Collisionless gas in the field of a plane gravitational wave

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An exact solution is obtained for the general-relativity collisionless kinetic equation that describes a gas in the field of a plane gravitational wave of arbitrary amplitude and polarization. The gas was at equilibrium prior to the arrival of the gravitational wave. All the macroscopic characteristics of the collisionless gas in the field of the gravitational wave are calculated for a Boltzmann gas, for a gas of massless particles, for a completely degenerate Fermi gas, and for a nonrelativistic gas of relict neutrinos. It is shown that the background tails of sufficiently strong gravitational waves with amplitude $h>6\times10^{-11}\ \omega^{-1/2}$ (ω is the gravitational-wave frequency in kHz) can be relatively simply revealed by the anomalies in the spectrum of the relict radiation. It is shown that plane gravitational waves induce in a plasma a longitudinal electric field. The induced field is described by the system of the Vlasov general-relativity equations, which in the case of a cold plasma with degenerate electrons reduces to a single nonlinear differential equation of second order. A solution of this equation is obtained for a nonrelativistic plasma and describes plasma-frequency field oscillations modulated by the square of the amplitude of the gravitational waves. The amplitude of the current induced by the gravitational wave in a metallic beam with cross section area 10^4 cm^2 reaches 10^{-9} A at a gravitational-wave amplitude 10^{-12}.

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§1. INTRODUCTION

The propagation of weak high-frequency gravitational waves (GW) in a collisionless gas has by now been investigated in sufficient detail. The propagation of GW in a gas was previously considered[1,2] against the background of a flat space. In Ref. 3, however, it was shown that when solving the propagation of gravitational waves in a medium one cannot neglect the proper background of the medium itself. The interaction of the GW with the medium's own background, which leads to a change in the GW phase velocity, turns out to be no weaker (and stronger in a nonrelativistic gas) than the interaction of the ordinary type, at which scattering of the GW by individual particles takes place. With this fact into account, a local dispersion equation was obtained[3] for high-frequency GW in a weakly inhomogeneous gas:

$$\omega^2=k^2c^2+\omega_g^2, \tag{1.1}$$

where $\omega_g^2\sim\varkappa\varepsilon c^2$. This formula is only an estimate. The numerical factor of ω_g^2, as well as the sign if ω_g^2, turned out subsequently to depend significantly on the structure of the gravitational background of the gas.

The local spectrum (1.1) is applicable to GW of sufficient high frequency:

$$\omega^2\gg\omega_g^2, \tag{1.2}$$

whose group velocity differs quite insignificantly from the speed of light. In Refs. 4–7, using the approximation (1.2), a covariant WKB theory was developed, which described the propagation of weak gravitational perturbations in a collisionless and weakly collisional gas, i.e., in the case

$$\min(\tau,\omega^{-1})\ll\tau_c, \tag{1.3}$$

where τ is the duration of the gravitational-perturbation pulse and τ_c is the average time between the particle collisions. The covariant WKB theory was used as the basis for investigations[4,8,9] of GW with astrophysical objects and collisional damping of GW (Refs. 4 and 10) in a nonrelativistic and an ultrarelativistic plasma. In addition, the propagation of cosmological GW in a collisionless gas was investigated.[9,11,12]

The foregoing studies, while casting some light on the dynamics of the interaction of GW with a gas, have a more pronounced astrophysical and cosmological rather than an experimental aspect. From the point of view of the problem of detection of gravitational waves, it is more important to study the behavior of a medium in the field of a given GW, when the reaction of the detector on the GW can be neglected. The necessary condition for this neglect is precisely the inequality (1.2), and more rigorous conditions will be obtained below. During the first stages we can confine ourselves to the simplest type of GW, a plane wave, all the more since we have at hand the corresponding exact solutions[13] (although it is still not clear whether a plane GW is physically real). It is precisely the high degree of symmetry of the plane GW (three-parameter group of motions) which makes it possible to construct an exact solution of the collisionless kinetic equation and by the same token provides a unique possibility of rigorously investigating the nonlinear effects that the GW initiate in a medium.

