^2}{\partial z^2}\right)\mathbf{u} - \sigma B^2\frac{\partial \mathbf{u}}{\partial t}=-[\mathbf{j}\times\mathbf{B}]-\mathbf{f}(\mathbf{r},\mathbf{r}-\mathbf{u}), \qquad (1)$$

where C_{11}, C_{44}, and C_{66} are the elastic moduli of the vortex lattice, σ is the conductivity of the superconductor in the absence of defects, $\mathbf{B}$ is the magnetic induction, $\mathbf{j}$ is the density of the transport current, and $\mathbf{f}$ is the force of interaction of the defects with the vortex lattice.

If the properties of the defect can be described by the change of the electron-electron interaction or by the change of the mean free path of the electrons, then the force $\mathbf{f}$ near the transition temperature can be represented in the form[3]

$$\mathbf{f}(\mathbf{r},\mathbf{r}-\mathbf{u})=-\delta(\delta F)/\delta\mathbf{u},$$
$$\delta F=\nu\int d^3r_1\left\{g_1(\mathbf{r}_1)|\Delta(\mathbf{r}_1-\mathbf{u})|^2+\frac{\pi}{8T}(\delta D)|\partial_-\Delta(\mathbf{r}_1-\mathbf{u})|^2\right\}, \qquad (2)$$

where $\nu=mp/2\pi^2$ is the state density on the Fermi surface, $D-vl_{tr}/3$ is the diffusion coefficient, and $\partial_-=\partial/\partial\mathbf{r}-2ie\mathbf{A}$ is the invariant derivative.

We shall consider defects of ellipsoidal form. For most defects, the interaction force can be written as a sum of two terms. The first is connected with the differences between the free energies of the matrix and of a region averaged over dimensions that are large with the lattice period, inside the defect. The second term is a periodic function having the period of the lattice and a zero mean value. The first term leads to a difference between the vortex densities inside the defect and in the superconducting matrix. We exclude this force and the elastic deformation it produces from Eq. (1). As a result, the function $\mathbf{f}$ in the right-hand side of (1) can be regarded as periodic with zero mean value.

3. CRITICAL CURRENT

At a low density of the defects, the critical current is proportional to the density of the defects for which the Labusch criterion is satisfied. It suffices therefore to calculate the average force exerted by one defect on the vortex lattice. Averaging (1) over the coordinates, we obtain

$$[\mathbf{j}_c\times\mathbf{B}]=-n\left\langle\int \mathbf{f}\,d^3r\right\rangle \qquad (3)$$

where n is the defect density and $\langle\int d^3r\mathbf{f}\rangle$ is the average force of interaction of one defect with the vortex lattice. On the other hand, the force produced when the defect is displaced relative to the lattice, is

$$-\partial F/\partial u(\infty), \qquad (4)$$

where F is the free energy of the superconductor with the defect. From formula (4) it follows that the average force differs from zero only if metastable states exist, and consequently F is not a single-valued function of $\mathbf{u}$. Averaging Eq. (4) over the displacements, we get

$$j_cB=n\delta F/\tilde{a}_1, \qquad (5)$$

where δF is the discontinuity of the free energy on going from one metastable state to another, and $\tilde{a}_1$ is the average distance over which such a transition takes place.

For a large defect in the form of an ellipsoid with sufficiently sharp boundary, the metastable states are due to the barrier that prevents the vortex layer from going into the region of the defect from the matrix. The vor-

 0038-5646/81/061221-06$02.40

tex layer is in contact with the defect surface. When the lattice moves relative to the defect, the contiguity region and the elastic-deformation force increase until the elastic forces exceed the force that prevents the vortices from entering the defect. At that instant the vortex layer becomes detached from the surface of the defect, and the free energy of the deformed vortex lattice decreases jumpwise. We shall consider below the case when the displacement of the vortex lattice is small compared with the radius of the defect. This is possible if the barrier to the entry of the vortices is not very high. The problem of the contiguity of the vortex layer and the defect surface has much in common with the Hertz problem of the contiguity of elastic bodies.[5] The difference is that the lattice vortex is not an elastic isotropic medium; furthermore, the vortex lattice fills all of space, and not a half-space as in the Hertz problem.

