Effect of boundaries in momentum space on energy and particle-number fluxes in a weak turbulence regime

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We consider the contributions to the total energy and particle-number fluxes which are connected with the boundedness of momentum space in the inertial range. We show that boundary effects make a macroscopically large contribution even in a local case.

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1. If the collision integral does not contain dimensional parameters, in the classical limit of occupation numbers $N \gg 1$ or $N \ll 1$ there are stationary power-law solutions with constant energy or particle-number fluxes in momentum space. This phenomenon is rather universal, it is met with in hydrodynamics[1-3], plasma physics,[4-6] gas kinetics,[7] and solid state physics[8] (see also the review article by Kadomtsev and Kontorovich[9]). By the nature of the problem (the presence of a source and a sink) the region in momentum space is bounded by some values $k_{\min}$ and $k_{\max}$. More precisely, the whole of momentum space is divided by the values $k_{\min}$ and $k_{\max}$ into three regions where the interaction between the degrees of freedom from the ranges $(0, k_{\min})$ and $(k_{\min}, k_{\max})$, and also $(k_{\min}, k_{\max})$ and $(k_{\max}, \infty)$ is effectively described as the occurrence of a source and a sink. However, the interaction of the degrees of freedom in the range $(k_{\min}, k_{\max})$ with one another is the inertial range. As only local solutions, in which the interaction between wave vectors of one order of magnitude is most important, are of physical interest, the effect of the boundaries is usually neglected completely. In the present paper we wish to show that allowance for boundary effects makes a macroscopically large contribution to the energy and particle number total fluxes.

2. We select in momentum space values k_l and k_u such that $k_{\min} \ll k_l \ll k_u \ll k_{\max}$ and we consider a "shortened" collision integral in which the integration is only over the region between k_l and k_u. As the collision integral does not contain dimensional parameters, we may assume without any essential restriction of generality, that it has, in the case of a power-law distribution function, the form

$$I_{coll} = f(\alpha) k^{\alpha} + \sum_i a_i(\alpha) k_l^{\lambda_{1i}} k^{\alpha-\lambda_{1i}} + \sum_i b_i(\alpha) k_u^{-\mu_{1i}} k^{\alpha+\mu_{1i}} + \sum_i c_i(\alpha) k_l^{\lambda_{2i}} k_u^{-\mu_{2i}} k^{\alpha-\lambda_{2i}+\mu_{2i}}, \quad (1)$$

where k is an external fixed momentum.

For local distributions all the exponents $\lambda_{1i}, \lambda_{2i}, \mu_{1i}, \mu_{2i} > 0$. Solutions which make the function $f(\alpha)$ vanish are stationary. In what follows we restrict ourselves for the sake of argument, to a growing spectrum:

$$\omega(\mathbf{k}) \propto k^{\rho}, \quad \rho > 0. \quad (2)$$

For solutions with an energy flux $f(\alpha)$ vanishes when

$$\alpha_0 + \rho + d = 0, \quad (3)$$

where d is the dimensionality of momentum space. For solutions with a particle number flux we have

$$\alpha_0 + d = 0. \quad (4)$$

We consider what contribution is made by boundary effects to the total energy flux. We multiply (1) by k^{ρ} and integrate over the region between k_l and k_u. Taking the limit in α in the case (3) and neglecting positive powers of the small ratio k_l/k_u we get

$$\dot{E}_{\text{bound}} \propto \sum_i \frac{a_i(\alpha_0)}{\lambda_{1i}} + \sum_i \frac{b_i(\alpha_0)}{\mu_{1i}}. \quad (5)$$

It follows from (5) that there is connected with the boundaries a universal energy flux which is independent of k_l and k_u and which must be taken into account when we take the physical limit $k_l \to k_{\min} \to 0$ and $k_u \to k_{\max} \to \infty$. Using (5), the total energy balance is now

$$\dot{E}_{\text{ext}} = \dot{E}_{\text{ext}} + \dot{E}_{\text{bound}}, \quad (6)$$

where $\dot{E}_{\text{in}}$ is the constant energy flux in the inertial range and is described by the first term in (1) and evaluated by the methods of the papers by Karas' and Kats.[10,11] Equation (6) is very important for an exact normalization of the distribution function and for joining the solutions in the different regions of k-space.