§2. VLASOV'S EQUATIONS

The motion of a collisionless gas of charged particles in a gravitational field is described by Vlasov's system of general-relativity equations[14]

$$[\mathscr{H}_a,F_a]\equiv\frac{\partial\mathscr{H}_a}{\partial P_i}\frac{\partial F_a}{\partial x^i}-\frac{\partial\mathscr{H}_a}{\partial x^i}\frac{\partial F_a}{\partial P_i}=0, \tag{2.1}$$

$$F^{ik}{}_{,k}=-4\pi j^i/c,\quad F^{*ik}{}_{,k}=0, \tag{2.2}$$

where $\mathscr{H}_a(x,P)$ is the invariant Hamiltonian of a charged particle of species a:

$$\mathscr{H}_a(x,P)=\frac{1}{2}g^{ik}\left(P_i-\frac{e_a}{c}A_i\right)\left(P_k-\frac{e_a}{c}A_k\right). \tag{2.3}$$

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A_i is the vector potential and P_i is the generalized momentum.

Since the Poisson brackets are antisymmetrical, the invariant Hamiltonian is an integral of the kinetic equations (2.1):

$$\mathcal{H}_a(x,P)=m_a^2c^2/2, \tag{2.4}$$

so that the distribution function of particles with fixed rest mass m_a is given by

$$F_a(x,P)=f_a(x,P)\delta(\mathcal{H}_a-m_a^2c^2/2), \tag{2.5}$$

where $f_a(x, P)$ is a function that is not singular on the mass shell (2.4).

We determine the macroscopic moments relative to the invariant distribution function:

$$n_a^i(x)=\int\frac{\partial\mathcal{H}_a}{\partial P_i}F_a(x,P)\,dP \tag{2.6}$$

is the particle-number flux-density vector

$$T_a^{ik}(x)=c\int\frac{\partial\mathcal{H}_a}{\partial P_i}\frac{\partial\mathcal{H}_a}{\partial P_k}F_a(x,P)\,dP \tag{2.7}$$

is the energy-momentum tensor of the a-th component of the gas,

$$j^i(x)=\sum e_acn_a^i(x) \tag{2.8}$$

is the current-density vector, where

$$dP=(-g)^{-1/2}dP_1dP_2dP_3dP_4 \tag{2.9}$$

is the invariant volume element in momentum space.

In the synchronous reference frame, the metric of a plane GW with arbitrary polarization is described by the expression

$$ds^2=2dudv-Adx^{2^2}-Bdx^{3^2}+2Cdx^2dx^3,$$
$$u=2^{-1/2}(x^4-x^1),\quad v=2^{-1/2}(x^4+x^1), \tag{2.10}$$

where A, B, and C are functions of u and are connected by a single differential equation.

We consider first a gas without a macroscopic electromagnetic field. Then the equations of the characteristics for (2.1) in the metric (2.10) take the form

$$\frac{du}{P_u}=dP_u\Big/\frac{-\partial\mathcal{H}}{\partial u}=\frac{dP_v}{0}=\frac{dP_2}{0}=\frac{dP_3}{0}, \tag{2.11}$$

for which follow immediately the independent integrals

$$C_1=P_v,\quad C_2=P_2,\quad C_3=P_3,\quad C_4=\mathcal{H}_a. \tag{2.12}$$

These integrals suffice to write down for (2.1) an exact solution that goes over, in the absence of GW, into an isotropic distribution. Indeed, a solution of (2.1) independent of the coordinates v, x^2, and x^3 is an arbitrary function of the integrals (2.12):

$$F_a(u,P)=F_a(P_v,P_2,P_3,\mathcal{H}_a). \tag{2.13}$$

Prior to the arrival of the GW ($u\leqslant 0$) we have

$$A(u)=B(u)=1,\quad C(u)=0;\quad u\leqslant 0. \tag{2.14}$$

Assume that prior to the arrival of the GW the distribution (2.13) was equilibrium and isotropic, i.e.,

$$F_a^+(u,P)|_{u\leqslant 0}=\frac{\rho}{(2\pi\hbar)^3}\delta\left(\mathcal{H}_a-\frac{1}{2}m_a^2c^2\right)U_+(p_4-m_ac)\Big/\left[\exp\left(-\frac{\mu_a+cp_4}{T}\right)\pm 1\right], \tag{2.15}$$

where $\rho=2S+1$, S is the particle spin, T is the temperature, the plus and minus signs correspond to fermions and bosons, respectively, μ_a is the chemical potential of the component and is determined from the condition

$$N_a=\int p_4F_a(u,P)|_{u\leqslant 0}\,dP; \tag{2.16}$$

and N_a is the density of the number of particles of species a prior to the arrival of the GW.