In places where the defect surface crosses a surface layer at an angle, the force averaged over dimensions that are large compared with the lattice period is zero. We assume that it differs from zero only in the places where the vortex layers and the defect surface are contiguous. For such a layer, the displacement in the contiguity region is

$$u_y=u_0-x^2/2R_1-z^2/2R_2, \tag{6}$$

where R_1 and R_2 are the curvature radii of the defect surface at the place of contiguity with the vortex layer. On the other hand, the value of the displacement is expressed in terms of the force with the aid of Eq. (1). As a result we obtain for the distribution of the forces the integral equation

$$\begin{gathered} u_0-x^2/2R_1-z^2/2R_2=M(P), \\ M(P)=\int\frac{d^3K}{(2\pi)^3}\int_S dx_1\,dz_1\{K_y^2[K_\perp^2(C_{11}K_\perp^2+C_{44}K_z^2)]^{-1} \\ +K_x^2[K_\perp^2(C_{66}K_\perp^2+C_{44}K_z^2)]^{-1}\}P(x_1,z_1)\exp[iK_x(x-x_1)+iK_z(z-z_1)], \end{gathered} \tag{7}$$

where

$$K_\perp^2=K_x^2+K_y^2, \quad P(x,z)=\int dy\, f_y(x,y,z), \tag{8}$$

S is the region of the contiguity of the vortex layer with the defect surface. We assume that the region S is an ellipse with semiaxes a and b, and the function $P(x,z)$ depends on only one argument:

$$P(x,z)=P(\rho^2), \quad \rho^2=(x/a)^2+(z/b)^2.$$

In this case the integral in the right-hand side of (7) takes the form

$$M(P)=\frac{b}{4\pi(C_{66}C_{44})^{1/2}}\int_0^{2\pi}\frac{d\varphi}{\sin^2\varphi}\frac{(\cos^2\varphi+C_{66}\sin^2\varphi/C_{11})^{1/2}-\cos^2\varphi}{(\cos^2\varphi+K^2\sin^2\varphi)^{1/2}} \tag{9}$$

$$\times\int_\mu^1 d\rho\,\rho P(\rho^2)(\rho^2-\mu^2)^{-1/2},$$

$$K=\frac{b}{a}\left(\frac{C_{66}}{C_{44}}\right)^{1/2}, \quad \mu^2=\frac{(x\cos\varphi/a+Kz\sin\varphi/b)^2}{\cos^2\varphi+K^2\sin^2\varphi}. \tag{10}$$

It follows from (9) that the solution of the integral equation (7) is of the form

$$P(x,z)=P_0(1-\rho^2)^{1/2}. \tag{11}$$

Substituting (11) in (9) and (7), we get

$$u_0-\frac{x^2}{2R_1}-\frac{z^2}{2R_2}=\frac{P_0 b}{16(C_{66}C_{44})^{1/2}}\left\{I_1\left(1-\frac{x^2}{a^2}\right)+I_2\left(\frac{x^2}{a^2}-\frac{z^2}{b^2}\right)\right\}, \tag{12}$$

where the integrals I_1 and I_2 are equal to

$$\begin{gathered} I_1=\int_0^{2\pi}\frac{d\varphi}{\sin^2\varphi(\cos^2\varphi+K^2\sin^2\varphi)^{1/2}}\left[\left(\cos^2\varphi+\frac{C_{66}}{C_{11}}\sin^2\varphi\right)^{1/2}-\cos^2\varphi\right], \\ I_2=K^2\int_0^{2\pi}\frac{d\varphi}{(\cos^2\varphi+K^2\sin^2\varphi)^{3/2}}\left[\left(\cos^2\varphi+\frac{C_{66}}{C_{11}}\sin^2\varphi\right)^{1/2}-\cos^2\varphi\right]. \end{gathered} \tag{13}$$

We present the values of the integrals I_1 and I_2 in limiting cases:

$$\left.\begin{aligned} I_1-I_2&=2K^{-1}(1+C_{66}/C_{11}) \\ I_2&=2K^{-1}(1+C_{66}/C_{11})\ln K \end{aligned}\right\} \quad K\gg 1, \tag{14}$$

$$\left.\begin{aligned} I_1&=4+2(C_{66}/C_{11})^{1/2}[-1+\ln(1+16C_{66}/C_{11}K^2)] \\ I_2&=4\left(K+\left(\frac{C_{66}}{C_{11}}\right)^{1/2}\right) \end{aligned}\right\} \quad K\ll 1. \tag{15}$$