In the case (4) we get a physically clear result: the energy flux is proportional to the particle flux multiplied by the maximum energy of the system:

$$\dot{E}_{\text{bound}} \sim \dot{N} \omega_u \propto \dot{N} k_u^{\rho}. \quad (7)$$

When evaluating the particle flux in case (4) we get an answer which is exactly analogous to (5) and in the case (3) we have

$$\dot{N}_{\text{bound}} \sim \dot{E}/\omega_l \propto \dot{E} k_l^{-\rho}. \quad (8)$$

Extrapolating (7), (8) to the region $k_l \to k_{\min}$ and $k_u \to k_{\max}$ is, of course, admissible only as an order of magnitude estimate, because the distribution function is distorted at the boundaries. It is clear from Eqs. (7) and (8) that the concept of a point source and an infinitely removed sink has a restricted applicability. Of basic interest is Eq. (8) since the phenomenon of the accumulation or outflow of particles with small k is a gross and easily registered experimental characteristic. It is not excluded that the accumulation of small $-k$

plasmons in the collapse case[12] and of magnons in ferromagnetic-resonance experiments[8,13] is to a large extent due to this effect.

3. We consider the concrete example of a collision integral in Landau's form[14] for the case of identical particles. One must then, however, recognize that integration by parts was used in deriving the final expression in Ref. 14 and a number of terms which play an important role in our considerations were dropped. It is therefore convenient to use a somewhat different scheme in which we do not integrate by parts:[15]

$$I_{coll}=\frac{4\pi ne^4}{m^2}\lambda\left[-\frac{\partial}{\partial v_i}\left(f\frac{\partial h}{\partial v_i}\right)+\frac{1}{2}\frac{\partial^2}{\partial v_i\,\partial v_j}\left(f\frac{\partial^2 g}{\partial v_i\,\partial v_j}\right)\right], \tag{9}$$

where $\lambda=\ln(T^{3/2}/n^{1/2}e^3)$ is the Coulomb logarithm,

$$g(\mathbf{v})=\int d\mathbf{v}'\,f(\mathbf{v}')\,|\mathbf{v}-\mathbf{v}'|, \tag{10}$$

$$h(\mathbf{v})=2\int\frac{d\mathbf{v}'}{|\mathbf{v}-\mathbf{v}'|}f(\mathbf{v}'). \tag{11}$$

Substituting the distribution function in the form $f=Av^\alpha$, we get

$$I_{coll}=\frac{16\pi^2ne^4}{m^2}\lambda A^2\left[\frac{2(2\alpha+3)(2\alpha+5)}{(\alpha+2)(\alpha+3)(\alpha+5)}v^{2\alpha}-\frac{\alpha}{\alpha+3}v^{\alpha-3}v_l^{\alpha+3}-\frac{\alpha(\alpha-2)}{3(\alpha+5)}v^{\alpha-5}v_l^{\alpha+5}+\frac{\alpha(\alpha+1)}{3(\alpha+2)}v^{\alpha-2}v_u^{\alpha+2}\right]. \tag{12}$$

The difference between this expression and the one given in Ref. 7 is connected with a somewhat different choice of normalization and with the already mentioned integration by parts. In Ref. 7 a local solution $f\propto v^{-5/2}$ was obtained for the given collision integral. Evaluation of the total energy flux for this distribution function gives

$$\dot{E}=\frac{32\pi^3ne^4}{m}\lambda A^2\left(-\frac{32}{5}+\frac{22}{5}\right). \tag{13}$$

The first term in (13) was found in Ref. 7, the second one is connected with boundary effects. For the particle flux we find that boundary effects operate on the side of the outflow of particles with small velocities:

$$\dot{N}\sim\dot{E}/mv_{min}^2. \tag{14}$$

4. The whole scheme can, of course, be used also for the case of small weakly turbulent deviations from an equilibrium distribution function. Thus, even for local distributions the effect of the boundaries makes an important contribution to the total energy and particle-number balance and in some cases [Eq. (7)] plays a decisive role. Experimental observation of storage or outflow of particles with small momenta may serve as a rough test for the registration of a transition into a weakly turbulent state with an energy flux across the spectrum.

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