Out of the independent integrals (2.12) we can construct the integral $\xi_a(u, P)$, which has the meaning of the total energy of the particle in the GW field:

$$\mathcal{E}_a(u,P)=\frac{c}{2^{3/2}P_v}(2\mathcal{H}_a+2P_v^2+P_2^2+P_3^2). \tag{2.17}$$

Indeed, in the absence of GW we have the relation

$$\mathcal{E}_a(u,P)|_{u\leqslant 0}=2^{-1/2}c(P_v+P_u)=cp_4. \tag{2.18}$$

Substituting in (2.17) the value of $\mathcal{H}_a$ from (2.4), we get

$$\mathcal{E}_a(u,P)=\frac{c}{2}\left(2^{1/2}P_v+\frac{m_a^2c^2+P_\perp^2}{2^{1/2}P_v}\right), \tag{2.19}$$

where $P_\perp^2=P_2^2+P_3^2$, from which it is seen that at $P_v\geqslant 0$ we have automatically $\xi_a(u,P)\geqslant m_ac^2$. We thus obtain the exact distribution function of a gas in the field of a plane GW (2.10), satisfying the initial condition (2.15):

$$F_a^+(u,P)=\frac{\rho}{(2\pi\hbar)^3}U_+(P_v)\delta\left(\mathcal{H}_a-\frac{1}{2}m_a^2c^2\right)\times\left\{\exp\left[\frac{-\mu_a+\mathcal{E}_a(u,P)}{T}\right]\pm 1\right\}^{-1}, \tag{2.20}$$

where T and μ_a are constants [μ_a is determined from (2.16)].

It is possible to construct similarly also the distribution function of a gas whose macroscopic velocity v prior to the arrival of the GW was different from zero, i.e.,

$$F_a^+(u,P)|_{u\leqslant 0}=\frac{\rho}{(2\pi\hbar)^3}U_+(p_4-m_4c)\delta\left(\mathcal{H}_a-\frac{1}{2}m_4^2c^2\right)\times\left\{\exp\left[\frac{-\mu_a+c(V,P)}{T}\right]\pm 1\right\}^{-1}, \tag{2.21}$$

where V^i is the macroscopic four-velocity of the gas. The distribution function satisfying the initial condition (2.21) is obtained from(2.20) by replacing $\xi_a(u, P)$ with expression

$$\mathcal{E}_a(u,P)(V^4-V^1)+c(2^{1/2}P_vV^1+P_2V^2+P_3V^3), \tag{2.22}$$

which goes over into $c(V, P)$ at $u\leqslant 0$.

§3. MACROSCOPIC MOMENTS

The Hamiltonian takes in the coordinates u, v, x^2, and x^3 the form

$$\mathcal{H}_a(u,P)=P_uP_v-S_\perp^2/2, \tag{3.1}$$

where

$$S_\perp^2=-g_\perp^{\alpha\beta}P_\alpha P_\beta=\frac{AP_3^2+BP_2^2+2CP_2P_3}{AB-C^2}. \tag{3.2}$$

It is easy to show that $S_1^2 \geq 0$ at $AB - C^2 \geq 0$. We substitute (3.1) in the mass-shell equation (2.4) and solve it with respect to P_u; the expressions for the macroscopic moments (2.6) and (2.7) are transformed into

$$n_i^+(u) = (-g)^{-1/2} \int_0^\infty \frac{dP_v}{P_v} \iint_{-\infty}^{+\infty} dP_2\, dP_3\, f_a(u,P) P_i, \qquad (3.3)$$

$$T_{ik}^+(u) = \frac{c}{(-g)^{1/2}} \int_0^\infty \frac{dP_v}{P_v} \iint_{-\infty}^{+\infty} dP_2\, dP_3\, f_a(u,P) P_i P_k. \qquad (3.4)$$