By equating the coefficients of x^2 and z^2 in (12), we obtain the semiaxes (a,b) of the ellipse as functions of the pressure P_0. With increasing u_0, the pressure P_0 reaches a critical value P_c. Integrating with respect to the coordinates in (3), we obtain for the critical current, taking (11) into account,

$$[\mathbf{j}_c\times\mathbf{B}]=-{}^2/_3\pi n\sum\langle ab\mathbf{P}_0\rangle, \tag{16}$$

where n is the bulk density of the defects. The averaging in (16) is over all possible displacements u_0, and the summation is over all the vortex rows in contact with the defect surface.

The main contribution to the critical current is made by the vortex rows located along the unit-cell vectors. Taking into account also the rows perpendicular to them, we find that the summation in (16) leads to the appearance, in the expression for the critical current, of a factor $G(\varphi)$ that depends on the angle φ between the direction of the current and one of the unit-cell vectors. If the distribution of the curvature radii of the defects does not depend on the angle φ, then

$$\begin{gathered} G(\varphi)=(|\cos\varphi|+|\cos(\varphi+\pi/3)|+|\cos(\varphi-\pi/3)|) \\ +\alpha(|\sin\varphi|+|\sin(\varphi+\pi/3)|+|\sin(\varphi-\pi/3)|). \end{gathered} \tag{17}$$

The coefficient α in (17) will be determined later on.

If the value of u_0 at which the break takes place is much larger than the lattice period, then the pressure P_0 in (16) is close to its critical value P_c, which can be obtained from Eqs. (2) and (8). The quantity P_c determines the critical current density for a plane infinite boundary. Its value is determined essentially by the thickness of the transition layer between the metal and the defect. If the thickness of this layer is large compared with the lattice period, then α is small.

Assuming P_c to be known, we obtain the dependence of the critical current on P_c and on the defect dimensions in various limiting cases. To this end we must express the contiguity area πab, which enters in (16), in terms of the curvature radii of the surface and of the value of P_c with the aid of formulas (12) and (13):

$$ab=P_c^2R_2^{3/2}R_1^{1/2}/4C_{44}C_{11}^{3/4}C_{66}^{1/4}, \quad R_1/R_2>(C_{66}C_{11})^{1/2}/C_{44}. \tag{18}$$

In most cases the elastic modulus C_{66} is small compared with the moduli C_{11} and C_{44}. Therefore Eq. (14)

is valid for defects whose curvature radii are of the same order, or for defects that are oblate or not too strongly prolate along the magnetic field. For defects that are strongly prolate along the magnetic field we get

$$ab=\frac{P_c^2R_2^3C_{66}^{1/2}}{4R_1C_{44}^{5/2}},\quad \frac{C_{66}}{C_{44}}<\frac{R_1}{R_2}<\frac{(C_{11}C_{66})^{1/2}}{C_{44}},$$
$$ab=\frac{P_c^2R_1^{3/2}R_2^{1/2}}{16C_{66}^2}\left(1+\frac{C_{66}}{C_{11}}\right)^2\ln^{1/2}\left(\frac{R_2C_{66}}{R_1C_{44}}\right)^{1/2},\quad \frac{R_1}{R_2}<\frac{C_{66}}{C_{44}}. \tag{19}$$

We obtain thus for the critical-current density

$$j_cB={}^2/_3\pi G(\varphi)nabP_c, \tag{20}$$

where the value of ab is determined by (18) and (19).

Using (12) we can study the pinning of vortices by large defects in a film. These defects have the same curvature radius R_1, and to find the length $2a$ of the contiguity region we must put in (12)

$$R_2=\infty,\quad b=\infty,\quad C_{11}=\infty,\quad a=P_cR_1/4C_{66}. \tag{21}$$

In analogy with (20), we obtain for the critical current density in a film

$$j_cB=G(\varphi)n\pi P_c^2R_1/8C_{66}, \tag{22}$$

where n is the number of defects per unit surface.

In the case of a diffuse boundary whose thickness exceeds the lattice period, the coefficient α in (17) is small. For a sharp boundary, in magnetic fields not close to H_{c2}, the coefficient α is determined by the distance between the vortices in the contiguous layer, and is equal to

$$\alpha=3^{-1/2},\quad \alpha={}^1/_3 \tag{23}$$

for a bulky superconductor and for a film, respectively. For a film it is necessary in this case to take into account in (17) the contribution from the inclined vortex layers.

Equation (20) for the critical-current density is valid if P_c is not too large, so that the dimensions of the contiguity regions are less than the corresponding curvature radii of the defect. Otherwise the deformation of the vortex lattice is large and plastic flow of the vortices may set in. An onset of plastic flow should be expected near the critical field H_{c2}, where the modulus C_{66} is small.

Formula (20) for the critical current density was obtained using the theory of elasticity for the vortex lattice. To this end it is necessary that the dimensions a and b of the contiguity region be large not only compared with the lattice period, but also compared with the depth of penetration of the magnetic field. Otherwise it is necessary to take into account the spatial dispersion of the elastic moduli[6]:

$$C_{44}=\frac{B^2}{4\pi}\frac{k_h^2}{K^2+k_h^2}. \tag{24}$$

Near the transition temperature we have

$$k_h^2=\frac{2e^2p^2l_{tr}}{3T}\langle|\Delta|^2\rangle.$$

If the Ginzburg-Landau parameter $\varkappa^2\gg1$, then the modulus

$$C_{11}=C_{44}. \tag{25}$$

This equality is violated only in the region where the momentum is large, $K>k_\psi$, where k_ψ^{-1} is the correlation length for the modulus of the order parameter.

The integral equation for the pressure $P(x,z)$ has the same form (7), in which Eq. (24) must be substituted for the moduli C_{11} and C_{44}. We do not know the exact solution of the resultant integral equation, and find therefore the order of magnitude of its solution for the case of strong dispersion:

$$ab=\frac{4\pi P(R_1R_2)^{1/2}}{B^2k_h^2a_1}\left[1+\left(\frac{B^2k_h^2a_1^2}{4\pi C_{66}}\frac{R_2}{R_1}\right)^{1/4}\right], \tag{26}$$
$$\frac{R_2}{R_1}<\frac{PR_1}{C_{66}a_1}+\left(\frac{4\pi PR_1}{B^2k_h^2a_1^3}\right)^{1/2},$$
$$\frac{R_2}{R_1}<\frac{PR_1}{C_{66}a_1}+\left(\frac{PR_1}{C_{66}a_1}\right)^3\left(\frac{B^2k_h^2a_1^2}{4\pi C_{66}}\right)^2$$
$$ab=\frac{R_2}{R_1}\frac{4\pi C_{66}^{1/2}(PR_2)^{1/2}}{(B^2k_h^2)^{5/4}},$$
$$\frac{PR_1}{C_{66}a_1}+\left(\frac{4\pi PR_1}{B^2k_h^2a_1^3}\right)^{1/2}<\frac{R_2}{R_1}<\frac{B^2k_h^2(PR_1)^2}{4\pi C_{66}^3}. \tag{27}$$

For a defect that is very strongly elongated along the z axis we obtain

$$ab=\left(\frac{R_2}{R_1}\right)^{1/2}\left(\frac{PR_1}{2C_{66}}\right)^2\ln^2\left[\frac{4\pi C_{66}}{B^2k_h^2}\frac{R_2}{R_1}\left(\frac{2C_{66}}{PR_1}\right)^2\right],$$
$$\frac{R_2}{R_1}>\frac{B^2k_h^2(PR_1)^2}{4\pi C_{66}^3} \tag{28}$$

in the derivation of (28) we have assumed that the dimension a of the contiguity region is larger than the lattice period a_1.