The integration in (3.3) and (3.4) is easiest to carry out by replacing P_v by a new variable

$$P_v = 2^{-1/2}[P_1^* + (m^2c^2 + P^2)^{1/2}],\quad P^2 = (P_1^*)^2 + P_\perp^2,$$

and then going over to a spherical coordinate system. In these coordinates

$$\mathscr{E}_a(u,P) = \mathscr{E}_a(P) = c(m_a^2c^2 + P^2)^{1/2}. \qquad (3.5)$$

Calculating the integrals (3.3) and (3.4) relative to the distribution function (2.21), we obtain after straightforward but cumbersome manipulations expressions for the nonzero components n_i and T_{ik}:

$$n_v = \frac{N}{2^{1/2}L^2},\quad n_u = \frac{N}{2^{1/2}L^2}[1 + (1-\mathscr{L}_n)(K^2-1)],$$

$$T_{uu} = \frac{1}{2L^2}\left\{\varepsilon - 3P - 4K^2(\varepsilon - 3P - \mathscr{L}_T) + 2\left(3K^4 - \frac{1}{L^2}\right)(\varepsilon - 2P - \mathscr{L}_T)\right\}.$$

$$T_{22} = T_{33} = \frac{P}{L^2},\quad T_{vv} = \frac{1}{2L^2}(P + \varepsilon),\quad T_{uv} = \frac{1}{2L^2}(\varepsilon - 3P + 2PK^2), \qquad (3.6)$$

where

$$L^2 = (AB - C^2)^{1/2},\quad K^2 = (A+B)/2L^4,$$

N, P, and ε are respectively the particle-number density, the pressure, and the energy density of each gas component in the unperturbed state (the subscript a has been left out for simplicity). The numbers $\mathscr{L}_n$ and $\mathscr{L}_T$ have no hydrodynamic analog. For them we have

$$\mathscr{L}_n = \frac{4\pi\rho m^2c^3}{(2\pi\hbar)^3N} \int_0^\infty \ln\left(\frac{\mathscr{E} + cp}{mc^2}\right) p\, dp \Big/ \mathscr{E}\left[\exp\left(-\frac{\mu + \mathscr{E}}{T}\right) \pm 1\right],$$

$$\mathscr{L}_T = \frac{4\pi\rho m^2c^3}{(2\pi\hbar)^3} \int_0^\infty \ln\left(\frac{\mathscr{E} + cp}{mc^2}\right) p\, dp \Big/ \left[\exp\left(-\frac{\mu + \mathscr{E}}{T}\right) \pm 1\right],$$

where ξ must be replaced by $\xi_a(p)$ from (3.5). In the ultrarelativistic limit $\xi_a = m_a c/\langle p\rangle \to 0$ these numbers tend to zero like $\xi_a^2 \ln \xi_a$, and for massless particles they are strictly equal to zero. We write down the expressions for these numbers in a number of important cases.

1. Boltzmann gas ($f_a = \text{const} \cdot \exp(-\xi_a/T)$):

$$\mathscr{L}_n = \frac{K_0(\lambda)}{K_2(\lambda)},\quad \mathscr{L}_T = \frac{Nmc^2}{K_2(\lambda)}\left[K_1(\lambda) + \frac{K_0(\lambda)}{\lambda}\right], \qquad (3.8)$$

where $\lambda = mc^2/T$, and $K_n(\lambda)$ are MacDonald functions.

2. Degenerate Fermi Gas ($\mu_F/T \to \infty$):

$$\mathscr{L}_n = \frac{\rho(mc)^3}{2\pi^2\hbar^3N}\left[\frac{\mu_F}{mc^2}\ln\frac{cp_F + \mu_F}{mc^2} - \frac{p_F}{mc}\right],$$
$$\mathscr{L}_T = \frac{\rho(mc)^4c}{8\pi^2\hbar^3}\left[\left(1 + \frac{2p_F^2}{m^2c^2}\right)\ln\frac{cp_F + \mu_F}{mc^2} - \frac{p_F\mu_F}{m^2c^3}\right], \qquad (3.9)$$

where p_F and μ_F are the Fermi momentum and energy (with allowance for the rest mass).