We consider now briefly the case of the presence of a barrier to the departure of the vortices from the defect region into the superconducting matrix. Assuming that the defect covers a region with dimensions larger than ξ, we find that the displacement is determined by the right-hand side of Eq. (7). Assuming also that the bending of the captured vortex layer is large and that the corresponding curvature radii are smaller than the curvature radii of the defect, we obtain an expression for the dimensions of the captured area:

a) with allowance for the dispersion of the elastic moduli:

$$a\sim C_{66}R_f/P,\quad b\sim R_f(C_{66}C_{44})^{1/2}/P,\quad u_0\sim -R_f,$$
$$j_cB\sim nC_{66}^{3/2}C_{44}^{1/2}R_f^3/Pa_1; \tag{29}$$

b) with allowance for the dispersion of the elastic moduli:

$$a\sim\frac{C_{66}R_f}{P},\quad b\sim a^2\left(\frac{B^2k_h^2}{4\pi C_{66}}\right)^{1/2},\quad u_0\sim -R_f,$$
$$j_cB\sim nP\frac{R_f}{a_1}\left(\frac{C_{66}R_f}{P}\right)^3\left(\frac{B^2k_h^2}{4\pi C_{66}}\right)^{1/2} \tag{30}$$

where R_f is the effective radius of the forces on the boundary between the defect and the matrix. It follows from (29) and (30) that an increase of the barrier to the emergence of the vortices from the defect leads to a decrease of the area of the capture region and to a decrease of the critical-current density. With decreasing P, the capture region increases and perturbation theory takes over at $R_1>a^2/R_f$ and $R_2>b^2/R_f$.

Since the displacement u_0 turned out to be of the order of the radius R_f of the interaction forces, estimates are incapable of providing a final answer to the question of formation of a metastable state in the presence of a barrier for the emergence of the vortices from the defect region into the superconducting matrix.

4. CURRENT-VOLTAGE CHARACTERISTIC AT LOW LATTICE VELOCITIES

We consider first low lattice-vortex velocities, such as to satisfy the condition

$$\sigma B^2 V/a_1 \ll \min(C_{44}/b^2,\ C_{66}/a^2), \tag{31}$$

where V is the average velocity of the vortex lattice. If condition (32) is satisfied, Eq. (1) can be solved by perturbation theory. At low velocities V, the motion of the lattice can be jumplike: the layers come in contact with the defect, followed by the jump. If the defect-induced lattice displacement u_0 is large in the static case compared with the lattice period a_1, then the jump is accompanied by landing of a new vortex layer on the defect, and the displacement at the instant of the jump changes by an amount equal tothe distance between the layers.

Taking the foregoing remarks into account, Eq. (1) for the Fourier component can be reduced, in the absence of dispersion of the elastic moduli, to the form

$$\frac{\tilde{a}_1}{2\pi iN}+\frac{1}{8\pi(C_{66}C_{44})^{1/2}}\left(\frac{-i\sigma B^2\omega_N}{C_{66}}\right)^{1/2} 1+\frac{C_{66}}{C_{11}} \times\int_S dx_1\,dz_1\,P_N(x_1,z_1)=M(P_N),\quad N\neq 0, \tag{32}$$

where $M(P)$ is defined in Eq. (7) and $\tilde{a}_1$ is the size of the jump, equal to the distance between the vortex layers,

$$\omega_N=2\pi VN/\tilde{a}_1.$$

The pressure $P(x,z,t)$ on the surface of the defect is equal to

$$P(x,z,t)=\sum_N P_N(x,z)\exp(-i\omega_N t). \tag{33}$$

The semi-axes a and b of the contiguity ellipse depend on the velocity V and are determined from the condition that at the instant of the break the pressure at the center of the defect reaches the maximum value P_c:

$$P_0(0)+\sum_{N\neq 0} P_N(0)\exp(i\omega_N\delta)=P_c, \tag{34}$$

with $\delta=+0$, since the break was taken by us to occur at $t=0$.

Using expression (9) for $M(P_N)$, it is easily seen that the solution of (32) is of the form

$$P_N(x,z)=P_N(1-x^2/a^2-z^2/b^2)^{-1/2}. \tag{35}$$

Substituting (35) in (9) and (32), we obtain

$$M(P_N)=\frac{bP_N}{8(C_{66}C_{44})^{1/2}}I_1,$$
$$P_N=-\frac{4i\tilde{a}_1(C_{66}C_{44})^{1/2}}{\pi NbI_1}\left[1+\frac{2a}{I_1}\left(1+\frac{C_{66}}{C_{11}}\right)\left(-\frac{i\sigma B^2\omega_N}{C_{66}}\right)^{1/2}\right],\quad N\neq 0. \tag{36}$$