3. Gas of nonrelativistic relict neutrinos torn away from matter during the ultrarelativistic expansion stage. In the calculation of the integrals (3.7) in this case, the distribution over the momenta must be taken to be ultrarelativistic with a relict-photon temperature $T_\gamma(t)(\mu_\nu = 0)$, and the expression for $\xi_\nu(P)$ should be taken nonrelativistic[15]:

$$N_\nu = \frac{3\rho\zeta(3)}{4\pi^2}\left(\frac{T_\gamma}{\hbar c}\right)^3,\quad P_\nu = N_\nu \frac{T_\gamma^2}{m_\nu c^2}\frac{\zeta(5)}{\zeta(3)};$$
$$\varepsilon_\nu = N_\nu m_\nu c^2 + \frac{3}{2}P_\nu,\quad \mathscr{L}_{T_\nu} = N_\nu m_\nu c^2 - \frac{1}{2}P_\nu; \qquad (3.10)$$
$$\mathscr{L}_{n_\nu} = 1 - \frac{45\zeta(5)}{4\zeta(3)}\left(\frac{T_\gamma}{m_\nu c^2}\right)^2.$$

We proceed now to an analysis of the obtained formulas. In Cartesian coordinates x^1 and x^4 we have from (3.6)

$$n^1(u) = \frac{1}{2L^2}N(K^2-1)(1-\mathscr{L}_n),\quad n^4(u) = \frac{N}{L^2} + n^1. \qquad (3.11)$$

Prior to arrival of the GW we have $K = 1$, i.e., $n^1|_{u\leq 0} = 0$, there is no gas flow. When the GW appears $K^2 - 1 \neq 0$ and the flow sets in. It is easily seen that this effect is nonlinear in the GW amplitude. For weak GW ($A = 1 + \beta$; $B = 1 - \beta$; β, $C \ll 1$) (Ref. 16) we have from (3.11).

$$n^1(u) \approx N(1 - \mathscr{L}_n)h^2/2,$$

where $h^2 = \beta^2 + C^2$. In addition, this flux has a purely relativistic character. In all the considered cases as $\xi_a \to \infty$ we have $\mathscr{L}_n \to 1$ and the flux vanishes, and at $\xi_a \to 0$ the flux is maximal. Since the particle-number flux depends only on the variable u, the GW transports the perturbation of the particle-number density at the speed of light. The kinematic velocity $u_k^\alpha = n^\alpha/n^4$ of the medium,[17] however, is equal to

$$v_k = \frac{c}{2}\frac{(1-\mathscr{L}_n)(K^2-1)}{1 + \frac{1}{2}(1-\mathscr{L}_n)(K^2-1)} \qquad (3.13)$$

and at $h^2 \ll 1$ we have

$$v_k = \tfrac{1}{2}ch^2(1 - \mathscr{L}_n) \ll c.$$

After passage of the GW packet[13]

$$A = B = L^2(u),\quad C = 0,\quad L = 1 - u/u_*,$$

therefore at $u < 2u_*$ we have $K^2 - 1 > 0$. From the cases considered above it follows that $\mathscr{L}_n \leq 1$ always: thus, at $u < 2u_*$ the GW drags the particles in the direction of its propagation ($v_k > 0$). At the point $u = u_* \sim 1/h'^2 c\tau$ (Ref. 13), a singular state ($v_c \to c$) sets in behind the GW front the gas is stopped at the point $u = 2u_*$, and at $u > 2u_*$ the gas moves in a direction opposite to the propagation of the GW, the gas velocity being constant at $u \to \infty (t \to +\infty)$ and equal to $-c$ for an ultrarelativistic gas. An analysis of the components of the energy-momentum tensor (3.6) also reveals the presence of a singularity at $L^2 = 0$. It is easily seen, e.g., that the dynamic gas velocity, defined as the eigenvector of the energy-momentum tensor,[17] also tends to the speed of light at $u = u_*$.

The presence of a singularity in the distribution of matter makes it necessary to analyze the main assumptions of §2. There were two such assumptions: 1) the gas is collisionless and 2) the GW propagates in vacuum. Inside the GW packet ($h \neq 0$) these two assumptions are valid if conditions (1.2) and (1.3) are satisfied, however, consideration of the wave at times $t \sim t_* = u_*/c$ presupposes a sufficiently large amplitude of the GW. An analysis of the Einstein equations in the medium yields a criterion for the vacuum and collisionless description of a GW in a gas at times $t \sim t_*$:

$$h^2 \gg 1/\tau_c \tau \omega^2, \quad h^2 \gg \omega_g/\omega^2 \tau. \tag{3.14}$$

These relations are not satisfied under laboratory conditions, but may be satisfied under cosmic conditions for sufficiently strong GW. For relict massive neutrinos, these conditions take the form

$$h > 6 \cdot 10^{-11} \omega_3^{-1} \tau_3^{-1/2}, \tag{3.15}$$

where ω_3 is the frequency of the GW in kHz, τ_3 is the duration of the GW pulse in microseconds. From Einstein's equations we can obtain also an estimate for $L_{\min}$, at which the vacuum character of the GW is violated:

$$L_{min} = (t_* - t_{max})/t_* \sim \omega_g^2/2h^4\omega^4\tau^2. \tag{3.16}$$

§4. RELICT RADIATION IN THE FIELD OF A STRONG GW

Strong GW with amplitude satisfying the conditions (3.14) are revealed by anomalies in the electromagnetic relict radiation (RR). Indeed, the condition (3.14), when applied to the cosmic background, means that the time of arrival of the singular front t_* should be much shorter than the cosmological time $t \sim 1/\omega_g$. Thus, GW with phonon tails should have local and not cosmological origin. We consider the gas of relict photons after the passage of a GW packet ($K = 1/L^2$). A geodesic observer at rest in the metric (2.10) will register, according to (3.6), the following orthogonal-reference projections of the energy-momentum tensor of the relict radiation:

$$T_{(2)(2)} = T_{(3)(3)} = \frac{\varepsilon}{3L^6}, \quad T_{(1)(4)} = -\frac{\varepsilon}{3L^2}\left(\frac{1}{L^4} - 1\right),$$
$$T_{(1)(1)} = \frac{\varepsilon}{3L^2}\left(\frac{1}{L^4} - \frac{1}{L^2} + 1\right), \quad T_{(4)(4)} = \frac{\varepsilon}{3L^2}\left(\frac{1}{L^4} + \frac{1}{L^2} + 1\right). \tag{4.1}$$

Directly behind the GW packet we have $L \approx = 1$ and the RR is isotropic; in the course of time, the observer will record an increase of the pressure, of the energy density, and of the anisotropy of the flux. At $L \ll 1 (t \to t_*)$ the RR is strongly anisotropic. Thus, when (3.15) is satisfied, the observer records an increase of the RR density by a factor

$$k = 1/3L_{min}^6. \tag{4.2}$$

Even under the weak condition (3.15), formula (4.2) will show discernible bursts of the RR. An observer with a high-directivity and narrow-band antenna will register, however, not the components of the RR energy-momentum tensor, but the spectral distribution of the RR energy. Putting $cP_4 = \omega\hbar$ in (2.20) and changing over to the orthogonal reference projections of the momentum, we write down the spectral distribution of the photon energies after passage of the GW packet:

$$d\mathscr{E}(\omega) = \frac{\hbar\omega^3 \sin\theta \, d\theta \, d\omega}{2\pi^2 c^3 [\exp(\hbar\omega'/T) - 1]}, \tag{4.3}$$

$$\omega' = \omega \frac{(1 - \cos\theta)^2 + L^2(\sin^2\theta)}{2(1 - \cos\theta)}, \tag{4.4}$$

where θ is the angle between the propagation direction of the GW and the momentum of the photon ($\pi - \theta$ is the angle between the directions of the GW and of the antenna). Directly behind the GW packet we have $L \approx 1$ and $\omega' \approx \omega$. At $\theta = \pi$ and at arbitrary L we have again $\omega' = \omega$, and the distribution does not differ in any way from the initial one. At $L \ll 1$, however, the distribution is strongly anisotropic:

$$\omega' \approx \omega(1 - \cos\theta) \text{ at } \theta \gg L \text{ and } \omega' \approx \omega L^2 \text{ at } \theta \lesssim 2L.$$