Substituting the expression for PN from (36) in (34), we obtain an expression for $P_0(0)$:

$$P_0(0)=P_c-\frac{4\tilde{a}_1(C_{66}C_{44})^{1/2}}{bI_1}\left[1+\frac{2^{1/2}aC}{\pi I_1}\left(1+\frac{C_{66}}{C_{11}}\right)\times\left(\frac{\sigma B^2\cdot 2\pi V}{C_{66}\tilde{a}_1}\right)^{1/2}\right]=\tilde{P}_c(1-\beta E^{1/2}),$$

$$\tilde{P}_c=P_c-\frac{4\tilde{a}_1(C_{66}C_{44})^{1/2}}{bI_1},\quad \beta=\frac{4\tilde{a}_1C_{44}^{1/2}}{bI_1^2}\frac{2^{1/2}aC}{\pi\tilde{P}_c}\left(1+\frac{C_{66}}{C_{11}}\right)\left(\frac{\sigma B\cdot 2\pi}{\tilde{a}_1}\right)^{1/2},$$
$$C=\lim_{N\to\infty}\left(2N^{1/2}-\sum_1^N N_1^{-1/2}\right)=-\zeta(1/2)=1.46. \tag{37}$$

where $\zeta(x)$ is the Riemann zeta function.

The quantity $P_0(x,z)$ satisfies the static equation (7), and the semi-axes a and b are determined, as before, from relations that follow from (12). These relations together with Eq. (37) determine the current-voltage characteristic of a superconductor with defects.

As follows from (20) and (22), j_c is proportional to P_c^3. Allowance for the lattice motion calls for replacement of P_c by $P_0(0)$, the latter being expressed in terms of P_c and E in accord with Eq. (37). At $E=0$ a renormalization of the critical current ($P_c\to\tilde{P}_c$) takes place, due to the fact that the average pinning force is somewhat smaller than the maximum. This renormalization is small if the static deformation is large compared with the lattice period. With increasing velocity, the average force decreases further. The friction in motion is less than the friction at rest. At small E

$$j=\sigma E+j_c(E),\quad j_c(E)=\tilde{j}_c(1-3\beta E^{1/2}). \tag{38}$$

The $j_c(E)$ dependence is not single-valued. This means that voltage jumps should take place in a circuit with a given current. The square-root singularity in the dependence of the current on the voltage or on the velocity is the result of the diffuse character of the lattice motion. The size of the deformation region is of the order of $V^{-1/2}$.

With increasing vortex-lattice velocity, condition (31) is violated and we obtain first the two-dimensional and then the one-dimensional case.

If the condition

$$C_{44}/b^2\ll\sigma B^2 2\pi V/\tilde{a}_1, \tag{39}$$

is satisfied, then the equation for $P_N(x,z)$ takes the form

$$\frac{\tilde{a}_1}{2\pi iN}=\frac{i}{2\sigma B^2\omega_N}\int\frac{dK_x}{2\pi}\exp(iK_x(x-x_1))\times\int_{-a(z)}^{a(z)}dx_1\,P_N(x_1,z)\left\{\left(K_x^2-\frac{i\sigma B^2\omega_N}{C_{11}}\right)^{1/2}-\frac{K_x^2}{(K_x^2-i\sigma B^2\omega_N/C_{66})^{1/2}}\right\}, \tag{40}$$

where $a(z)=a(1-z^2/b^2)^{1/2}$. We solve the integral equation (40) in the following limiting cases (a, b, and c):

$$\text{a.}\quad a^{-2}\gg\sigma B^2\cdot 2\pi V/\tilde{a}_1C_{66}. \tag{41}$$

If condition (41) is satisfied, the solution of (40) takes the form

$$P_N(x,z)=\frac{P_N(z)}{(a^2(z)-x^2)^{1/2}}. \tag{42}$$

We substitute (42) in (40) and obtain an explicit expression for the coefficients $P_N(z)$. We obtain next from (34) an expression for the coefficient $P_0(0)$:

$$P_0(0)=P_c-\frac{2\tilde{a}_1}{a}\left\{\left(-\frac{1}{2}+\ln\left(\frac{4}{a\gamma(\sigma B^2\cdot 2\pi V/C_{66}\tilde{a}_1)^{1/2}}\right)\right)\Big/C_{66}+\left(\frac{1}{2}+\ln\left(\frac{4}{a\gamma(\sigma B^2\cdot 2\pi V/C_{11}\tilde{a}_1)^{1/2}}\right)\right)\Big/C_{11}\right\}^{-1}, \tag{43}$$

where $\ln\gamma=0.577$ is the Euler constant;

b. $\sigma B^2\cdot 2\pi V/C_{11}\tilde{a}_{11} \ll a^{-2} \ll \sigma B^2\cdot 2\pi V/C_{66}\tilde{a}_1$. (44)