Thus, at $\theta \lesssim 2L$ the maximum of the RR energy shifts towards higher frequencies:

$$\hbar\omega_{max} = 2.82 T_\gamma / L^2. \tag{4.5}$$

Since the characteristic time scale t_* of the background tail of the GW is usually, as can be readily seen, larger than the terrestrial time scales, passage of a strong GW in the past ($t = -t_*$) will lead at present to establishment of anisotropy of the RR.

§5. ELECTRIC FIELD INDUCED IN A PLASMA BY A STRONG GW

We consider the action of a GW on a collisionless plasma that was isotropic and electrically neutral prior to the arrival of the GW. When the GW enters such a plasma there is produced, according to (2.8) and (3.6), a longitudinal electric current.

$$j^1(u) = j^4(u) = -\frac{c}{2L^2}(K^2 - 1) \sum e_a \mathscr{L}_{n_a} N_a, \tag{5.1}$$

which vanishes in a nonrelativistic ($\xi_a \to \infty$) and ultrarelativistic plasma (all $\xi_a \to 0$), and is maximal for a plasma with ultrarelativistic electrons and nonrelativistic ions. It follows from (3.8) and (3.9) that at $0 < u < 2u_*$ the GW induces in a collisionless plasma a negative charge density and carries the latter at the speed of light.[18]

Since an electric current and the associated longitudinal electric field are produced in the plasma, it is necessary to solve the self-consistent system of Vlasov's equations (2.1) and (2.2). We seek stationary solutions of this system, assuming

$$F_{41} = E(u). \tag{5.2}$$

As seen from Maxwell's equations (2.2), the necessary and sufficient condition for the satisfaction of (5.2) is isotropy of the current-density vector, i.e.,

$$j_v(u) = 0. \tag{5.3}$$

The continuity equation takes in the coordinates u and v the form

$$(-g)^{-1/2} \frac{d}{du}((-g)^{1/2} j_v) = 0,$$

from which we get

$$j_v(u) = \text{const}(-g)^{-1/2}.$$

But since $j_v = 0$ prior to the arrival of the GW, the constant is zero and consequently (5, 3) is automatically satisfied on account of the kinetic equations (2. 1). We put $E(u)=dA_v/du$, where A_v is the potential of the electric field in the Hamiltonian

$$\mathcal{H}_a=\left(P_v-\frac{e_a}{c}A_v\right)P_u-\frac{1}{2}S_\perp^2. \qquad (5.4)$$

This potential does not satisfy the Lorentz gauge condition, but by means of the gauge transformation

$$A_i'=A_v\delta_i^v+\partial_i\varphi,\quad \varphi=-\frac{v}{2}\left(A_v+\frac{1}{L}\int L'A_v\,du\right)$$

a Lorentz gauge can be attained.

It is easily seen that an exact solution of (2. 1) with the Hamiltonian (5. 4) is again (2. 20). Just as before, the condition that selects the states in momentum space is $P_v\geqslant 0$, since P_v is an integral of the motion. Thus, we write down the exact value for the current-density vector of the charged collisionless particles in the GW field (2. 10) and in a longitudinal electric field $E(u)$:

$$n_{au}=\frac{\pi\rho}{2(2\pi\hbar)^3L^2}\int_0^\infty dP_\perp^2(m_a^2c^2+K^2P_\perp^2)\int_0^\infty \operatorname{sgn}\left(P_v-\frac{e_a}{c}A_v\right)dP_v\left\{\left(P_v-\frac{e_a}{c}A_v\right)^2\left[\exp\left(\frac{-\mu_a+\mathcal{E}_a}{T}\right)\pm1\right]\right\}^{-1}. \qquad (5.5)$$

As follows from the results of §4, the flux of nonrelativistic particles in the GW is proportional to $1/\xi_a^2\to 0$, therefore the ion current can be neglected in the expression for the total current.