If condition (44) is satisfied, the solution of the integral equation (40) is

$$P_N(x,\ z)=P_N(a^2(z)-x^2)^{1/2},\quad P_N=-2\sigma B^2 V. \qquad (45)$$

The expression for $P_0(0)$ takes the form

$$P_0(0)=P_c-2\sigma B^2 Va, \qquad (46)$$

c. $a^{-2}\ll\sigma B^2\cdot 2\pi V/C_{11}\tilde{a}_1$. (47)

When condition (37) is satisfied we arrive at the one-dimensional case:

$$P_N=\frac{C_{11}\tilde{a}_1}{i\pi N}\left(-\frac{i\sigma B^2\omega_N}{C_{11}}\right)^{1/2}, \qquad (48)$$

$$P_0(0)=P_c-\frac{2^{1/2}C}{\pi}(C_{11}\tilde{a}_1\sigma B^2\cdot 2\pi V)^{1/2}.$$

We consider now the situation when the condition

$$C_{11}/a^2\ll\sigma B^2\cdot 2\pi V/\tilde{a}_1. \qquad (49)$$

is satisfied. In this case the equation for $P_N(x,z)$ takes the form

$$\frac{\tilde{a}_1}{2\pi iN}=\frac{1}{2\pi(C_{11}C_{44})^{1/2}}\int\limits_{-b(x)}^{b(x)}dz_1P_N(x,z_1)K_0\left(|z-z_1|\left(-\frac{i\sigma B^2\omega_N}{C_{44}}\right)^{1/2}\right), \qquad (50)$$

where K_0 is a Bessel function.

In the region

$$b^{-2}\gg\sigma B^2\cdot 2\pi V/C_{44}\tilde{a}_1 \qquad (51)$$

the solution of (50) is

$$P_N(x,\ z)=P_N(x)/(b^2(x)-z^2)^{1/2},\quad b(x)=b(1-x^2/a^2)^{1/2}, \qquad (52)$$

$$P_N(x)=-\frac{i\tilde{a}_1(C_{11}C_{44})^{1/2}}{\pi N}\Big/\ln\left(\frac{4}{\gamma b(x)(-i\sigma B^2\omega_N/C_{44})^{1/2}}\right). \qquad (53)$$

Substituting Eq. (53) for $P_N(x)$ in (34), we obtain

$$P_0(0)=P_c-\frac{\tilde{a}_1(C_{11}C_{44})^{1/2}}{b}\Big/\ln\left(\frac{4}{\gamma b(\sigma B^2\cdot 2\pi V/C_{44}\tilde{a}_1)^{1/2}}\right). \qquad (54)$$

In the other limiting case

$$b^{-2}\ll\sigma B^2\cdot 2\pi V/C_{44}\tilde{a}_1, \qquad (55)$$

the Kernel K_0 in (50) can be replaced by a δ-function, and the result is the one-dimensional case c.

It follows from (1) that the motion of the vortex lattice cuts off the interaction with the defect at a distance proportional to $V^{-1/2}$. A similar screening of the interaction results also from the presence of other randomly distributed defects. Estimating the radius of this screening, we find that the previously obtained formulas for the current-voltage characteristic are valid under the condition

$$V>2\pi n^2b^2a^2\tilde{a}_1C_{44}/\sigma B^2. \qquad (56)$$

Equations (31)–(56) were obtained without allowance for the spatial dispersion of the elastic moduli. Similar formulas can be obtained also under conditions of a strongly pronounced spatial dispersion of the elastic moduli. We confine ourselves to consideration of the current-voltage characteristic in the one-dimensional case.

Equation (1) near the extremum of the function f takes, under conditions of strong spatial dispersion of the elastic modulus C_{11}, the form

$$\frac{\partial u}{\partial t}+\frac{k_h^2}{4\pi\sigma}u-\frac{\mu}{\sigma B^2}u^2-\frac{k_h^2V}{4\pi\sigma}t=0, \qquad (57)$$

where $f(u)=-f_m+\mu u^2$, and $\mu=\frac{1}{2}\partial^2f/\partial x^2$ is a derivative taken at the extremum point of the function f. Putting

$$u=\frac{B^2k_h^2}{8\pi\mu}+y\left(\frac{B^4k_h^2\sigma V}{4\pi\mu^2}\right)^{1/3},$$

$$t=\left(\frac{4\pi\sigma^2B^2}{k_h^2V\mu}\right)^{1/3}\left[t'-\frac{\mu^{1/3}}{(B^4k_h^2\sigma V/4\pi)^{1/3}}\frac{B^4k_h^4}{64\pi^2\mu}\right], \qquad (58)$$

we reduce (57) to the form

$$\partial y/\partial t'-y^2-t'=0. \qquad (59)$$

A solution of Eq. (59), satisfying the condition

$$y(-t')\to-(-t')^{1/2}$$

at $-\infty$, is

$$y=\Phi'(-t')/\Phi(-t'), \qquad (60)$$

where $\Phi(x)$ is an Airy function.

The current surface density is given by

$$J(V)B=V\int f(u)\,dt,\quad J(V)=\int\limits_{-\infty} j(V)\,dx. \qquad (61)$$

From (60) and (61) we get

$$J(V)B=J(0)B-V\left(\frac{\sigma^2B^2\cdot 4\pi}{k_h^2V\mu}\right)^{1/3}\bar{t}\left(f_m-\frac{B^4k_h^4}{64\pi^2\mu}\right), \qquad (62)$$

where $\bar{t}=2.338$ ($-\bar{t}$ is the first zero of the Airy function). The second term in (62) is connected with the delay of the vortex lattice near the extremum of the function f. It is necessary to add to the right-hand side of (62) the contribution from regions far from the extremal point. For a function f with a sharp extremum, this contribution can be obtained by the method described above:

$$-\sigma B^2V\tilde{a}_1^2. \qquad (63)$$

The critical density of the surface current is then

$$\frac{J(0)}{\tilde{a}_1}=f_m-\frac{B^2k_h^2}{8\pi}\tilde{a}_1. \qquad (64)$$

5. CONCLUSION

The value of the critical current and the shape of the current-voltage characteristic depend strongly on the character of the pinning centers. In the limiting case of weak pinning centers, when the pinning area is small compared with the lattice period, the Labusch criterion is usually not satisfied, and an individual center does not produce metastable lattice states. The critical current is small in this case and is proportional to the center density raised to a high power. The current-voltage characteristic has the universal form (1), and the pinning force increases as the velocity is increased. This case is apparently not realized in experiment, since a small number of strong pinning centers is always present. In the opposite limiting case of sufficiently large centers with properties that differ greatly from those of the matrix, plastic deformation of the vortex lattice takes place. The character of the motion of the vortices and the form of the current-voltage characteristic in this case are not clear.

We have considered above an intermediate case when the dimension of the pinning center is large, but its properties differ little from those of the matrix, for example a dislocation cluster or regions with increased impurity density. The vortex-lattice deformation due to such a defect is small and can be described by elasticity theory. However, owing to the large size of the defect, the lattice displacement can exceed the lattice period, so that metastable states can exist for an individual pinning center and the critical current has therefore a term linear in the defect concentration. It is assumed that an energy barrier to the entry of vortex layer into the region occupied by the defect exists (the barrier to the departure of the vortices from the defect into the matrix is much less effective). The maximum pressure at the barrier P_c can be expressed in terms of microscopic characteristics of the defect and of the matrix, and can be regarded as a phenomenological parameter. When the vortex lattice moves, the vortex layer becomes contiguous with the defect surface before it enters the defect. The contiguity region is an ellipse whose area is proportional to P_c^2, so that the average force and the critical current are proportional to P_c^3. With increasing lattice velocity, the average force decreases, and at low velocities this decrease is larger than the viscosity force. Voltage jumps should therefore appear on the current-voltage characteristic. This was observed in experiments.[7,8]

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Translated by J. G. Adashko