To simplify the procedure, we consider a plasma consisting of fully degenerate electrons and cold ions (metal at $T\to 0$). [At $T=0$ the GW does not displace the ion core of the plasma $(n^2_{(i)}=0)$]. Under condition of complete degeneracy $(\mu/T\to\infty)$, all the states with $\xi(P)-\mu>0$ are unoccupied. Putting

$$(\mu_F-2^{1/2}eA_v)/cp_F\equiv\psi>1 \qquad (5.6)$$

and integrating in (5. 5) inside the Fermi surface, we reduce the only nontrivial Maxwell's equation to the form

$$\frac{d}{du}\left(L^2\frac{dA_v}{du}\right)=-2^{1/2}\pi Ne\left\{K^2-1+3\left[\frac{m^2c^2}{p_F^2}-K^2(\psi^2-1)\right]\left(\frac{\psi}{2}\ln\frac{\psi+1}{\psi-1}-1\right)\right\}. \qquad (5.7)$$

The solution of this equation under the initial conditions

$$A_v(0)=0,\quad \left.\frac{dA_v}{du}\right|_{u=0}=0 \qquad (5.8)$$

describes exactly the electric field induced by a plane GW in an unbounded electrically neutral plasma with degenerate electrons.

We consider now a nonrelativistic Fermi gas ($p_F\ll mc$) and a weak electric field $|eA_v|\ll mc^2$. Expanding the right-hand side of (5. 7) in a Taylor series in powers of $1/\psi$ and retaining terms up to $1/\psi^5$, and then linearizing the equation with respect to A_v, we reduce it to the form

$$\frac{d}{du}\left(L^2\frac{dA_v}{du}\right)+\frac{2\omega_p^2}{c^2}A_v=-\frac{2^{5/2}\pi Nep_F^2}{5m^2c^2}(K^2-1), \qquad (5.9)$$

where ω_p is the plasma frequency. The solution of this equation, satisfying the initial conditions (5. 8) inside the GW packet ($L\approx 1$), is of the form

$$A_v(\bar u)=\frac{2^{1/2}\pi Nep_F^2c}{5m^2c^2\omega_p}\int_0^{\bar u}[K^2(\bar u')-1]\sin\frac{\omega_p}{c}(\bar u'-\bar u)\,d\bar u', \qquad (5.10)$$

$$E(u)=-\frac{4\pi Nep_F^2}{5m^2c^2}\int_0^{\bar u}[K^2(\bar u')-1]\cos\frac{\omega_p}{c}(\bar u'-\bar u)\,d\bar u', \qquad (5.11)$$

where $\bar u=ct-x$. Substituting the value of $A_v(u)$ from (5. 10) in the value of the induced current [right-hand side of (5. 7)], we obtain the current induced by the GW, with allowance for the plasma response.

$$j^1=\frac{Necp_F^2}{5m^2c^2}[K^2(\bar u)-1]\cos\frac{\omega_p\bar u}{c}, \qquad (5.12)$$

where we have put $\omega\ll\omega_p$. Thus, the current induced by the GW constitutes oscillations with plasma frequency, propagating at the speed of light and modulated by the function $K^2(\bar u)-1\approx h^2(\bar u)$. The amplitude of the total current excited in a metal beam with cross section area 10^4 cm^2, at $N=10^{24}$ cm^{-3} and $h=10^{-12}$, amounts to 1.4×10^{-9} A. Although the amplitude of the current is appreciable, its frequency is also high ($\omega_p\sim 6\times10^{16}$ sec^{-1}), so that the possibility of its detection remains uncertain.

It is easy to obtain also an exact solution of Eq. (5. 9) after the passage of the GW packet:

$$E(u)=\frac{\bar A}{L^3}+\frac{1}{L^{1/2}}\left(\bar B\cos a\ln\frac{L_0}{L}+\bar C\sin a\ln\frac{L_0}{L}\right)$$

$$a=\left[\left(\frac{\omega_p u_*}{c}\right)^2-\frac{1}{4}\right]^{1/2},$$

where $\bar A$, $\bar B$, and $\bar C$ are certain constants.

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